A numerical algorithm for the explicit calculation of SU(*N*) and SL(*N*, ℂ) Clebsch–Gordan coefficients

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We present an algorithm for the explicit numerical calculation of SU(N) and SL(N, ℂ) Clebsch–Gordan coefficients, based on the *Gelfand–Tsetlin pattern* calculus. Our algorithm is well suited for numerical implementation; we include a computer code in an appendix. Our exposition presumes only familiarity with the representation theory of SU(2). © *2011 American Institute of Physics*. [doi:10.1063/1.3521562]

I. INTRODUCTION

Clebsch–Gordan coefficients (CGCs) arise when decomposing the tensor product $\mathbb{V}^S \otimes \mathbb{V}^{S'}$ of the representation spaces of two irreducible representations (*irreps*) S and S' of some group into a direct sum $\mathbb{V}^{S_1''} \oplus \cdots \oplus \mathbb{V}^{S_r''}$ of irreducible representation spaces. They describe the corresponding basis transformation from a tensor product basis $\{|M \otimes M'\rangle\}$ to a basis $\{|M''\rangle\}$ which explicitly accomplishes this decomposition.

CGCs are familiar to physicists in the context of angular momentum coupling, in which the direct product of two irreps of the SU(2) group is decomposed into a direct sum of irreps. SU(3) Clebsch–Gordan coefficients arise, for example, in the context of quantum chromodynamics, while SU(N) CGCs, for general N, appear in the construction of unifying theories whose symmetries contain the SU(3) × SU(2) × $U(1)$ standard model as a subgroup.[1] SU(N) CGCs are also useful for the numerical treatment of models with SU(N) symmetry where they arise when exploiting the Wigner–Eckart theorem to simplify the calculation of matrix elements of the Hamiltonian. Such a situation arises, for example, in the numerical treatment of SU(N)-symmetric quantum impurity models using the numerical renormalization group.[2] Such models can be mapped onto SU(N)-symmetric, half-infinite quantum chains, with hopping strengths that decrease exponentially along the chain. The Hamiltonian is diagonalized numerically in an iterative fashion, requiring the explicit calculation of matrix elements of the Hamiltonian of subchains of increasing length. The efficiency of this process can be increased dramatically by exploiting the Wigner–Eckart theorem, which requires knowledge of the relevant Clebsch–Gordan coefficients. (Details of how to implement SU(N) symmetries within the context of the numerical renormalization group will be published elsewhere.) Similarly, tremendous gains in efficiency would result from developing SU(N)-symmetric implementations of the density matrix renormalization group for treating generic quantum chain models,[3,4] or generalizations of this approach for treating two-dimensional tensor network models.[5]

For explicit calculations with models having SU(N) symmetry, explicit tables of SU(N) Clebsch–Gordan coefficients are needed. Their calculation is a problem of applied representation theory of Lie groups that has been solved, in principle, long ago.[6–10] For example, for SU(2), Racah[11] has found an explicit formula that gives the CGCs for the direct product decomposition of two arbitrary irreps S and S'. For SU(N), explicit CGC formulas exist for certain special cases, e.g., where S' is the defining representation.[12–14] Moreover, symbolic packages such as the program "Lie" (Ref. 15) also allow the computation of certain CGCs but rather have been conceived as

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a general-purpose software for manipulating Lie algebras than a high-speed implementation for calculating CGCs. However, for the general case no explicit CGC formulas are known that would constitute a generalization of Racah's results to arbitrary N, S, and S'.

The present paper describes a numerical solution to this problem, by presenting an elementary but efficient algorithm (and a computer implementation thereof) for producing explicit tables of CGCs arising in the direct product decomposition of two arbitrary SU(N) irreps, for arbitrary N. (Since SU(N) and SL(N, $\mathbb{C}$) have the same CGCs, our algorithm also applies to the latter, but for definiteness we shall usually refer only to the former.) Our work is addressed at a readership of physicists. Our algorithm uses only elementary facts from SU(N) representation theory, which we introduce and summarize as needed, presuming only knowledge of SU(2) representation theory at a level conveyed in standard quantum mechanics textbooks. Previous attempts at formulating an algorithm for calculating SU(N) CGCs are either not sufficiently general for our purposes[16,17] or require mathematical methods[18] much more advanced than ours, far beyond the scope of a standard physics education.

We begin in Sec. II by formulating the problem to be solved in rather general terms. To set the scene for its solution, Secs. III–VII summarize the various elements of SU(N) representation theory (without proofs, since this is all textbook material). First, in Sec. III we review the calculation of SU(2) CGCs using a strategy that can readily be generalized to the case of SU(N). Then we proceed to SU(N) representation theory and review in Secs, IV–VII a scheme, due to Gelfand and Tsetlin (GT),[19] for labeling the generators of the corresponding Lie algebra $\mathfrak{su}(N)$, its irreps and the states in each irrep. The GT-scheme is convenient for our purposes since it yields explicit matrix representations for any SU(N) irrep (Eqs. (28) and (29) below). With these in hand, we are finally in a position to formulate, in Secs. VIII–XII, our novel algorithm for computing SU(N) CGCs: it is simply a suitably generalized version of the SU(2) strategy of Sec. III.

The main text is supplemented by several technical appendices. Appendix A reviews the relation between the GT-patterns used in the text and Young tableaux, with which physicists are perhaps somewhat more familiar. Appendix B deals with the Littlewood–Richardson rule for determining which irreps $\mathbb{V}^{S''}$ occur in the decomposition $\mathbb{V}^S \otimes \mathbb{V}^{S'}$. Appendix C describes two algorithms, needed for indexing purposes, which map the labels of irreps and of carrier states, respectively, onto natural numbers. Finally, Appendix D, which is available in electronic form,[20] gives the source code for our computer implementation, written in C++. As a service to potential users, we have set up a website[21] containing an interactive "CGC-generator." It allows visitors to perform a number of tasks on input data of their own choice, such as finding all irreps S'' occuring in the decomposition of $S \otimes S'$ or finding the complete set of CGCs arising in the decomposition of $S \otimes S'$.

II. STATEMENT OF THE PROBLEM

To fix notation, let us state the problem we wish to solve for a general matrix Lie group $\mathcal{G}$. (In subsequent sections, we restrict our attention to $\mathcal{G} = \mathrm{SU}(N)$ or $\mathrm{SL}(N, \mathbb{C})$.) Let S be an irrep label that distinguishes different irreps of $\mathcal{G}$ and d_S the dimension of irrep S. Let $\mathbb{V}^S = \mathrm{span}\{|M\rangle\}$ denote the carrier space for S, spanned by d_S carrier states $|M\rangle$, where the label M will be understood to specify both the irrep S and a particular state in its carrier space. (This will be made explicit in subsequent sections.) Note that, throughout this paper, we adopt the viewpoint of quantum mechanics, where we consider only representations on *complex* vector spaces. Besides, a state is to be understood as a one-dimensional subspace, not a vector. However, we pick a representative vector $|M\rangle$ of each such subspace and subsequently treat a state as a vector. We assume the inner product of two such normalized vectors $|M\rangle$ and $|M'\rangle$ to be given by $\langle M|M'\rangle = \delta_{M,M'}$ unless noted otherwise.

The action of a group element $g \in \mathcal{G}$ can be represented on $\mathbb{V}^S$ as a linear transformation,

$$g : |M\rangle \rightarrow \sum_{M'} (U_g^S)_{MM'} |M'\rangle, \tag{1}$$

where the U_g^S are $d_S \times d_S$ dimensional unitary matrices respecting the group structure $U_{g_1}^S U_{g_2}^S = U_{g_1 g_2}^S$.

Now consider the direct product of two carrier spaces, $\mathbb{V} \otimes \mathbb{V}' = \text{span}\{|M \otimes M'\rangle\}$, of dimension $d_S \cdot d_{S'}$. We are interested in its decomposition into a direct sum of carrier spaces $\mathbb{V}^{S''}$ of irreps S'',

$$\mathbb{V}^S \otimes \mathbb{V}^{S'} = \bigoplus_{S''} \bigoplus_{\alpha=1}^{N_{SS'}^{S''}} \mathbb{V}^{S'',\alpha} \equiv \bigoplus_{S''} N_{SS'}^{S''} \mathbb{V}^{S''}. \tag{2}$$

Here, the integer $N_{SS'}^{S''} \geq 0$, called the *outer multiplicity* of S'', specifies the number of times the irrep S'' occurs in this decomposition, and for a given S'', the outer multiplicity index $\alpha = 1, \ldots, N_{SS'}^{S''}$ distinguishes multiple occurrences of S''. Correspondingly, let $\{|M'', \alpha\rangle\}$ be a basis for the direct sum decomposition, i.e., $\mathbb{V}^{S'',\alpha} = \text{span}\{|M'', \alpha\rangle\}$. Carrier space dimensions add up according to $d_S \cdot d_{S'} = \sum_{S''} N_{SS'}^{S''} d_{S''}$.

The decomposition (2) implies that a basis transformation C can be found from the direct product basis to the direct sum basis which block-diagonalizes the matrix representations of all group elements (p. 100 of Ref. 22),

$$C(U_g^S \otimes U_g^{S'})C^\dagger = \begin{pmatrix} U_g^{\tilde{S}_1} & & & \\ & U_g^{\tilde{S}_2} & & \\ & & U_g^{\tilde{S}_3} & \\ & & & \ddots \end{pmatrix}, \tag{3a}$$

where each $\tilde{S}_j$ is a shorthand for a certain (S'', α).

Since $\mathcal{G}$ is a matrix Lie group (SU(N) or SL($N, \mathbb{C}$)), it is convenient to work with its associated Lie algebra $\mathfrak{g}$ ($\mathfrak{su}(N)$ or $\mathfrak{sl}(N, \mathbb{C})$). It is obtained by considering the infinitesimal action of $\mathcal{G}$ on $\mathbb{V}^S$, i.e., by taking derivatives of the group at the identity. This derivative acts on the direct product of two group representations according to the product rule, so that the basis transformation C could equally be defined by the property that it block-diagonalizes the following algebra representation:

$$C(U_A^S \otimes \mathbb{I}^{S'} + \mathbb{I}^S \otimes U_A^{S'})C^\dagger = \begin{pmatrix} U_A^{\tilde{S}_1} & & & \\ & U_A^{\tilde{S}_2} & & \\ & & U_A^{\tilde{S}_3} & \\ & & & \ddots \end{pmatrix}. \tag{3b}$$

When projected to the subspace $\mathbb{V}^{S'',\alpha}$ (denote the corresponding projector by $P^{S'',\alpha}$), the action of the algebra in the direct product representation can thus be written as

$$C(U_A^S \otimes \mathbb{I}^{S'} + \mathbb{I}^S \otimes U_A^{S'})C^\dagger \quad \xrightarrow{P^{S'',\alpha}} \quad U_A^{S'',\alpha}. \tag{4}$$

Concretely, the basis transformation C can be expressed in the form

$$|M'', \alpha\rangle = \sum_{M,M'} C_{M,M'}^{M'',\alpha} |M \otimes M'\rangle, \tag{5}$$

where the $C_{M,M'}^{M'',\alpha}$ are the Clebsch–Gordan coefficients of present interest. They are understood to be defined only for $N_{S,S'}^{S''} \neq 0$ and express the carrier states of $\mathbb{V}^{S'',\alpha}$ in terms of linear combinations of product basis states from $\mathbb{V}^S \otimes \mathbb{V}^{S'}$. The CGCs encode the so-called *selection rules*, in that $C_{MM'}^{M'',\alpha} \neq 0$ only for a limited number of combinations of M, M', and M''.

Since the CGCs are the entries of the unitary matrix C, they satisfy the following orthonormality conditions:

$$\sum_{M,M'} C_{M,M'}^{M'',\alpha} (C_{M,M'}^{\tilde{M}'',\tilde{\alpha}})^* = \delta_{M'',\tilde{M}''} \delta_{\alpha,\tilde{\alpha}}, \tag{6a}$$

$$\sum_{M'',\alpha} C_{M,M'}^{M'',\alpha} (C_{\tilde{M},\tilde{M}'}^{M'',\alpha})^* = \delta_{M,\tilde{M}} \delta_{M',\tilde{M}'}. \tag{6b}$$
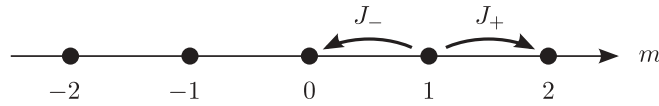
FIG. 1. SU(2) weight diagram for $S = 2$. Arrows show the action of $J_\pm$ on the state $|S = 2, m = 1\rangle$.

Actually, the $C_{M,M'}^{M'',\alpha}$ can always be chosen to be real, and we shall do so throughout.

The goal of the present work is to present (and implement on a computer) an efficient algorithm for $\mathcal{G} = \mathrm{SU}(N)$ or $\mathrm{SL}(N, \mathbb{C})$, which, for any specified N and any specified irrep labels S and S', produces explicit tables of all CGCs arising in the direct product decomposition (2).

III. REVIEW OF SU(2) CLEBSCH–GORDAN COEFFICIENTS

Before considering the general SU(N) case, we first review a method for calculating SU(2) CGCs. While there are various ways to accomplish this task, the particular approach presented below illustrates the general strategy to be used for SU(N) in later sections. The discussion is structured as follows: First, we recall the Lie algebra associated with SU(2), then its irreducible representations, then move on to product representation decompositions, and finally set up equations specifying the CGCs.

The Lie algebra associated with SU(2), denoted by $\mathfrak{su}(2)$, consists of all real linear combinations of three basis elements, J_x, J_y, and J_z, obeying the commutation relation $[J_x, J_y] = i J_z$ (plus cyclic permutations of the indices). However, it will be more convenient to deal with complex linear combinations of these, which constitute the algebra $\mathfrak{sl}(2, \mathbb{C})$. As a basis for the latter, it is common to choose three elements, $J_+ = J_x + i J_y$, $J_- = J_x - i J_y$, and J_z, obeying the following commutation relations:

$$[J_z, J_\pm] = \pm J_\pm, \tag{7a}$$

$$[J_+, J_-] = 2J_z. \tag{7b}$$

Each $\mathfrak{su}(2)$ irrep, and correspondingly, each SU(2) irrep, can be uniquely (up to an isomorphism) identified by a nonnegative half-integer, $S = 0, 1/2, 1, \ldots$. The carrier space $\mathbb{V}^S$ of such an irrep has an orthonormal basis where the states, denoted by $|S, m\rangle$, are labeled by a half-integer, $m = S, S - 1, \ldots, -S$, such that the action of J_z and $J_\pm$ is given by

$$J_z |S, m\rangle = m |S, m\rangle, \tag{8a}$$

$$J_\pm |S, m\rangle = \sqrt{(S \pm m + 1)(S \mp m)} \, |S, m \pm 1\rangle. \tag{8b}$$

The J_z-eigenvalue m will be called the *z-weight* of the state $|S, m\rangle$ (in anticipation of similar nomenclature to be used for SU(N) below). The action of $J_\pm$ can be visualized in a so-called *weight diagram*, which represents each carrier state $|S, m\rangle$ by a mark on an axis at the corresponding m-value. For example, the carrier space of $S = 2$ is shown in Fig. 1. In anticipation of the generalization to SU(N), we label basis states from now on by a composite index $M = (S, m)$, which includes both the irrep label S and the basis index m.

Each carrier space $\mathbb{V}^S$ contains a unique (up to normalization) *highest-weight state*, $|H''\rangle$, defined by the property that

$$J_+ |H\rangle = 0. \tag{9}$$

For $\mathfrak{su}(2)$, it carries the labels $|H\rangle = |S, m = S\rangle$.

In the direct product decomposition of two $\mathfrak{su}(2)$ irreps S and S', the outer multiplicity $N_{S,S'}^{S''}$, in the notation of Eq. (2) is given by

$$N_{S,S'}^{S''} = \begin{cases} 1, & \text{for } |S - S'| \le S'' \le S + S' \\ 0, & \text{otherwise} \end{cases}. \tag{10}$$

Since $N_{S,S'}^{S''} \leq 1$ for $\mathfrak{su}(2)$, we shall, throughout this section, omit the index α appearing in Eq. (5). In particular, Eq. (5) now takes the form

$$|M''\rangle = \sum_{M,M'} C_{M,M'}^{M''} |M \otimes M'\rangle, \tag{11}$$

where the CGCs $C_{M,M'}^{M''}$ satisfy the following selection rule:

$$m'' \neq m + m' \quad \implies \quad C_{M,M'}^{M''} = 0. \tag{12}$$

It reflects the fact that $|M''\rangle$, $|M\rangle$, and $|M'\rangle$ are eigenstates of $J_z^{S''}$, J_z^S, and $J_z^{S'}$, respectively, where the superscripts on J_z indicate which carrier space the respective operators act on.

To obtain the CGCs for given S and S' explicitly, we consider each S'' for which $N_{S,S'}^{S''} > 0$ separately. Let us make the following ansatz for the expansion of $|H''\rangle$ in terms of product basis states:

$$|H''\rangle = \sum_{M,M'} C_{M,M'}^{H''} |M \otimes M'\rangle, \tag{13}$$

where $C_{M,M'}^{H''}$ are the CGCs of $|H''\rangle$ and the sum runs only over values of m and m' that satisfy the selection rule (12). Inserting (13) into (9), we obtain

$$\sum_{M,M'} C_{M,M'}^{H''}(J_+^S \otimes \mathbb{I}^{S'} + \mathbb{I}^S \otimes J_+^{S'})|M \otimes M'\rangle = 0. \tag{14}$$

After evaluating the action of the raising operators on $|M \otimes M'\rangle$ using Eq. (8b) and requiring the coefficients in front of each state $|M \otimes M'\rangle$ to vanish independently, we obtain a homogeneous linear system of equations. We solve for $C_{M,M'}^{H''}$ and fix a solution by the normalization condition (6a) and by requiring $C_{M,M'}^H$ to be real and positive for the largest value of m for which $C_{M,M'}^H$ is nonzero.

The CGCs of lower-weight states (i.e., states other than the highest-weight state) are found by noting that

$$\begin{aligned}|M''\rangle = |S'', m''\rangle &= \mathcal{N}(J_-)^{S''-m''} |H''\rangle \\ &= \mathcal{N} \sum_{M,M'} C_{M,M'}^{H''}(J_-^S \otimes \mathbb{I}^{S'} + \mathbb{I}^S \otimes J_-^{S'})^{S''-m''} |M \otimes M'\rangle.\end{aligned} \tag{15}$$

($\mathcal{N} = \sqrt{(S''+m'')!/(S''-m'')!(2S'')!}$ is a normalization constant.) The right-hand side of this equation is fully known from Eq. (8b). By rewriting it into the form of Eq. (11), the desired $C_{M,M'}^{M''}$ can readily be identified.

For given S'', S, and S' it is possible to write Eq. (15) as a recursion relation relating CGCs with different m''.[23] Moreover, for $\mathfrak{su}(2)$, there exists a closed formula for $C_{M,M'}^{M''}$.[11] Nevertheless, for present purposes, the approach presented here is the most convenient as its key steps can readily be generalized to calculate $\mathfrak{su}(N)$ Clebsch–Gordan coefficients. The differences in comparison to $\mathfrak{su}(2)$ will lie in (i) the more complex structure of raising and lowering operators, (ii) the labeling schemes for irreps and states, and (iii) the method for finding the irreps occurring in a product representation decomposition, all of which we tackle in the following sections.

IV. THE LIE ALGEBRA ASSOCIATED WITH SU(*N*)

Instead of working with the group SU(*N*) itself, it will be more convenient for our purposes to consider its associated Lie algebra, $\mathfrak{su}(N)$ [ch. 13 of Ref. 24] The latter consists of all traceless anti-Hermitian $n \times n$ matrices, while the ordinary commutator serves as its Lie bracket. Most results obtained for representations of $\mathfrak{su}(N)$ carry over to SU(*N*) one-to-one, with the elements of the Lie algebra representing the generators of the Lie group. Notably, the Clebsch–Gordan coefficients of their representations are identical.

We begin by specifying a basis for the $\mathfrak{su}(N)$ algebra, in order to illustrate its structure. Let $E^{p,q}$ be the single-entry matrices, i.e., $E_{r,s}^{p,q} = \delta_{p,r}\delta_{q,s}$. A possible choice of basis is given by the matrices

$i(E^{k,l} + E^{l,k})$ and $E^{k,l} - E^{l,k}$ for $1 \leq k < l \leq N$, and $i(E^{l,l} - E^{l+1,l+1})$ for $1 \leq l \leq N - 1$. $\mathfrak{su}(N)$ is spanned by *real* linear combinations of these matrices. Just as for $\mathfrak{su}(2)$ and $\mathfrak{sl}(2, \mathbb{C})$, however, it will be convenient to work with a basis for $\mathfrak{sl}(N, \mathbb{C})$. To this end, define for $1 \leq l \leq N - 1$ the *complex* linear combinations,

$$J_z^{(l)} = \tfrac{1}{2}(E^{l,l} - E^{l+1,l+1}), \tag{16a}$$

$$J_+^{(l)} = E^{l,l+1}, \tag{16b}$$

$$J_-^{(l)} = E^{l+1,l}, \tag{16c}$$

which satisfy, for each l, the familiar $\mathfrak{su}(2)$ commutation relations of Eq. (7),

$$\left[J_z^{(l)}, J_\pm^{(l)} \right] = \pm J_\pm^{(l)}, \tag{17a}$$

$$\left[J_+^{(l)}, J_-^{(l)} \right] = 2 J_z^{(l)}. \tag{17b}$$

The $N - 1$ matrices $J_z^{(l)}$ form a maximal set of mutually commuting matrices, $[J_z^{(l)}, J_z^{(l')}] = 0$ (thus, the $i J_z^{(l)}$ span the Cartan subalgebra of $\mathfrak{su}(N)$). Thus, none of the $J_\pm^{(l)}$ commutes with all elements of this set, or with all other $J_\pm^{(l')}$ operators.

The matrices $J_z^{(l)}$ and $J_\pm^{(l)}$ are not anti-Hermitian and thus do not belong to $\mathfrak{su}(N)$ but rather to $\mathfrak{sl}(N, \mathbb{C})$. However, it is sufficient to restrict our attention to $J_\pm^{(l)}$ because, from these, we can recover an anti-Hermitian basis using

$$E^{p,q} = [J_-^{(p-1)}, [J_-^{(p-2)}, \ldots [J_-^{(q+1)}, J_-^{(q)}]] \ldots] \quad \text{for} \quad p > q, \tag{18a}$$

$$E^{p,q} = [J_+^{(p)}, [J_+^{(p+1)}, \ldots [J_+^{(q-2)}, J_+^{(q-1)}]] \ldots] \quad \text{for} \quad p < q. \tag{18b}$$

In other words, once we know representations for all $J_\pm^{(l)}$ on a given carrier space, the representations of all other elements of both the algebras $\mathfrak{sl}(N, \mathbb{C})$ and $\mathfrak{su}(N)$ are also known. For definiteness, we shall refer to $\mathfrak{su}(N)$ below, although the constructions apply equally to $\mathfrak{sl}(N, \mathbb{C})$.

V. LABELING OF IRREPS AND STATES

The $\mathfrak{su}(N)$ basis, defined in Sec. IV, has a feature that makes it particularly convenient for our purposes: if one also adopts a specific labeling scheme, devised by GT,[19] for labeling $\mathfrak{su}(N)$ irreps and the basis states of their carrier spaces, these basis states are simultaneous eigenstates of all the matrices $J_z^{(l)}$, and explicit formulas exist for the matrix elements of the $J_\pm^{(l)}$ with respect to these basis states. The next three sections are devoted to summarizing the GT labeling scheme without dwelling on its mathematical roots—the mere knowledge of its rules is sufficient for our purposes. (The relation of the GT-scheme labeling scheme to a frequently used alternative but equivalent labeling scheme, employing Young diagrams and Young tableaux, is summarized, for convenience, in Appendix A.)

Up to equivalent representations, each $\mathfrak{su}(N)$ irrep can be identified uniquely by a sequence of N integers,[25]

$$S = (m_{1,N}, \ldots, m_{N,N}), \tag{19}$$

or $S = (m_{k,N})$, in short, fulfilling $m_{k,N} \geq m_{k+1,N}$ for $1 \leq k \leq N - 1$. We shall call such a sequence an *irrep weight* or *i-weight*, in short. The second index, N, identifies the algebra, $\mathfrak{su}(N)$; the reasons for displaying this index explicitly will become clear below. Two i-weights S and S' for which all components differ only by a k-independent constant, i.e., $m'_{k,N} = m_{k,N} + c$ with $c \in \mathbb{Z}$, designate the *same* $\mathfrak{su}(N)$ irrep. This fact can be used to bring any i-weight into a "normalized" form having $m_{N,N} = 0$, which will be assumed below, unless otherwise specified.

GT exploited the fact that the carrier space of any $\mathfrak{su}(N)$ irrep splits into disjoint carrier spaces of $\mathfrak{su}(N - 1)$ irreps to devise a labeling scheme with a very convenient property: it yields a remarkably simple rule for enumerating which $\mathfrak{su}(N - 1)$ irreps occur in the decomposition of $S = (m_{k,N})$, namely, all those with i-weights $(m_{1,N-1}, \ldots, m_{N-1,N-1})$ that satisfy the condition $m_{k,N} \geq m_{k,N-1} \geq m_{k+1,N}$ for $1 \leq k \leq N - 1$. Note that, here, it is crucial *not* to set $m_{N-1,N-1} = 0$ so that we can distinguish between multiple occurrences of the same $\mathfrak{su}(N - 1)$ irrep.

Recursively, the carrier spaces of $\mathfrak{su}(N - 1)$ irreps give rise to $\mathfrak{su}(N - 2)$ irreps and so on, down to $\mathfrak{su}(1)$, the carrier spaces of which are one-dimensional. This sequence of decompositions can be exploited to label the basis states $|M\rangle$ of a given $\mathfrak{su}(N)$ irrep $S = (m_{k,N})$ using the so-called *Gelfand–Tsetlin patterns* (GT-patterns). These are triangular arrangements of integers, to be denoted by $M = (m_{k,l})$, with the structure

$$
M = \begin{pmatrix} m_{1,N} \; m_{2,N} & \cdots & m_{N,N} \\ m_{1,N-1} & \cdots & m_{N-1,N-1} \\ & \ddots \qquad \iddots & \\ m_{1,2} \; m_{2,2} & & \\ m_{1,1} & & \end{pmatrix}, \tag{20}
$$

i.e., the first index labels diagonals from left to right, and the second index labels rows from bottom to top. The top row contains the i-weight $(m_{k,N})$ that specifies the irrep, and the entries of lower rows are subject to the so-called *betweenness condition*,

$$
m_{k,l} \geq m_{k,l-1} \geq m_{k+1,l}, \qquad (1 \leq k < l \leq N). \tag{21}
$$

The dimension of an irrep $S = (m_{k,N})$ is equal to the number of valid GT-patterns having S as their top row. There exists a convenient formula for this number,

$$
\dim(S) = \prod_{1 \leq k < k' \leq N} \left(1 + \frac{m_{k,N} - m_{k',N}}{k' - k} \right). \tag{22}
$$

Note that the SU(2) basis state conventionally labeled as $|j, m\rangle$ corresponds to the GT-pattern $\binom{2j \quad 0}{j-m}$, and the above formula reduces to $\dim(j) = 2j + 1$.

To obtain a complete description of SU(N) irreps, we need to specify how the Lie algebra $\mathfrak{su}(N)$ acts on states labeled by Gelfand–Tsetlin patterns. The following two sections are devoted to this task, Sec. VI with $J_z^{(l)}$ and Sec. VII dealing with $J_\pm^{(l)}$.

VI. WEIGHTS AND WEIGHT DIAGRAMS

A very convenient property of the GT-labeling scheme is that every state $|M\rangle$ is a simultaneous eigenstate of all $J_z^{(l)}$ generators,

$$
J_z^{(l)} |M\rangle = \lambda_l^M |M\rangle, \qquad (1 \leq l \leq N - 1), \tag{23}
$$

with eigenvalues

$$
\lambda_l^M = \sigma_l^M - \frac{1}{2}(\sigma_{l+1}^M + \sigma_{l-1}^M), \qquad (1 \leq l \leq N - 1), \tag{24}
$$

where the *row sum* $\sigma_l^M = \sum_{k=1}^{l} m_{k,l}$ denotes the sum over all entries of row l of GT-pattern M ($\sigma_0^M = 0$ by convention). We shall call the sequence of $N - 1$ $J_z^{(l)}$ eigenvalues the *z-weight* of the

state $|M\rangle$ and denote it by $W_z(M) = (\lambda_1^M, \ldots, \lambda_{N-1}^M)$. The *z-weight* of $|M\rangle$ is a straightforward generalization of the quantum number m in quantum angular momentum.

As will be elaborated below, the notion of weights of states is useful for elucidating the structure of carrier spaces of $\mathfrak{su}(N)$ irreps, and in particular for visualizing the action of raising and lowering operators. The above way of introducing weights is, however, not unique. We shall often find it convenient to employ an alternative definition of the weight of states, which has the convenient property that it always yields nonnegative *integer* elements (in contrast to $W_z(M)$). This alternative weight, to be called *pattern weight* or *p-weight*, and denoted by $W(M)$, is defined to be a sequence of N integers, $W(M) = (w_1^M, \ldots, w_N^M)$, where

$$w_l^M = \sigma_l^M - \sigma_{l-1}^M, \qquad (1 \le l \le N), \tag{25}$$

is the difference between summing up rows l and $l-1$ of the GT-pattern M. Note that the number of *independent* elements of $W(M)$ is the same as that of $W_z(M)$, namely $N-1$, since the w_l^M satisfy the relation $\sum_{l=1}^N w_l^M = \sigma_N^M$. The two types of weights are directly related to each other: via Eq. (24), we obtain $\lambda_l^M = (w_l^M - w_{l+1}^M)/2$. For definiteness, we will mostly refer to p-weights below (noting here that most statements involving p-weights can be translated into equivalent statements involving z-weights).

At this point, the first of several fundamental differences between $\mathfrak{su}(2)$ and $\mathfrak{su}(N)$ with $N \ge 3$ appears. While for $\mathfrak{su}(2)$, there always exists exactly one state with a given p-weight, this is not the case for $\mathfrak{su}(N)$ in general; for $N \ge 3$, several linearly independent states in the carrier space can have the same p-weight. Indeed, two states have the same p-weight, $W(M) = W(M')$, if and only if they have the same set of row sums ($\sigma_l^M = \sigma_l^{M'}$ for $1 \le l \le N-1$) (i.e., they differ only in the way in which the "weight" of the row sums is distributed among the entries of each row). For a given p-weight W, the number of states $|M\rangle$ having the same p-weight, $W(M) = W$, is called the *inner multiplicity* of that p-weight, to be denoted by $I(W)$. Consequently, p-weights or z-weights are not suited for uniquely labeling states of a carrier space (which is why GT-patterns are used for this purpose).

z-weights nevertheless do provide a convenient way to visualize the carrier space of an $\mathfrak{su}(N)$ irrep. To this end, consider $W_z(M) = (\lambda_1^M, \ldots, \lambda_{N-1}^M)$ as a vector in $(N-1)$-dimensional space and, for each state, mark the endpoint of its weight vector in an $(N-1)$-dimensional lattice. The resulting diagram is called a *weight diagram*. For the $\mathfrak{su}(2)$ irrep j, weight diagrams consist of a coordinate axis with markings at $-j, -j+1, \ldots, j$ (see Fig. 1); for $\mathfrak{su}(3)$, weight diagrams are two-dimensional (see Fig. 2); for $N \ge 4$, weight diagrams cannot be readily drawn on paper because the corresponding lattices have more than two dimensions.

Note that, in Fig. 2, the z-weight $W_z = (0, 0)$ has inner multiplicity two, since the two states $\begin{pmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}$ have the same row sums.

VII. RAISING AND LOWERING OPERATORS

Weight diagrams are also very convenient for visualizing the action of the raising and lowering operators $J_\pm^{(l)}$. The action of $J_\pm^{(l)}$ on a given state $|M\rangle$ produces a linear combination of all states of the form $|M \pm M^{k,l}\rangle$ with arbitrary k, where this notation implies element-wise addition and subtraction of the single-entry pattern $M^{k,l}$ having one at position k, l and zeros elsewhere,

$$M^{k,l} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ & 0 & \ldots & 0 \\ & \ddots & 1_{k,l} & \cdot^{\cdot} \\ & 0 & & 0 \\ & & 0 & \end{pmatrix}. \tag{26}$$

(Note that $M^{k,l}$ on its own is not a valid GT-pattern.) Thus the resulting patterns differ from M only in row l. All states $|M \pm M^{k,l}\rangle$ that are generated in this fashion have the same row sums, z-weights,
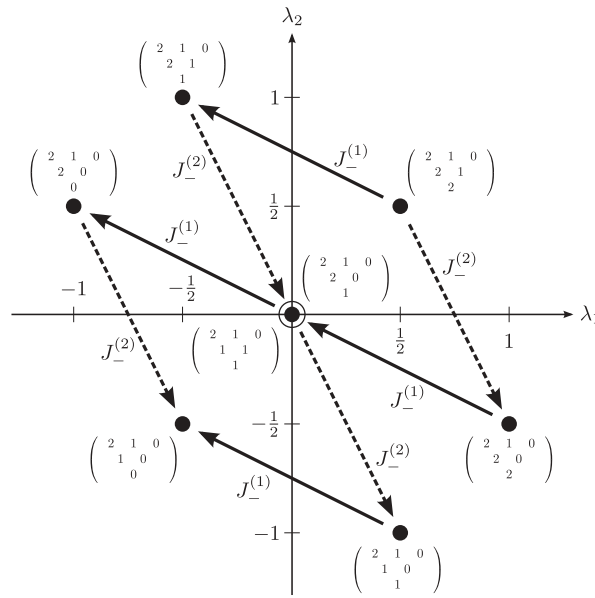
FIG. 2. Weight diagram of the $\mathfrak{su}(3)$ irrep $(2, 1, 0)$. Each dot represents a z-weight; we also indicate the GT-patterns of the corresponding states. The double circle around $(0, 0)$ indicates that there are two states with this weight. The solid and dashed arrows represent the action of $J_-^{(1)}$ and $J_-^{(2)}$, respectively. ($J_+^{(l)}$ could be represented by arrows pointing in directions opposite to those of $J_-^{(l)}$.) Note that both $J_-^{(1)}$ acting on $\left(\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 \\ & & 2 \end{smallmatrix}\right)$ and $J_-^{(2)}$ acting on $\left(\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 1 \\ & & 1 \end{smallmatrix}\right)$ produce linear combinations of $\left(\begin{smallmatrix} 2 & & 1 & & 0 \\ & 2 & & 0 \\ & & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & & 1 & & 0 \\ & 1 & & 1 \\ & & 1 \end{smallmatrix}\right)$, albeit different ones. (In the literature it is not uncommon to choose a different $\mathfrak{su}(3)$ basis that renders this weight diagram more symmetric.)

and p-weights (independent of k),

$$W_z(M \pm M^{k,l}) = \left(\lambda_1^M, \ldots, \lambda_{l-2}^M, \lambda_{l-1}^M \mp 1/2, \lambda_l^M \pm 1, \lambda_{l+1}^M \mp 1/2, \lambda_{l+2}^M, \ldots, \lambda_{N-1}^M\right), \quad \text{(27a)}$$

$$W(M \pm M^{k,l}) = \left(w_1^M, \ldots, w_{l-1}^M, w_l^M \pm 1, w_{l+1}^M \mp 1, w_{l+2}^M, \ldots, w_N^M\right), \quad \text{(27b)}$$

unless states with this weight do not exist, in which case the result vanishes.

The weight-shifting action of lowering operators is illustrated in Fig. 2 for the weight diagram of the $\mathfrak{su}(3)$ irrep $S = (2, 1, 0)$. Since the weight diagram is two-dimensional, there are two lowering operators, $J_-^{(1)}$ and $J_-^{(2)}$, which shift in different directions (indicated by solid/dashed lines). ($J_+^{(l)}$ produces a shift in the opposite direction of $J_-^{(l)}$.) Note that there are two different "paths" to reach the z-weight $(0,0)$ from the z-weight $(\frac{1}{2}, \frac{1}{2})$, namely, via either $J_-^{(1)}J_-^{(2)}$ or $J_-^{(2)}J_-^{(1)}$. Since $J_-^{(1)}$ and $J_-^{(2)}$ do not commute, these paths are inequivalent; indeed, they produce two different linear combinations of the two states with z-weight $(0, 0)$. Generally, the fact that inner multiplicities larger than one arise for $\mathfrak{su}(N)$ representations with $N > 2$ is a direct consequence of the fact that there are, in general, several different ways of reaching one weight from another via a chain of raising and lowering operators, and that these ways are not equivalent, because $J_\pm^{(l)}$ and $J_\pm^{(l')}$ do not commute for $l \neq l'$.

Very conveniently, closed expressions have been found by Gelfand and Tsetlin[19] for the matrix elements of all raising and lowering operators with respect to the basis of GT-patterns. Explicitly,

the only nonzero matrix elements of $J_-^{(l)}$ are given, for any $1 \leq k \leq l \leq N-1$, by [p. 280 of Ref. 26]

$$\langle M - M^{k,l}|J_-^{(l)}|M\rangle = \left(-\frac{\prod\limits_{k'=1}^{l+1}(m_{k',l+1} - m_{k,l} + k - k' + 1)\prod\limits_{k'=1}^{l-1}(m_{k',l-1} - m_{k,l} + k - k')}{\prod\limits_{\substack{k'=1 \\ k' \neq k}}^{l}(m_{k',l} - m_{k,l} + k - k' + 1)(m_{k',l} - m_{k,l} + k - k')} \right)^{\frac{1}{2}}. \tag{28}$$

These matrix elements are real and nonnegative, and the right-hand side vanishes if $M - M^{k,l}$ is not a valid pattern. As $J_+^{(l)}$ is the Hermitian transpose of $J_-^{(l)}$, we can obtain its nonzero matrix elements by taking the complex conjugate of the preceding formula and replacing $|M\rangle$ by $|M + M^{k,l}\rangle$,

$$\langle M + M^{k,l}|J_+^{(l)}|M\rangle = \left(-\frac{\prod\limits_{k'=1}^{l+1}(m_{k',l+1} - m_{k,l} + k - k')\prod\limits_{k'=1}^{l-1}(m_{k',l-1} - m_{k,l} + k - k' - 1)}{\prod\limits_{\substack{k'=1 \\ k' \neq k}}^{l}(m_{k',l} - m_{k,l} + k - k')(m_{k',l} - m_{k,l} + k - k' - 1)} \right)^{\frac{1}{2}}. \tag{29}$$

These formulas generalize Eq. (8b) to $\mathfrak{su}(N)$.

Each irrep has a unique state $|H\rangle$, called its *highest-weight state*, that is annihilated by all $N-1$ raising operators,

$$J_+^{(l)}|H\rangle = 0, \qquad (1 \leq l \leq N-1). \tag{30}$$

Since $|H\rangle$ is a unique state, the inner multiplicity of its p-weight $W(H)$ is one, and the irrep can be identified by specifying $W(H)$. Our labeling scheme indeed exploits this fact: the i-weight of an irrep is equal to the p-weight of its highest-weight state $|H\rangle$, i.e., $S = W(H)$. Conveniently, the GT-pattern $H = (h_{k,l})$ has the highest possible entries fulfilling Eq. (21), i.e., $h_{k,l} = h_{k,N}$ for $1 \leq k \leq l \leq N-1$ (all entries on the k-th diagonal are equal to $m_{k,N}$).

This concludes our exposition of those elements of SU(N) representation theory in the GT-scheme that are needed in this work. In the following sections we discuss the decomposition of direct product representations and the calculation of the associated CGCs. The specific details of the strategy described below are, to the best of our knowledge, original.

VIII. PRODUCT REPRESENTATION DECOMPOSITIONS

The product of two irreps, say $S \otimes S'$, is, in general, reducible to a sum of irreps (Eq. (2)). While it is well-known for $\mathfrak{su}(2)$ that which irreps occur in such a decomposition (see Eq. (10)), the corresponding result for $\mathfrak{su}(N)$ relies on a relatively simple but hard to prove the method based on the *Littlewood–Richardson rule.*[27] This method involves writing down all possible GT-patterns for the irrep S and using each of these to construct, starting from S', a new irrep S''. As the outcome of this method is the same when interchanging S and S', it is preferable to take the irrep with the smaller dimension of the two as S.

For given irreps $S = (m_{k,N})$ and $S' = (m'_{k,N})$, and a particular GT-pattern $M = (m_{k,l})$ associated with S, let us introduce some auxiliary notation. For $l = 1, \ldots, N$ and $k = 1, \ldots, l$, we set $b_{k,l} = m_{k,l} - m_{k,l-1}$ (where $m_{k,l} \equiv 0$ if $k > l$, for ease of notation) and $B_{k,l} = m'_{l,N} + \sum_{k'=1}^{k} b_{k,l}$ (note that here, $m'_{k,l}$ carries a prime, while $b_{k,l}$ does not). Then, the irrep $S'' = (m''_{k,N}) \equiv (B_{k,k})$ occurs in the

decomposition of $S \otimes S'$ if and only if

$$B_{k-1,1} \geq B_{k-1,2} \geq \cdots \geq B_{k-1,l-1} \geq B_{k,l} \geq B_{k,l+1} \geq \cdots \geq B_{k,N} \quad \text{for all} \quad 1 \leq k \leq l \leq N.$$

(31)

(We emphasize that this condition must hold for *each* value of k and l.) By checking whether (31) holds for all GT-patterns associated with S, all S'' in the decomposition of $S \otimes S'$ can be identified.

There exists a more efficient way to validate Eq. (31) than to check each value of k and l independently. For a given GT-pattern $M = (m_{k,l})$ associated with S, proceed as follows:

(1)   Initialize $(t_1, \ldots, t_N) = (m'_{1,N}, \ldots, m'_{N,N})$ by the i-weight of S'.
(2)   Step through the pattern M along the diagonals from top to bottom and from left to right, i.e., in the order $m_{1,N}, m_{1,N-1}, \ldots, m_{1,1}, m_{2,N}, m_{2,N-1}, \ldots, m_{2,2}, \ldots, m_{N,N}$.
(3)   At each position, say $m_{k,l}$, replace t_l by $t_l + b_{k,l}$.
(4)   If $l > 1$, check whether $t_{l-1} \geq t_l$. If this condition is violated, discard this GT-pattern, construct the next one, and commence again from step (1).
(5)   If we reach the end of the pattern M, the current value of $(t_1, \ldots, t_N)$ specifies the weight of an irrep S'' that occurs in the decomposition of $S \otimes S'$.

For $N > 2$, this procedure, in general, can produce several occurrences of the same irrep S''. The number of such occurrences, denoted by $N_{SS'}^{S''}$ in Eq. (2), is the outer multiplicity of S''. (For SU(2), the outer multiplicity is either 0 or 1.)

Let us illustrate this procedure by an example (individual steps are shown in Table I),

$$(2,1,0) \otimes (2,1,0) = (4,2,0) \oplus (3,3,0) \oplus (4,1,1) \oplus (3,2,1) \oplus (3,2,1) \oplus (2,2,2) \quad (32a)$$

$$= (4,2,0) \oplus (3,3,0) \oplus (3,0,0) \oplus (2,1,0) \oplus (2,1,0) \oplus (0,0,0). \quad (32b)$$

(For the second line, we adopted "normalized" i-weights with $m_{N,N} = 0$.) To check that the dimensions are correct, use Eq. (22) to verify the dimensions of the irreps in this equation are $8 \times 8 = 27 + 10 + 10 + 8 + 8 + 1$, respectively. Note that the irrep $(2,1,0)$ occurs twice in the decomposition, in other words, its outer multiplicity is two.

IX. SELECTION RULE FOR SU(N) CLEBSCH–GORDAN COEFFICIENTS

The fact that all states labeled by GT-patterns are eigenstates of $J_z^{(l)}$ operators implies a selection rule for SU(N) CGCs. Explicitly, let us consider a state $|M''\rangle$ occurring in a decomposition of a product representation. On the one hand, we have

$$J_z^{(l)} |M'', \alpha\rangle = \lambda_l^{M'', \alpha} |M'', \alpha\rangle, \quad (33a)$$

and on the other hand, by Eqs. (4) and (5),

$$J_z^{(l)} |M'', \alpha\rangle = \sum_{M,M'} C_{M,M'}^{M'', \alpha} (J_z^{(l),S} \otimes \mathbb{I}^{S'} + \mathbb{I}^S \otimes J_z^{(l),S'}) |M \otimes M'\rangle$$

$$= \sum_{M,M'} C_{M,M'}^{M'', \alpha} (\lambda_l^M + \lambda_l^{M'}) |M \otimes M'\rangle. \quad (33b)$$

These equations can only be fulfilled if $C_{M,M'}^{M'', \alpha}$ vanishes whenever $\lambda_l^{M'', \alpha} \neq \lambda_l^M + \lambda_l^{M'}$ for any l. Defining an element-wise addition on weights, we write, in short,

$$W_z(M'') \neq W_z(M) + W_z(M') \implies C_{M,M'}^{M'', \alpha} = 0. \quad (34)$$

This equation (or a transcription thereof involving p-weights) represents the generalization of Eq. (12) to $\mathfrak{su}(N)$.

TABLE I. Application of the Littlewood–Richardson rule according to the steps of Sec. VIII, for the decomposition of $S \otimes S' = (2, 1, 0) \otimes (2, 1, 0)$.

All possible $\begin{pmatrix} m_{1,3}\ m_{2,3}\ m_{3,3} \\ m_{1,2}\ m_{2,2} \\ m_{1,1} \end{pmatrix}$	Corresponding $\begin{pmatrix} b_{1,3}\ b_{2,3}\ b_{3,3} \\ b_{1,2}\ b_{2,2} \\ b_{1,1} \end{pmatrix}$	Initial value = $(m'_{1,3}, m'_{2,3}, m'_{3,3})$	Previous + $(0, 0, b_{1,3})$	Previous + $(0, b_{1,2}, 0)$	Previous + $(b_{1,1}, 0, 0)$	Previous + $(0, 0, b_{2,3})$	Previous + $(0, b_{2,2}, 0)$	Previous + $(0, 0, b_{3,3})$	Final irrep
$\begin{pmatrix} 2\ 1\ 0 \\ 2\ 1 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0\ 0\ 0 \\ 0\ 1 \\ 2 \end{pmatrix}$	$(2,1,0)$	$(2,1,0)$	$(2,1,0)$	$(4,1,0)$	$(4,1,0)$	$(4,2,0)$	$(4,2,0)$	$(4,2,0)$
$\begin{pmatrix} 2\ 1\ 0 \\ 2\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0\ 0\ 0 \\ 1\ 1 \\ 1 \end{pmatrix}$	$(2,1,0)$	$(2,1,0)$	$(2,2,0)$	$(3,2,0)$	$(3,2,0)$	$(3,3,0)$	$(3,3,0)$	$(3,3,0)$
$\begin{pmatrix} 2\ 1\ 0 \\ 2\ 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 0\ 1\ 0 \\ 0\ 0 \\ 2 \end{pmatrix}$	$(2,1,0)$	$(2,1,0)$	$(2,1,0)$	$(4,1,0)$	$(4,1,1)$	$(4,1,1)$	$(4,1,1)$	$(4,1,1) = (3,0,0)$
$\begin{pmatrix} 2\ 1\ 0 \\ 2\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0\ 1\ 0 \\ 1\ 0 \\ 1 \end{pmatrix}$	$(2,1,0)$	$(2,1,0)$	$(2,2,0)$	$(3,2,0)$	$(3,2,1)$	$(3,2,1)$	$(3,2,1)$	$(3,2,1) = (2,1,0)$
$\begin{pmatrix} 2\ 1\ 0 \\ 2\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0\ 1\ 0 \\ 2\ 0 \\ 0 \end{pmatrix}$	$(2,1,0)$	$(2,1,0)$	$(2,3,0)$					discarded
$\begin{pmatrix} 2\ 1\ 0 \\ 1\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1\ 0\ 0 \\ 0\ 1 \\ 1 \end{pmatrix}$	$(2,1,0)$	$(2,1,1)$	$(2,1,1)$	$(3,1,1)$	$(3,1,1)$	$(3,2,1)$	$(3,2,1)$	$(3,2,1) = (2,1,0)$
$\begin{pmatrix} 2\ 1\ 0 \\ 1\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1\ 1\ 0 \\ 0\ 0 \\ 1 \end{pmatrix}$	$(2,1,0)$	$(2,1,1)$	$(2,1,1)$	$(3,1,1)$	$(3,1,2)$			discarded
$\begin{pmatrix} 2\ 1\ 0 \\ 1\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1\ 1\ 0 \\ 1\ 0 \\ 0 \end{pmatrix}$	$(2,1,0)$	$(2,1,1)$	$(2,2,1)$	$(2,2,1)$	$(2,2,2)$	$(2,2,2)$	$(2,2,2)$	$(2,2,2) = (0,0,0)$

X. CLEBSCH–GORDAN COEFFICIENTS OF HIGHEST-WEIGHT STATES

After determining which kinds of irreps S'' appear in the decomposition of a product representation, we are ready to construct their Clebsch–Gordan coefficients. For each S'', we start by finding the CGCs of its highest-weight state, $|H'', \alpha\rangle$, as defined in Eq. (30). The index $\alpha = 1, \ldots, N_{S,S'}^{S''}$ distinguishes between the instances of irreps with outer multiplicity. Nevertheless, we determine the CGCs of $|H'', \alpha\rangle$ with given S'' for all values of α in a single run.

For this purpose, we make an ansatz of the form (5) for the highest-weight state (compare Eq. (13)),

$$|H'', \alpha\rangle = \sum_{\substack{M, M' \\ W(M)+W(M')=W(H'',\alpha)}} C_{M,M'}^{H'',\alpha} |M \otimes M'\rangle , \qquad (35)$$

with CGCs $C_{M,M'}^{H'',\alpha}$, where the sum is restricted to those combinations of states $|M \otimes M'\rangle$ that respect the selection rule (34). Now insert Eq. (35) into Eq. (30) to obtain (compare Eq. (14)),

$$\sum_{\substack{M, M' \\ W(M)+W(M')=W(H'',\alpha)}} C_{M,M'}^{H'',\alpha}(J_+^{(l),S} \otimes \mathbb{I}^{S'} + \mathbb{I}^S \otimes J_+^{(l),S'})|M \otimes M'\rangle = 0, \quad (1 \le l \le N-1). \quad (36)$$

After evaluating the action of the raising operators on the product basis states via Eq. (29), we obtain a homogeneous linear system of equations in the CGCs $C_{M,M'}^{H'',\alpha}$. It has $N_{S,S'}^{S''}$ linearly independent solutions, one for each value of α. Thus, an outer multiplicity larger than one leads to an ambiguity among the CGCs of the highest-weight states of all irreps of the same kind S'': a unitary transformation $|H, \alpha\rangle \to \sum_{\alpha'} U_{\alpha,\alpha'} |H, \alpha'\rangle$ among the highest-weight states will produce different but equally acceptable highest-weight CGCs $C_{M,M'}^{H'',\alpha}$. The full set of CGCs of the irreps S'' will change accordingly, too. For some applications, there is no need to uniquely resolve this ambiguity. For applications where it must be resolved, we will adopt the following convention, suggested by Zaránd:[28] Write down the independent solutions in the form of a matrix with elements $C_{MM'}^{H'',\alpha}$, where $\alpha = 1, \ldots, N_{S,S'}^{S''}$ serves as row index and $(M, M') = 1, \ldots, I(H)$ as composite column index (where $I(H)$ is the inner multiplicity of $W(H)$ in the product representation). Then use Gaussian elimination to bring this matrix into a normal form, namely, the reduced row echelon form,

$$\begin{pmatrix} \cdot & \cdots & \cdot \\ \vdots & C_{M,M'}^{H'',\alpha} & \vdots \\ \cdot & \cdots & \cdot \end{pmatrix} \rightarrow \begin{pmatrix} 0 & \cdots & 0 & + & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & + & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & + & * & \cdots & * \end{pmatrix}, \qquad (37)$$

where $+$ and $*$ denote positive and arbitrary matrix elements, respectively. This normal form is the same for all equivalent matrices. To obtain orthonormal highest-weight states, we then do a Gram–Schmidt orthonormalization of the rows of the resulting matrix from top to bottom. This procedure uniquely specifies the CGCs for the highest-weight states.

As an aside, we note that the above mentioned ambiguity does not arise for the case of $S' = (1, 0, \ldots, 0)$ (the defining representation of $\mathfrak{su}(N)$) and arbitrary S, since then all outer multiplicites are either zero or one, i.e., then $N_{S,S'}^{S''} = 0$ or 1. (However, there would still be a sign ambiguity for the CGCs, and the above procedure constitutes one way of fixing it.) We note that for this case, explicit formulas for SU(N) CGCs can be found.[14]

XI. CLEBSCH–GORDAN COEFFICIENTS OF LOWER-WEIGHT STATES

Let us now turn to the CGCs of states of S'' other than its highest-weight state. These are obtained by acting on the both sides of Eq. (35) with lowering operators, using Eq. (28) for the matrix representations of $J_-^{(l)}$ for the carrier space $\mathbb{V}^{S'',\alpha}$ on the left-hand side and for the direct product carrier space $\mathbb{V}^S \otimes \mathbb{V}^{S'}$ on the right-hand side. However, according to Eq. (28), the action of $J_-^{(l)}$ in general produces not a unique basis state, but a linear combination of basis states of $\mathbb{V}^{S'',\alpha}$. We shall therefore calculate, in parallel, the CGCs of *all* basis states with a given α and given p-weight $W = (w_l)$, i.e., of all $|M'', \alpha\rangle$ having $W(M'') = W$.

To this end, assume that we have already determined all "parent states" of the desired p-weight W within $\mathbb{V}^{S'',\alpha}$. By parent states we mean those which, when acted upon by a single $J_-^{(l)}$, yield (linear combinations of) states of weight W. For a given $J_-^{(l)}$ (with $1 \leq l \leq N-1$), the relevant parent states have p-weight $(w_1, \ldots, w_{l-1}, w_l + 1, w_{l+1} - 1, w_{l+2}, \ldots, w_N)$ and consist of all states of the form $|M'' + M^{k,l}, \alpha\rangle$ with $W(M'') = W$ and $1 \leq k \leq l$, for which $M'' + M^{k,l}$ is a valid GT-pattern. Each parent state can be expressed as

$$|M'' + M^{k,l}, \alpha\rangle = \sum_{M,M'} C_{M,M'}^{M''+M^{k,l},\alpha} |M \otimes M'\rangle, \tag{38}$$

where the CGCs are, by assumption, already known. Now, the action of $J_-^{(l)}$ on any parent state can be written as a linear combination of all states $|M''', \alpha\rangle$ with $W(M''') = W$,

$$J_-^{(l),S''} |M'' + M^{k,l}, \alpha\rangle = \sum_{M'''} b_{M'',k,l}^{M'''} |M''', \alpha\rangle, \tag{39}$$

where the coefficients $b_{M'',k,l}^{M'''}$ are determined by the matrix representation of $J_-^{(l)}$ within $\mathbb{V}^{S'',\alpha}$, as given by Eq. (28). Combining Eqs. (38) and (39) and using the direct product representation of $J_-^{(l)}$ on $\mathbb{V}^S \otimes \mathbb{V}^{S'}$, we obtain a linear system of equations of the form (compare Eq. (15))

$$\sum_{M'''} b_{M'',k,l}^{M'''} |M''', \alpha\rangle = \sum_{M,M'} C_{M,M'}^{M''+M^{k,l},\alpha} (J_-^{(l),S} \otimes \mathbb{I}^{S'} + \mathbb{I}^S \otimes J_-^{(l),S'}) |M \otimes M'\rangle. \tag{40}$$

Each combination of indices M'', k, and l specifies a separate equation, where M'' runs over all GT-patterns such that $W(M'') = W$, l runs from 1 to $N-1$, and k runs from 1 to l, provided that $M'' + M^{k,l}$ is a valid GT-pattern. Actually, only $I(W)$ of these equations are linearly independent; as we do not know in advance which ones these are, we include them all, i.e., the system of equations (40) is, in general, overdetermined. Since the action of the $J_-^{(l)}$s on the right-hand side is known from Eq. (28), the sought-after CGCs $C_{M,M'}^{M'',\alpha}$ can now be readily obtained by inverting the matrix of the coefficients $b_{M'',k,l}^{M'''}$ in order to bring Eq. (40) into the familiar form of Eq. (5).

XII. ALGORITHM FOR COMPUTER IMPLEMENTATION

Having gathered in the preceding sections all necessary ingredients, we are now ready to formulate the sought-after algorithm for calculating SU(N) CGCs. Given two SU(N) irreps S and S', perform the following steps:

(1) Find the irreps S'' appearing in the decomposition of $S \otimes S'$, as described in Sec. VIII.
(2) For each irrep S'', find the Clebsch–Gordan coefficients of the $N_{S,S'}^{S''}$ highest-weight states $|H'', \alpha\rangle$. Resolve outer multiplicity ambiguities, as described in Sec. X.
(3) From each highest-weight state $|H'', \alpha\rangle$, construct the lower-weight states by repeated application of $J_-^{(l)}$ operators, treating each weight of S'' separately, as described in Sec. XI.

An explicit computer implementation of this strategy is presented in Appendix D. To check that our algorithm works correctly, we have verified that it satisfies the following consistency checks:

- For SU(2) and SU(3), the results coincide with known formulas and tables, up to sign conventions.
- The selection rule (34) is fulfilled.

- The matrix C of Clebsch–Gordan coefficients (see Sec. II) is unitary.
- The matrix C block-diagonalizes the representation matrices (Eq. (3)).

The speed of the algorithm depends polynomially on the dimensions of the irreps S and S'. On a modern computer (2 GHz CPU clock speed), smaller $\mathfrak{su}(3)$ cases (e.g., dim $S = 6$, dim $S' = 15$) run instantly, while medium-sized $\mathfrak{su}(5)$ cases (e.g., dim $S = 35$, dim $S' = 224$) take a few minutes, and larger $\mathfrak{su}(5)$ cases (e.g., dim $S = 280$, dim $S' = 420$) require several hours of computing time.

As an outlook, we note that it should be possible to greatly speed up our algorithm by exploiting the fact that the weight diagrams are symmetric under the Weyl group, which in this context can be thought of as the group of all permutations of the elements of the p-weights, $(w_1^M, \ldots, w_N^M) \to (w_{\sigma(1)}^M, \ldots, w_{\sigma(N)}^M)$. Exploiting this symmetry is a nontrivial task, since the Gelfand–Tsetlin basis is not stable under the operation of the Weyl group. Nevertheless, we expect that it should be possible to do within the general framework of our algorithm, by adopting a suitably modified state labeling scheme that exploits the Weyl symmetry. Work along these lines is currently in progress.

ACKNOWLEDGMENTS

APPENDIX A: CORRESPONDENCE BETWEEN GELFAND–TSETLIN PATTERNS AND YOUNG TABLEAUX

There exists a one-to-one correspondence between i-weights and Young diagrams, and between GT-patterns and semistandard Young tableaux. Thus, our algorithm could equally well have been formulated in terms of Young diagrams and Young tableaux. Since the latter are easy to visualize and are perhaps more widely known in the physics community than the GT-scheme, this appendix summarizes the relation between the two schemes. Our reason for preferring GT-patterns to Young tableaux lies in the complexity of the computer implementation: GT-patterns can be stored in a simpler data structure and allow for a simpler evaluation of the matrix elements (28) and (29).

Note that Young tableaux can also be used to label bases that differ from the GT basis used in this work, notably the one constructed via Young symmetrizers [ch. 7 of Ref. 29]. Thus, the correspondence between GT-patterns and Young tableaux set forth below is purely of combinatorial nature.

1. Definition of Young diagrams and Young tableaux

A Young diagram is an arrangement of boxes in rows and columns conforming to the following rules: (YD.1) there is a single, contiguous cluster of boxes; (YD.2) the left borders of all rows are aligned; and (YD.3) each row is not longer than the one above.

Note that the empty Young diagram consisting of no boxes is a valid Young diagram. For the purpose of describing an $\mathfrak{su}(N)$ irrep, we additionally require that (YD.4) there are at most N rows; and (YD.5) columns with N boxes are dropped, i.e., diagrams which differ only by such columns are identified with each other.

Every Young diagram D satisfying rules (YD.1) to (YD.5) uniquely labels an $\mathfrak{su}(N)$ irrep (or $\mathfrak{sl}(N, \mathbb{C})$ irrep), i.e., the label S used in the main text can be associated with a Young diagram D. Some $\mathfrak{su}(3)$ examples are shown in Table IIa. A further example is given by the Young diagrams specifying $\mathfrak{su}(2)$ irreps: the irrep $S = j$ (describing total angular momentum j) corresponds to a Young diagram with $2j$ boxes in a single row.

A *(semistandard) Young tableau* is a Young diagram, of which the boxes are filled according to the following rules: (YT.1) Each box contains a single integer between 1 and N, inclusive; (YT.2)

TABLE II. (a) Examples of Young diagrams of 𝔰𝔲(3) irreps. Since columns with three boxes can be deleted, the last example is effectively equal to the first one. (b) Set of all of valid 𝔰𝔲(3) Young tableaux of shape ⊟.

the numbers in each row of boxes weakly increase from left to right (i.e., each number is equal to or larger than the one to its left); and (YT.3) the numbers in each column strictly increase from top to bottom (i.e., each number is strictly larger than the one above it).

The basis states of an 𝔰𝔲(N) representation identified by a given Young diagram D can be uniquely labeled by the set of all associated valid semistandard Young tableaux (satisfying rules YT.1 to YT.3), i.e., the label M used in the main text can be associated with a valid Young tableau T. We shall denote the corresponding state by $|T\rangle$. For example, all eight Young tableaux for the diagram ⊟ with respect to 𝔰𝔲(3) are shown in Table IIb. As another example, let us give the correspondence between states $|S, m\rangle$ of an 𝔰𝔲(2) irrep and Young tableaux: $|S, m\rangle$ corresponds to a Young tableau with $2S$ boxes in a single row, containing 1 in the leftmost $S + m$ boxes and 2 in the remaining $S - m$ boxes.

The dimension of a carrier space labeled by a Young diagram is given by the number of valid Young tableaux with the same shape as the Young diagram.

2. Translating GT-patterns to Young tableaux

Each GT-pattern $M = (m_{k,l})$ uniquely specifies a corresponding Young tableau (p. 526 of Ref. 30), which can be constructed as follows. Start with an empty Young tableau (no boxes at all), and step through the entries of the pattern using either of the following two stepping orders, illustrated in Table IIIa and IIIb, respectively:

TABLE III. Conversion of a GT-pattern to a Young tableau, stepping (a) along rows, and (b) along diagonals.

(a)   Proceed from the bottom to top, one row at a time (increasing l from 1 to n), and within each row from left to right (increasing k from 1 to l) or

(b)   Proceed from left to right, one diagonal at a time (increasing k from 1 to n), and within each diagonal from bottom to top (increasing l from k to n).

For each step to a new entry in the pattern, say $m_{k,l}$ located in diagonal k and row l, extend the length of the k-th tableau row to a total of $m_{k,l}$ boxes, by adding to its right boxes containing the number l.

According to the above procedure, the topmost row of the GT-pattern specifies the number of boxes in the rows of the corresponding Young diagram: for the latter, row k of the latter contains $m_{k,N}$ boxes. In this way, the information specifying the irrep S, which for a GT-pattern resides in its topmost row, specifies the shape of the corresponding Young diagram. Moreover, the number of l-boxes (i.e., boxes containing the number l) in tableau row k, say $d_{k,l}$, is given by

$$d_{k,l} = m_{k,l} - m_{k,l-1} \,, \qquad (\text{where} \quad m_{k,l} \equiv 0 \text{ if } k > l). \tag{A1}$$

Since both stepping orders ensure that pattern entries in the same diagonal k are visited in order of increasing l, they yield the same final Young tableau. Order (b) has the feature that an entire tableaux row is completed before the next row is begun. As a result, (b) is more convenient for transcribing the Littlewood–Richardson rule for decomposing a product representation from the language of Young tableaux to that of GT-patterns.

The converse process of transcribing a Young tableau to a GT-pattern can be achieved by using the tableau's kth row, read from left to right, to fill in the pattern's kth diagonal, from bottom to top, in such way as to respect the above rules.

3. Remarks about Young tableaux

In order to aid our intuition for the $\mathfrak{su}(N)$ representation theory presented in the main text, this section restates some of the properties discussed there in terms of Young tableaux.

The p-weight $W(M)$ of a GT-pattern M, as introduced in Sec. VI, has an illustrative interpretation; w_l^M is the number of l-boxes (i.e., boxes containing l) in the tableau corresponding to M. Thus, for the highest-weight Young tableau (the GT-pattern of the corresponding state $|H\rangle$ is specified at the end of Sec. VII), row l from the top contains only l-boxes (i.e., $w_l^M = m_{l,N}$), e.g., $\begin{smallmatrix}\boxed{1}\boxed{1}\boxed{1}\boxed{1}\\ \boxed{2}\boxed{2}\\ \boxed{3}\boxed{3}\end{smallmatrix}$. Furthermore, if the states $|T\rangle$ and $|T'\rangle$ have the same p-weight, the tableaux T and T' contain the same set of entries (i.e., the same number of l-boxes) but arranged in different ways. For example, for $\mathfrak{su}(3)$ $\begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}$ and $\begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}$ have the same p-weight $W = (1, 1, 1)$.

The action of the raising and lowering operators $J_{\pm}^{(l)}$ on p-weights is given by Eq. (VII). The corresponding action of $J_{+}^{(l)}$ on a state labeled by a Young tableau T produces a linear combination of states labeled by tableaux containing one more l-box and one less $(l+1)$-box. Analogously, $J_{-}^{(l)}$ has the reverse effect on Young tableaux: it produces a linear combination of tableaux containing one less l-box and one more $(l+1)$-box.

APPENDIX B: DERIVATION OF OUR FORMULATION OF THE LITTLEWOOD–RICHARDSON RULE

Our formulation of the Littlewood–Richardson rule in Sec. VIII is based on a version by van Leeuwen,[27] formulated in terms of Young tableaux, which we outline here. We then rephrase this in the language of Gelfand–Tsetlin patterns to derive the method presented in Sec. VIII, in particular Eq. (31).

Given two Young diagrams D and D', write down all possible semistandard Young tableaux for D, and for each such tableau (to be called the *current tableau* below), construct a corresponding

Young diagram (to be called the *trial diagram* below) in the following manner:

(1) Start the trial diagram as a fresh copy of D'.
(2) Step through the boxes of the current tableau from right to left, from top to bottom.
(3) If the box encountered at a given step is an *l*-box, add a box at the right end of row *l* of the trial diagram.
(4) If this produces a trial diagram that is no longer a valid Young diagram (having a row longer than the one above), discard it and start a new with the next tableau.
(5) If, however, a valid Young diagram is constructed during each step, the final Young diagram obtained after the last step represents an irrep occurring in the decomposition of $D \otimes D'$.

Let us now translate the above steps into the GT-scheme, thus deriving the rules set forth in Sec. VIII. There, we assume two i-weights S and S' to be given instead of two Young diagrams. Naturally, taking a fresh copy of D' corresponds to initializing $(t_1, \ldots, t_N) = (m'_{1,N}, \ldots, m'_{N,N})$ and stepping through the current tableau in the reading order of step (2) corresponds to stepping through the GT-pattern M associated with S along the diagonals from top to bottom and from left to right. (This follows from the rules for translating GT-patterns to Young tableaux given in Sec. A 2; recall that the *k*th diagonal of a GT-pattern specifies the content of the *k*th row of the corresponding Young tableau.)

Instead of processing one box of the current tableau at a time, we treat all identical boxes of a given row at once when stepping through the corresponding GT-pattern. Recalling that $b_{k,l}$ of Eq. (A1) gives the number of *l*-boxes in row *k* of the current tableau, it follows that $B_{k,l} \equiv m'_{l,N} + \sum_{k'=1}^{k} b_k$ then gives the number of boxes in row *l* of the trial diagram after having processed all boxes of type *l* in row *k* of the current tableau.

The condition (31), which must be fulfilled for all $1 \leq k \leq l \leq N$, finally assures that the trial diagram is a valid Young diagram after each step.

APPENDIX C: IDENTIFYING IRREPS AND STATES BY A SINGLE INTEGER

For numerical codes dealing with i-weights, it is useful to identify each i-weight by a unique number. To this end, we need a one-to-one mapping between the set of all $\mathfrak{su}(N)$ i-weights (for given N) and the set of nonnegative integers. We shall construct such a mapping by devising an ordering rule for i-weights, using this rule to arrange all possible diagrams in a list of increasing order, and labeling each i-weight by its position in this list.

Similarly, we would like to map GT-patterns to matrix indices, so we also need a one-to-one mapping between the set of all GT-patterns belonging to a given irrep and the integers from one to the dimension of that irrep. Therefore, we also define an order on GT-patterns of a given irrep and proceed analogously.

1. Identifying i-weights with a single number

We adopt throughout the convention for an i-weight $S = (m_{k,N})$ that $m_{N,N} = 0$ (Sec. V). For i-weights we choose the following ordering rule: the "smaller" of two i-weights is taken to be the one with the smaller first element; in case of a tie, compare the second element, and so on. Formally, given two i-weights S and S', we assign the order

$$S < S' \text{ if and only if, for the smallest index (say } k \text{) for which } m_{k,N} \neq m'_{k,N}, \text{ we have } m_{k,N} < m'_{k,N}.$$

(C1)

Table IV shows the first few i-weights of SU(4), arranged in increasing order.

Using this ordering rule, all possible $\mathfrak{su}(N)$ i-weights can be arranged in a list of increasing order and uniquely labeled by a nonnegative integer, say $P(S)$, giving its position in this list,

$$P(S) = \#\{S'|S' < S\}.$$

(C2)

TABLE IV. The first few i-weights of $\mathfrak{su}(4)$ (excluding weights with $m_{4,4} \neq 0$), arranged in increasing order.

$P(S)$	$S = (m_{1,4}, m_{2,4}, m_{3,4}, m_{4,4})$	$P(S)$	$S = (m_{1,4}, m_{2,4}, m_{3,4}, m_{4,4})$
0	$(0,0,0,0)$	5	$(2,1,0,0)$
1	$(1,0,0,0)$	6	$(2,1,1,0)$
2	$(1,1,0,0)$	7	$(2,2,0,0)$
3	$(1,1,1,0)$	8	$(2,2,1,0)$
4	$(2,0,0,0)$	9	$(2,2,2,0)$

To determine $P(S)$ for a given i-weight S, we simply count the number of smaller weights S': this number is given by the number (say $P_1(S)$) of all weights S' with $m'_{1,N} < m_{1,N}$, plus the number of all S' with $m'_{1,N} = m_{1,N}$ but $m'_{2,N} < m_{2,N}$ (say $P_2(S)$), etc. Thus,

$$P(S) = \sum_{k=1}^{N-1} P_k(S), \tag{C3}$$

where $P_k(S)$ is the number of weights S' whose first $k-1$ entries are the same as those of S ($m'_{k',N} = m_{k',N}$ for all $k' < k$), while the kth entry is arbitrary but smaller than that of S ($m'_{k,N} < m_{k,N}$), and the remaining entries arbitrary (but subject to S' being a valid i-weight, with $m'_{N,N} = 0$). The nontrivial "free" (though constrained) entries of S', namely $(m'_{k,N}, m'_{k+1,N}, \ldots, m'_{N-1,N})$, can be viewed as an i-weight $\tilde{S} = (\tilde{m}_{\tilde{k},\tilde{N}})$ of length $\tilde{N} = N - k$, whose entries $\tilde{m}_{\tilde{k},\tilde{N}} = m'_{k-1+\tilde{k},N}$ (for $1 \leq \tilde{k} \leq \tilde{N}$) satisfy

$$m_{k,N} - 1 \geq \tilde{m}_{1,\tilde{N}} \geq \tilde{m}_{2,\tilde{N}} \geq \ldots \geq \tilde{m}_{\tilde{N},\tilde{N}} \geq 0 . \tag{C4}$$

$P_k(S)$ thus is the number of allowed weights $\tilde{S}$ that satisfy (C4).

To calculate $P_k(S)$, we note that it is equal to the number of ways to draw or "strike out," from the set of integers $\{1, \ldots, m_{k,N} - 1 + \tilde{N}\}$, an ordered subset $\{d_{\tilde{k}}\}$ of $\tilde{N}$ integers, $d_1 < d_2 < \ldots < d_{\tilde{N}}$ [see Fig. 3(a)], since there is a one-to-one correspondence between the set of all possible such strikeouts and the set of all i-weights $\tilde{S}$ satisfying (C4): for a given struck-out set $\{d_{\tilde{k}}\}$, with $1 \leq \tilde{k} \leq \tilde{N}$, set $\tilde{m}_{\tilde{k},\tilde{N}}$ equal to the number of nonstruck-out integers smaller than $d_{\tilde{N}+1-\tilde{k}}$ (i.e., $\tilde{m}_{\tilde{k},\tilde{N}} = d_{\tilde{N}+1-\tilde{k}} - (\tilde{N} + 1 - \tilde{k})$). For example, the weight $\tilde{S}$ that is largest (w.r.t. to the ordering rule
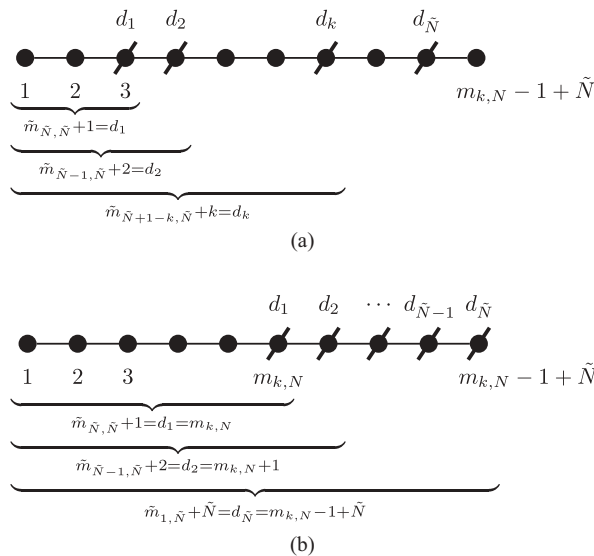

FIG. 3. Enumeration scheme of i-weights. (a) Illustration of the combinatorics underlying $P_k(S)$. (b) Striking out items such that each $\tilde{m}_{\tilde{k},\tilde{N}}$ takes on the largest possible value.

TABLE V. (a) Illustrating the row-by-row rule chosen in Appendix. C 2 to define an ordering scheme for the indices of GT-patterns: $(k, l) < (k', l')$ if $l = l'$ and $k < k'$, or if $l > l'$. (b) Ordering of all GT-patterns belonging to the SU(3) irrep $(2, 1, 0)$, together with the corresponding pattern indices $Q(M)$.

$$(a) \begin{pmatrix} 1 & 2 & 3 \\ & 4 & 5 \\ & & 6 \end{pmatrix}$$

$$(b) \quad \begin{pmatrix} 2 & 1 & 0 \\ & 1 & 0 \\ & & 0 \end{pmatrix} < \begin{pmatrix} 2 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} < \begin{pmatrix} 2 & 1 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} < \begin{pmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 0 \end{pmatrix} < \begin{pmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 1 \end{pmatrix} < \begin{pmatrix} 2 & 1 & 0 \\ & 2 & 0 \\ & & 2 \end{pmatrix} < \begin{pmatrix} 2 & 1 & 0 \\ & 2 & 1 \\ & & 1 \end{pmatrix} < \begin{pmatrix} 2 & 1 & 0 \\ & 2 & 1 \\ & & 2 \end{pmatrix}$$

$Q(M)$:   1   2   3   4   5   6   7   8

(C1)), namely, having all elements equal to $m_{k,N} - 1$, is obtained by choosing the struck-out integers $d_{\tilde{k}}$ to be as large as possible (see Fig. 3(b)). Thus, we have

$$P_k(S) = \binom{m_{k,N} - 1 + \tilde{N}}{\tilde{N}} = \binom{N - k + m_{k,N} - 1}{N - k} \tag{C5}$$

and, consequently,

$$P(S) = \sum_{k=1}^{N-1} \binom{N - k + m_{k,N} - 1}{N - k}. \tag{C6}$$

2. Mapping of Gelfand–Tsetlin patterns to matrix indices

In analogy to the ordering we have defined on i-weights, we introduce an ordering on the set of Gelfand–Tsetlin patterns of a given irrep (i.e., given top row of the pattern). Let $M = (m_{k,l})$ and $M' = (m'_{k,l})$ (where, $1 \leq k \leq l \leq N$) denote two patterns with $m_{k,N} = m'_{k,N}$ for $k = 1, \ldots, N$. We define a row-by-row ordering of indices (see Table Va), increasing from left to right within a row, and from top row to bottom row, i.e., $(k, l) < (k', l')$ if $l = l'$ and $k < k'$, or if $l > l'$. We then define $M' < M$ if and only if for the smallest index for which $m'_{k,l} \neq m_{k,l}$, we have $m'_{k,l} < m_{k,l}$. An example of this ordering is given in Table Vb.

We map each Gelfand–Tsetlin pattern M to a nonnegative integer $Q(M)$ by counting the number of smaller Gelfand–Tsetlin patterns, i.e.,

$$Q(M) = \#\{M' | M' \leq M\}. \tag{C7}$$

This number can be determined by generating the pattern (say $\tilde{M}(\{\tilde{m}_{k,l}\})$) located directly preceding M in the ordered list of patterns, then the pattern preceding $\tilde{M}$, and so on, until we arrive at the beginning of this list. To construct the predecessor of the pattern M, we start by finding the largest index $(\tilde{k}, \tilde{l})$ whose entry $m_{\tilde{k},\tilde{l}}$ can be decreased without violating the betweenness condition (21), rewritten here as

$$m_{k,l+1} \geq m_{k,l} \geq m_{k+1,l+1}, \ (1 \leq k < l + 1 \leq N), \tag{C8}$$

with respect to smaller indices while disregarding it with respect to larger indices (i.e., without violating the second inequality but disregarding the first). Thus, $(\tilde{k}, \tilde{l})$ is the index for which $m_{k,l} = m_{k+1,l+1}$ for all $(k, l) > (\tilde{k}, \tilde{l})$ but $m_{\tilde{k},\tilde{l}} > m_{\tilde{k}+1,\tilde{l}+1}$. We then decrease $m_{\tilde{k},\tilde{l}}$ by one and reset the entries of all larger indices to the maximal values that satisfy the new betweenness condition. Concretely,

$$\tilde{m}_{k,l} = \begin{cases} m_{k,l} & \text{for } (k, l) < (\tilde{k}, \tilde{l}) \quad \text{(keep entries with smaller indices unchanged)} \\ m_{k,l} - 1 & \text{for } (k, l) = (\tilde{k}, \tilde{l}) \quad \text{(decrease by 1 the entry with largest index} \\ & \qquad \qquad \qquad \qquad \text{for which this is possible)} \\ \tilde{m}_{k,l+1} & \text{for } (k, l) > (\tilde{k}, \tilde{l}) \quad \text{(give entries with larger indices their} \\ & \qquad \qquad \qquad \qquad \text{largest possible value)} \end{cases} . \tag{C9}$$

The number $Q(M)$ is, of course, the number of times we can repeat the process of constructing a preceding pattern. This procedure maps the lowest-weight and highest-weight states of an irrep S to the numbers 1 and $\dim(S)$, respectively.

APPENDIX D: SOURCE CODE

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