Bosonization for Beginners — Refermionization for Experts

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Abstract. This tutorial review gives an elementary and self-contained derivation of the standard identities ($\psi_\eta(x) \sim F_\eta e^{-i\phi_\eta(x)}$, etc.) for abelian bosonization in 1 dimension in a system of finite size L, following and simplifying Haldane's constructive approach. As a non-trivial application, we rigorously resolve (following Furusaki) a recent controversy regarding the tunneling density of states, $\rho_{dos}(\omega)$, at the site of an impurity in a Tomonaga-Luttinger liquid: we use finite-size refermionization to show exactly that for $g = \frac{1}{2}$ its asymptotic low-energy behavior is $\rho_{dos}(\omega) \sim \omega$. This agrees with the results of Fabrizio & Gogolin and of Furusaki, but not with those of Oreg and Finkel'stein (probably because we capture effects not included in their mean-field treatment of the Coulomb gas that they obtained by an exact mapping; their treatment of anti-commutation relations in this mapping is correct, however, contrary to recent suggestions in the literature). — The tutorial is addressed to readers with little or no prior knowledge of bosonization, who are interested in seeing "all the details" explicitly; it is written at the level of beginning graduate students, requiring only knowledge of second quantization, but not of field theory (which is not needed here). At the same time, we hope that experts too might find useful our explicit treatment of certain subtleties that can often be swept under the rug, but are crucial for some applications, such as the calculation of $\rho_{dos}(\omega)$ – these include the proper treatment of the so-called Klein factors that act as fermion-number ladder operators (and also ensure the anti-commutation of different species of fermion fields), the retention of terms of order $1/L$, and a novel, rigorous formulation of finite-size refermionization of both $Fe^{-i\Phi(x)}$ and the boson field $\Phi(x)$ itself.

Changes relative to first version of cond-mat/9805275: We have substantially revised our discussion of the controversy regarding the tunneling density of states ρ_{dos} at the site of an impurity in a Luttinger liquid, with regard to the following points: (1) In a new Appendix K, we confirm explicitly that Oreg and Finkel'stein's treatment of fermionic anti-commutation relations is *correct*, contrary to recent suggestions (including our own). (2) To try to understand why their result for ρ_{dos} differs from that of Fabrizio & Gogolin, Furusaki and (for g=1/2) ourselves, we make a new suggestion in Sections 1.B and 10.D: this is probably because of effects not captured by their *mean-field* treatment of their Coulomb gas. (3) In Sections 10.C and 10.D we have replaced the first version of our calculation of ρ_{dos} by a more explicit one (the result is unchanged), in which we refermionize not only the exponential $e^{i\Phi}$ but, for the first time, also the field Φ itself (Section 10.C.4); this allows us to calculate various correlation functions involving Φ explicitly in terms of *fermion* operators (a new Appendix J contains several detailed examples, and a new Figure 4 showing the corresponding Feynman diagrams).

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Contents

1 Introduction

1-Dimensional abelian[1] bosonization is a technique for representing 1-D fermion fields $\psi_\eta(x)$, where η is a species (e.g. spin) index, in terms of bosonic fields $\phi_\eta(x)$ through a relation of the form[2] $\psi_\eta \sim F_\eta \, e^{-i\phi_\eta}$, where F_η is a so-called Klein factor which lowers the number of η-fermions by one. Over the years, it has become a rather popular tool for treating certain strongly-correlated electron systems in 1 dimension. The reason for its popularity is that some problems which appear intractable when formulated in terms of fermions turn out to become easy, even trivial, when formulated in terms of boson fields – successful applications include Tomonaga-Luttinger liquid theory (dealing with a quantum wire of interacting 1-D electrons), quantum Hall edge states and quantum impurity problems such as the Kondo problem.

Once one has learnt the "bosonization rules", the formalism is very user-friendly – one seldom needs to know more than how to work with free boson fields. However, as often happens, actually *proving* the validity of the bosonization formalism in explicit detail and ironing out all the subtleties is substantially harder than simply applying it. The successive efforts of quite a number of pioneers was required to piece together the puzzle, of which we mention just a few milestones (brief overviews of historical developments are also given in Refs. [3] and [4]). Tomonaga [5] was the first to identify boson-like behavior of certain elementary excitations in a 1-D theory of interacting fermions. A precise definition of these bosonic excitations in terms of bare fermions was given by Mattis and Lieb [6], who took the first step towards a correct solution of a model of interacting 1-D fermions earlier proposed by Luttinger [7]. A bosonic representation of a fermion field at a single point, essentially of the form $\psi_\eta(x=0) \sim e^{-i\phi_\eta(x=0)}$, was first introduced by Schotte and Schotte [8] to calculate x-ray edge transition rates. The extension of their relation to arbitrary x, $\psi_\eta(x) \sim e^{-i\phi_\eta(x)}$, was discovered simultaneously by Mattis [9] and by Luther and Peschel [10], which made the systematic calculation of general correlation functions very simple. However, their expressions for $\psi_\eta(x)$ were not operator identities in Fock space, since they did not discuss the number-lowering Klein factors F_η. The first completely precise bosonization relation in the solid-state literature (though from a field-theoretical viewpoint) was given by Heidenreich [11] when discussing the model of [10] (there was an entirely parallel development in the field-theoretical literature on the the related "massless Thirring model" [12, 13], a review of which lies beyound our scope). The first explicit construction of the Klein factors F_η in terms of bare fermionic operators was given by Haldane [14], whose detailed discussion in [4] essentially completed the development of the bosonization formalism.

These advances resulted in two somewhat different and not entirely equivalent approaches, which we shall call "field-theoretical" and "constructive", respectively.

1.A Field-theoretical versus constructive bosonization

We recapitulate the differences between field-theoretical and constructive bosonization, and explain why we strongly prefer the latter, which is more rigorous and, we believe, more user-friendly.

The field-theoretical approach, examples of which are summarized in Appendix A (following Shankar [15]) or in Section 10.B.3 (following Kane and Fisher [16]), has a somewhat formal character: typically one starts by defining bosonic fields $\phi_\eta(x)$ with a set of prescribed properties, usually in a system of infinite size, and then uses field-theoretical machinery to calculate the commutation relations and Green's functions of $e^{-i\phi_\eta(x)}$ and $e^{i\phi_\eta(x)}$. These turn out to be the same as those of fermion fields $\psi_\eta(x)$ and

[1] We discuss only abelian bosonization. For a review of non-abelian bosonization, see [1] or [2].

[2] Alternative notations are discussed in Sections 10.A.4, 10.B.3, A.3 and D.2.

$\psi^\dagger_\eta(x)$, suggesting their formal correspondence, so that one writes $\psi_\eta(x) \sim F_\eta e^{-i\phi_\eta(x)}$. The so-called *Klein factors* F_η (often denoted by $e^{-i\theta_\eta}$, or viewed as Majorana fermions) are included formally to guarantee appropriate anti-commutation relations ($\{\psi_\eta, \psi^\dagger_{\eta'}\} = 0$ if $\eta \neq \eta'$). Some treatments also get by completely without Klein factors, using instead appropriately contrived definitions of the $\phi_\eta(x)$ fields (see Appendix A for an example).

Such field-theoretical approaches are completely adequate to prove *that* bosonization works; however, they do not really clarify *why* it works. Moreover, $\phi_\eta(x)$ and F_η do not appear naturally from first principles, but instead seem to have the status of mere auxiliary quantities that are introduced somewhat artificially, with properties that must be fine-tuned to make things work. Newcomers may be left with the sense that $\psi_\eta(x) \sim F_\eta e^{-i\phi_\eta(x)}$, though demonstrably true, is a somewhat arbitrary coincidence, without clearly understanding its origin and how it could possibly have been discovered.

These issues are clarified in the more rigorous "constructive" approach, used e.g. by Mattis and Lieb [6], Luther and Peschel [10] and Emery [3], and nurtured to maturity by Haldane, whose 1981 paper [4] is the standard reference. This approach refrains from using any formal field-theoretical machinery. Instead, it takes as starting point a fermion field $\psi_\eta(x) = \left(\frac{2\pi}{L}\right)^{1/2} \sum_k e^{-ikx} c_{k\eta}$ in a system of finite size L (this quantizes the momenta, yielding a Hilbert space with a countable set of states, which is crucial), and constructs (hence "constructive") *all* further operators and fields explicitly and naturally in terms of the initially given $c_{k\eta}$-operators (these constructions are summarized in Table 1 on p. 8). The entire bosonization formalism can then be derived deductively at a most elementary level as a set of operator identities in Fock space, simply by judiciously employing standard operator identities like the Baker-Hausdorff lemma to manipulate functions of the electron operators $c_{k\eta}$.

At present, the field-theoretical approach seems to be in much wider use than the constructive one (perhaps because Haldane's discussion of his construction of Klein factors [4] can appear complicated and hard to follow, though unneccessarily so). For example, it is used extensively in the path-breaking work of Kane and Fisher on impurities in Tomonaga-Luttinger liquids [16] (we review their notation in Section 10.B.3). Nevertheless, this tutorial reviews and strongly advocates the constructive approach, since in our opinion it is significantly superior, for a number of reasons (admittedly the last three are subjective):

1. Constructive bosonization is more rigorous: $\psi_\eta \sim F_\eta e^{-i\phi_\eta}$ has the status of an *operator identity in Fock space,* and since its ingredients are constructed explicitly from the $c_{k\eta}$'s, their physical meaning becomes explicit: $\partial_x \phi_\eta(x)$ represents local density fluctuations (at fixed total fermion number) of the Fermi sea, and the Klein factor F_η lowers the total number of η-fermions by one. In contrast, in the field-theoretical approach the Fock space of states is not explicitly defined, and $\psi_\eta \sim F_\eta e^{-i\phi_\eta}$ merely has the status of a formal correspondence. Though $\partial\phi_\eta(x)$ can also be related to density fluctuations, the Klein factors F_η usually are viewed merely as formal tools ensuring proper anti-commutation relations, and the fact that they lower the number of η-electrons is ignored. This is particularly evident in papers in which F_η are viewed as a Majorana fermions, which is imprecise: in fact $F_\eta^2 \neq 1$, since removing two η-electrons is not equivalent to unity.

2. In the constructive approach, refermionization is more rigorous too: the refermionization identity, in which the bosonization identity is "read backwards", so to speak, also has the status of an operator identity in Fock space.

3. The constructive approach is more "user-friendly" (because of, not in spite of, its higher rigor): since all needed operators arise naturally and have a physical interpretation, the formalism is easier to learn and to work with. In contrast, field-theoretical bosonization is, due to its formal nature, littered with formal pitfalls. Of course it also yields correct results when used with sufficient care – however, regarding Klein factors it is quite easy to make mistakes (Ref. [27] discusses an example).

4. Constructive bosonization offers a very simple answer to the question: "*Why* is it possible at all to represent a fermionic field as the exponential of a bosonic field?" The essence of the answer is that the state $\psi_\eta(x)|\vec{0}\rangle_0$, where $|\vec{0}\rangle_0$ is the Fermi ground state, turns out to be an *eigenstate* of the bosonic operators $b_{q\eta}$ from which the boson field $\phi_\eta(x) = -\sum_{q>0}\left(\frac{2\pi}{qL}\right)^{1/2}(e^{-q(ix+a/2)}b_{q\eta} + \text{h.c.})$. is constructed. Therefore it must have a *coherent state representation* in terms of the $b^\dagger_{q\eta}$'s, which turns out to be precisely $\sim F_\eta e^{-i\phi_\eta(x)}|\vec{0}\rangle_0$. Thus, one discovers that the trick that makes bosonization work, its *raison d'etre,* so to speak, is that it cleverly exploits some very convenient properties of bosonic coherent states!

5. In constructive bosonization, regularizing infinities is easier: one simply consistently normal-orders all c_k's and b_q's. Instead, field-theoretical treatments customarily employ a point-splitting prescription that becomes rather cumbersome when terms of order $1/L$ are to be retained, as here (or [17, 18]). Of course, normal-ordering and point-splitting are equivalent regularization schemes, see Appendix G.

In this tutorial we give a (at times very) detailed account of the constructive approach to bosonization, at a level accessible to beginning graduate students with a knowledge of second quantization, but not of field theory. Our development of the formalism is a simplified version of that given by Haldane [4] (the relation between his and our notation is given in Section 10.A.4). Ours differs from Haldane's mainly in that we use only left-moving fields (it is easy to rewrite some of them as right-movers, if required, see Section 10.A), and in that we exploit to the full the above-mentioned connection to the properties of boson coherent states. (Another discussion of constructive bosonization similar in spirit to ours was recently written by Schönhammer and Meden [19, 20].)

1.B Application to Tomonaga-Luttinger liquid with impurity

To give a non-trivial example of how the formalism is used, we discuss impurity scattering in a Tomonaga-Luttinger liquid; this requires not only bosonization but also refermionization, a rigorous, novel treatment of which allows us to resolve a recent controversy.

In Section 10 we consider the tunneling density of states, $\rho_{dos}(\omega)$, at the site of an impurity in a Tomonaga-Luttinger liquid [5, 7], i.e. a quantum wire of interacting 1-D electrons characterized by the dimensionless electron-electron interaction parameter $g > 0$ (for free electrons $g = 1$). The exponent ν governing the low-energy behavior of $\rho_{dos}(\omega) \sim \omega^{\nu-1}$ as $\omega \to 0$, was the subject of a recent controversy: Without impurities, it is known that $\nu_{free} = (g + 1/g)/2$. In the presence of an impurity, Oreg and Finkel'stein (OF) [21] found $\nu = 1/(2g)$, using an exact mapping to a Coulomb gas problem, which they treated in a mean-field approximation; this would imply that $\nu < \nu_{free}$, i.e. $\rho_{dos}(\omega)$ is *enhanced,* and actually diverges for $\omega \to 0$ if $g > 1/2$. In contrast, Fabrizio and Gogolin (FG) [23] and Furusaki [22] found $\nu = 1/g$, which would imply that for repulsive interactions ($g < 1$) one has $\nu > \nu_{free}$, i.e. $\rho_{dos}(\omega)$ is *suppressed,* with $\rho_{dos}(0) = 0$ (reminiscent of the classical RG conclusion of Kane and Fisher [16] that the conductance accross a backscattering impurity vanishes at zero temperature if $g < 1$). Furusaki [22] checked his result for the exactly solvable case $g = 1/2$ (using refermionization, the inverse, so to speak, of bosonization), and indeed found $\nu = 2$. To explain why OF had obtained a different result, FG [23] suggested that OF had neglected the effects of Klein factors. OF disputed this [24] and in turn alledged that FG had incorrectly replaced the impurity by "open boundary conditions", although the two are in general *not* equivalent: a "cut wire" suppresses both current *and* density fluctuations, whereas an impurity suppresses *only* current fluctuations.

Our opinion of these matters is explained in detail in Section 10.D.3 and Appendix J.1. In brief, we believe (i) that FG's analysis of the role Klein factors is correct, but not their criticism of OF; (ii) that

OF did treat Klein factors correctly (see our Appendix K), showing that they produce a Coulomb gas with a certain "sign problem", but that OF's mean-field treatment of the latter is not sufficiently accurate; (iii) that OF's assertion about the inequivalence of an impurity and a "cut wire" is correct, but that their criticism of FG is misguided nevertheless, since FG do incorporate density fluctuations and use the cut wire only to find the effects of *current* fluctuations; (iv) that the relevant issues become *much* clearer when reformulated using *constructive* instead of the field-theoretic bosonization used by FG, OF and Furusaki. Using the former, we give an appealingly simple yet more rigorous version of Furusaki's $g = 1/2$ calculation (he nonrigorously treats Klein factors as Majorana fermions), which resolves the controversy in favor of FG and Furusaki.

To set the stage for this calculation, we discuss *refermionization* in pedagogical detail in Section 10.C.2. We refermionize *at finite* L, since then the requisite Klein factors can be introduced as naturally and rigorously as during bosonization. Moreover, we refermionize not only the usual combination $Fe^{-i\Phi(x)}$, but, for the first time, also the bosonic field $\Phi(x)$ itself; this enables us to calculate general bosonic correlation functions in terms of fermionic ones. Since refermionization is usually implemented less rigorously, our treatment of this topic might be of interest to experts too, hence the second part of the review's title.

1.C Outline and bosonization dictionary

The outline of the review is as follows: The main ingredients of the constructive approach to bosonization are summarized in Table 1 below, for ease of reference and to survey what is to be proven in subsequent sections. In Section 2 we state the properties required to make a 1-D fermion theory amenable to bosonization, and in Section 3 define the standard fermion fields $\psi_\eta(x)$ as Fourier sums over a given set of $c_{k\eta}$'s. In Section 4 we show that the fermionic Fock space spanned by the $c_{k\eta}$'s can also be reorganized in terms of the electron number operators $\widehat{N}_\eta$, their raising and lowering operators F_η, $F_\eta^\dagger$ and bosonic particle-hole operators $b_{q\eta}$, $b_{q\eta}^\dagger$, and construct from the latter the boson fields $\phi_\eta(x)$ in Section 5. The heart of this review is Section 6, where we give a very simple yet rigorous and detailed derivation of the bosonization identity. In Section 7 we consider fermions with linear dispersion and bosonize the Hamiltonian, in Section 8 derive a remarkable relation between free fermion and boson Green's functions, and in Section 9 derive some general properties of the so-called vertex operators $V_\lambda^{(\eta)} \sim e^{i\lambda\phi_\eta}$. In Section 10 we illustrate the formalism by calculating the tunneling density of states $\rho_{dos}(\omega)$ at an impurity site in a $g = \frac{1}{2}$ Tomonaga-Luttinger liquid.

In Appendix A we make explicit the connection between the constructive and field-theoretical approaches to bosonization (as described by Shankar [15]), by showing how the operators used in the latter can be constructed in terms of the former. The remaining appendices contain details somewhat too arduous to appear in the main text.

starting point:	$\{c_{k\eta}, c^\dagger_{k'\eta'}\} = \delta_{\eta\eta'}\delta_{kk'} \qquad (\eta = 1, \dots, M)$	(1)
k-quantization:	$k = \frac{2\pi}{L}(n_k - \frac{1}{2}\delta_b) \qquad (\delta_b \in [0,2), \quad n_k \in \mathbb{Z})$	(2)
vacuum state:	$c_{k\eta}\lvert\vec{0}\rangle_0 \equiv 0 \quad \text{for} \quad k > 0, \qquad c^\dagger_{k\eta}\lvert\vec{0}\rangle_0 \equiv 0 \quad \text{for} \quad k \le 0$	(10)
number operator:	$\widehat{N}_\eta \equiv \sum_k {}^*_* c^\dagger_{k\eta} c_{k\eta} {}^*_* = \sum_k \left[c^\dagger_{k\eta} c_{k\eta} - {}_0\langle\vec{0}\lvert c^\dagger_{k\eta} c_{k\eta}\rvert\vec{0}\rangle_0 \right]$	(13)
boson creator:	$b^\dagger_{q\eta} \equiv \frac{i}{\sqrt{n_q}} \sum_k c^\dagger_{k+q\ \eta} c_{k\eta} \quad (q = \frac{2\pi}{L} n_q > 0,\ n_q \in \mathbb{Z}^+)$	(16)
boson commutator:	$[b_{q\eta}, b^\dagger_{q'\eta'}] = \delta_{\eta\eta'}\delta_{qq'}$	(18)
$\vec{N}$-part. ground st.:	$\widehat{N}_\eta\lvert\vec{N}\rangle_0 = N_\eta\lvert\vec{N}\rangle_0\ , \qquad b_{q\eta}\lvert\vec{N}\rangle_0 = 0$	(15, 20)
def. of Klein factor:	$F^\dagger_\eta f(b^\dagger)\lvert\vec{N}\rangle_0 \equiv f(b^\dagger) c^\dagger_{N_\eta+1}\lvert\vec{N}\rangle_0$	(25) [or $(D15)$]
F commutators:	$[F, b] = 0, \quad \{F^\dagger_\eta, F_{\eta'}\} = 2\delta_{\eta\eta'}, \quad [\widehat{N}_\eta, F_{\eta'}] = -\delta_{\eta\eta'}F_\eta$	(24, 30, 32)
fermion field:	$\psi_\eta(x) \equiv \left(\frac{2\pi}{L}\right)^{1/2} \sum_k e^{-ikx} c_{k\eta}$	(3)
ψ_η commutator:	$\{\psi_\eta(x), \psi^\dagger_\eta(x')\} = \delta_{\eta\eta'}\, 2\pi\, \delta(x - x') \quad (\text{for } \lvert x - x'\rvert < L)$	(8)
boson field:	$\phi_\eta(x) \equiv -\sum_{q>0} \frac{1}{\sqrt{n_q}} \left(e^{-iqx} b_{q\eta} + e^{iqx} b^\dagger_{q\eta}\right) e^{-aq/2}$	(34)
$\phi_\eta, \partial_x\phi_\eta$ commutator:	$[\phi_\eta(x), \partial_{x'}\phi_{\eta'}(x')] = \delta_{\eta\eta'}\, 2\pi i \left(\delta(x - x') - \frac{1}{L}\right)$	(50)
<u>bosonization identity:</u>	$\psi_\eta(x) = F_\eta\, a^{-1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\phi_\eta(x)}$	(63)
(2π) density:	$\rho_\eta(x) \equiv {}^*_* \psi^\dagger_\eta(x)\psi_\eta(x) {}^*_* = \partial_x\phi_\eta(x) + \frac{2\pi}{L}\widehat{N}_\eta$	(35)
free ferm. Hamilton:	$H_{0\eta} \equiv \sum_k k\, {}^*_* c^\dagger_{k\eta} c_{k\eta} {}^*_* = \int_{-L/2}^{L/2} \frac{dx}{2\pi}\, {}^*_* \psi^\dagger_\eta(x) i\partial_x \psi_\eta(x) {}^*_*$	(65, 66)
bosonized Hamilton:	$= \sum_{q>0} q\, b^\dagger_{q\eta} b_{q\eta} \ +\ \frac{2\pi}{L}\frac{1}{2}\widehat{N}_\eta(\widehat{N}_\eta + 1 - \delta_b)$	(69)
	$= \int_{-L/2}^{L/2} \frac{dx}{2\pi}\frac{1}{2}\, {}^*_* (\partial_x\phi_\eta(x))^2 {}^*_* \ +\ (\frac{2\pi}{L})\frac{1}{2}\widehat{N}_\eta(\widehat{N}_\eta + 1 - \delta_b)$	(70)
Green's functions:	$\langle \mathcal{T}\psi_\eta(z)\psi^\dagger_{\eta'}(0)\rangle = \delta_{\eta\eta'}\ a^{-1}\, \mathrm{sign}(\tau)\, e^{\langle \mathcal{T}\phi_\eta(z)\phi_\eta(0) - \phi_\eta(0)\phi_\eta(0)\rangle}$	(78)
$(z \equiv \tau + ix)$	$= \delta_{\eta\eta'} \left(\frac{\beta}{\pi}\sin[\frac{\pi}{\beta}(z + a\,\mathrm{sign}(\tau))]\right)^{-1} \stackrel{T=0}{\longrightarrow} \frac{1}{z + a\,\mathrm{sign}(\tau)}$	(73)
vertex operator:	$V^{(\eta)}_\lambda(z) \ \equiv\ \left(\frac{L}{2\pi}\right)^{-\lambda^2/2}\ {}^*_* e^{i\lambda\phi_\eta(z)} {}^*_* \ =\ a^{-\lambda^2/2} e^{i\lambda\phi_\eta(z)}$	(85)
$V^{(\eta)}_\lambda$ Green's funct.:	$\langle V^{(\eta)}_\lambda(z) V^{(\eta')}_{\lambda'}(0)\rangle = \frac{\delta_{\eta\eta'}\ (L/2\pi)^{-(\lambda+\lambda')^2/2}}{\left(\frac{\beta}{\pi}\sin[\frac{\pi}{\beta}(z + a)]\right)^{-\lambda\lambda'}} \stackrel{L=\beta=\infty}{\longrightarrow} \frac{\delta_{\eta\eta'}\delta_{-\lambda,\lambda'}}{(z+a)^{\lambda^2}}$	(87)

Table 1: Bosonization Dictionary: A survey of the main ingredients and results of the constructive bosonization formalism (with the equation numbers used below), listed here for ease of reference.

2 Bosonization prerequisites

It is possible to constructively bosonize a theory involving M species of fermions whenever the following prerequisites are met: The theory can be formulated in terms of a set of fermion creation and annihilation operators with canonical anti-commutation relations

$$\{c_{k\eta}, c^\dagger_{k'\eta'}\} = \delta_{\eta\eta'}\delta_{kk'} \ , \qquad k \in [-\infty, \infty], \qquad \eta = 1, \ldots, M, \tag{1}$$

which are labelled by a *species* index $\eta = 1, \ldots, M$ distinguishing the M different species from each other, and a discrete, unbounded momentum (or wave-number) index k of the form

$$k = \tfrac{2\pi}{L}(n_k - \tfrac{1}{2}\delta_b) \ , \qquad \text{with} \quad n_k \in \mathbb{Z} \quad \text{and} \quad \delta_b \in [0, 2) \ . \tag{2}$$

Here the n_k are integers, L is a length to be associated with the system size, and δ_b is a parameter that will determine the boundary conditions of the fermion fields defined below [see Eq. (5)].

For example, η can denote electron spin: $\eta = (\uparrow, \downarrow)$, $M = 2$; or it can distinguish left-moving from right-moving spinless electrons, e.g. in a one-dimensional wire: $\eta = (L, R)$, $M = 2$, see Section 10; or both: $\eta = (L\uparrow, R\uparrow, L\downarrow, R\downarrow)$, $M = 4$, etc. The momentum index k typically labels the eigenergies ε_k of the free, non-interacting system (with ε_0 corresponding to the Fermi-energy ε_F), and hence could equally well have been called an "energy" index. That k be *discrete* and *unbounded* is an essential prerequisite of a detailed and systematic derivation of the bosonization identities. Its discreteness is needed to allow systematic book keeping of states, its unboundedness to allow the definition of proper bosonic operators, see Eq. (16) to (18) below.

The manipulations required to cast a given problem in a form that meets the above prerequisites depends, of course, on the details of the problem. However, they are a prelude to bosonization, not part of the technique itself. Therefore, we do not discuss them here, but refer the reader to Section 10.A for an example, a 1-D quantum wire containing spinless left- and right-moving electrons (for another example, the Kondo problem, see Refs. [17, 18]). Suffice it here to state the main ideas: To ensure that k is discrete, one considers a system of finite size L and definite boundary conditions, thus quantizing the momenta k and energies ε_k. If the continuum limit $L \to \infty$ is required, it is taken only at the end, after the bosonization rules have been established. If the dispersion relation does not automatically imply that k is unbounded, one can make it so by adding a set of (unphysical) negative-energy "positron" states (see Section 10.A).

For definiteness, the reader may think in terms of a linear dispersion relation, $\varepsilon_k = \hbar v_F(k - k_F)$ (i.e. energies and momenta are measured relative to ε_F and k_F), with an infinite bandwidth, so that $k \in [-\infty, \infty]$. However, we emphasize that the bosonization identity $\psi_\eta = F_\eta e^{-i\phi_\eta}$ can be derived *without* specifying the dispersion relation (though the nomenclature to be used, like "ground state" and "particle-hole excitations" only makes sense when the dispersion is monotonic, i.e. $|k| > |k'| \Rightarrow \varepsilon_k > \varepsilon_{k'}$). This is possible because the bosonization identity is an *operator identity,* i.e. valid when acting on any state in the full Fock space. Hence it is independent of the Hamiltonian, whose detailed form only becomes relevant when one calculates correlation functions. Therefore, we shall refrain from specifying a Hamiltonian until Section 7, after all bosonization formalities have been dealt with.

3 Fermion fields – definition and properties

Starting from given a set of electron annihilation operators $c_{k\eta}$ with the properties (1) and (2) specified in Section 2, a set of M fermion fields $\psi_\eta(x)$ can be defined as follows:[3]

$$\psi_\eta(x) \equiv \left(\tfrac{2\pi}{L}\right)^{1/2} \sum_{k=-\infty}^{\infty} e^{-ikx} c_{k\eta} \; , \tag{3}$$

$$\text{with inverse} \qquad c_{k\eta} = (2\pi L)^{-1/2} \int_{-L/2}^{L/2} dx \, e^{ikx} \, \psi_\eta(x) \; . \tag{4}$$

Though in applications one usually takes $x \in [-L/2, L/2]$ (and often $L \to \infty$), the formalism developed below holds for arbitrary $x \in [-\infty, \infty]$. The physical meaning of the $\psi_\eta(x)$-fields and the variable x depends on the manipulations required to formulate a given model in terms of the $c_{k\eta}$'s. For present purposes, the $\psi_\eta(x)$'s are to be regarded simply as mathematical constructs that have the useful property, to be proven below, of being expressable in terms of bosonic fields.

Given a set of discrete k's of the form (2), the ψ_η obey the following periodicity condition [the simplest cases are $\delta_b = 0$ (or 1) for complete periodicity (or anti-periodicity)]:

$$\psi_\eta(x + L/2) = e^{i\pi\delta_b} \, \psi_\eta(x - L/2) \; . \tag{5}$$

(Alternatively, one can view Eq. (5) as a boundary condition that is purposefully imposed on the ψ_η in order to obtain discrete k's satisfying Eq. (2).) Furthermore, Eqs. (1) and (2) and the identity [26]

$$\sum_{n \in \mathbb{Z}} e^{iny} = 2\pi \sum_{\bar{n} \in \mathbb{Z}} \delta(y - 2\pi\bar{n}) \tag{6}$$

immediately imply the anti-commutation relations

$$\{\psi_\eta(x), \psi^\dagger_{\eta'}(x')\} = \delta_{\eta\eta'} \tfrac{2\pi}{L} \sum_{n \in \mathbb{Z}} e^{-i(x-x')(n-\delta_b/2)2\pi/L} \tag{7}$$

$$= \delta_{\eta\eta'} \, 2\pi \sum_{\bar{n} \in Z} \delta(x - x' - \bar{n}L) e^{i\pi\bar{n}\delta_b} \; ; \tag{8}$$

$$\{\psi_\eta(x), \psi_{\eta'}(x')\} = 0 \; . \tag{9}$$

For $x, x' \in [-L/2, L/2]$, these are just the standard[4] relations obeyed by fermion fields. For unrestricted values of x, x', one obtains a more general δ-function with appropriate periodicity.

4 Bosonic reorganization of Fock space

"Bosonizing" a fermionic theory means rewriting it in terms of bosonic degrees of freedom. The "deep reason" why this is possible for 1-D fermion theories is that the Fock space $\mathcal{F}$ of states spanned by the $c_{k\eta}$ operators can be reorganized as a direct sum, $\mathcal{F} = \sum_{\oplus \vec{N}} \mathcal{H}_{\vec{N}}$ over Hilbert spaces $\mathcal{H}_{\vec{N}}$ characterized

[3]There is no particular reason for the choice of phase in Eq. (3), namely e^{-ikx} instead of e^{ikx}; the former defines so-called left-moving fields, the latter right-moving fields (this nomenclature is explained in footnote 10), and the two are related simply by $x \leftrightarrow -x$. In Section 10.A we shall use both kinds of fields.

[4]Note though, that many authors use normalization $(1/L)^{1/2}$ instead of our $(2\pi/L)^{1/2}$ in Eq. (3), so that in Eq. (8) their fields are normalized to 1 instead of our 2π. The advantage of our normalization (used e.g. in conformal field theory), is that correlation functions are normalized to unity, $\langle\psi(x)\psi^\dagger(0)\rangle = 1/(ix)$, see Eq. (73).

by a **fixed particle number** $\vec{N}$*, within each of which all excitations are are particle-hole-like and hence have* **bosonic** *character. In this section, we define the concepts and operators needed to accomplish this reorganization.*

4.A Vacuum state $|\vec{0}\rangle_0$

Let $|\vec{0}\rangle_0$ be the state defined by the properties

$$c_{k\eta}|\vec{0}\rangle_0 \equiv 0 \qquad \text{for} \qquad k > 0, \quad (\text{i.e. } n_k > 0)\,, \tag{10}$$

$$c^\dagger_{k\eta}|\vec{0}\rangle_0 \equiv 0 \qquad \text{for} \qquad k \le 0, \quad (\text{i.e. } n_k \le 0)\,. \tag{11}$$

and illustrated for $M = 1$ in Fig. 1(a). In other words, for all η, the highest filled level of $|\vec{0}\rangle_0$ is by definition labeled by $n_k = 0$ and the lowest empty level by $n_k = 1$ (irrespective of $\delta_b \in [0, 2)$). We shall call $|\vec{0}\rangle_0$ the *vacuum state* (*Fermi sea* would be equally appropriate) and use it as reference state relative to which the occupations of all other states in $\mathcal{F}$ are specified. In particular, we define the operation of *fermion-normal-ordering*, to be denoted by ${}^*_*\ \ {}^*_*$, with respect to this vacuum state: to fermion-normal-order a function of c and $c^\dagger$'s, all $c_{k\eta}$ with $k > 0$ and all $c^\dagger_{k\eta}$ with $k \le$ are to be moved to the right of all other operators (namely all $c^\dagger_{k\eta}$ with $k > 0$ and $c_{k\eta}$ with $k \le 0$) , so that

$${}^*_* ABC \ldots {}^*_* = ABC \ldots \; - \; {}_0\langle\vec{0}|ABC \ldots |\vec{0}\rangle_0 \qquad \text{for} \quad A, B, C, \ldots \in \{c_{k\eta}; c^\dagger_{k\eta}\}\,. \tag{12}$$

4.B $\vec{N}$-particle ground states $|\vec{N}\rangle_0$

Let $\widehat{N}_\eta$ be the operator that counts the number of η-electrons relative to $|\vec{0}\rangle_0$:

$$\widehat{N}_\eta \equiv \sum_{k=-\infty}^{\infty} {}^*_* c^\dagger_{k\eta} c_{k\eta} {}^*_* = \sum_{k=-\infty}^{\infty} \left[c^\dagger_{k\eta} c_{k\eta} - {}_0\langle\vec{0}|c^\dagger_{k\eta} c_{k\eta}|\vec{0}\rangle_0 \right]. \tag{13}$$

The set of all states with the same $\widehat{N}_\eta$-eigenvalues $\vec{N} = (N_1, \ldots, N_M) \in \mathbb{Z}^M$ will be called the $\vec{N}$-particle Hilbert space $H_{\vec{N}}$. It contains infinitely many states, corresponding to different configurations of particle-hole excitations, all of which will generically be denoted by $|\vec{N}\rangle$.

Furthermore, for given $\vec{N}$, let $|\vec{N}\rangle_0$ be that particular $\vec{N}$-particle state which has *no* particle-hole excitations; since it is the lowest-energy state in $H_{\vec{N}}$, we shall call it the $\vec{N}$*-particle ground state*. To resolve ambiguities in its phase, we define it by specifying a particular ordering of operators (for ease of notation, we here use n_k instead of k as index):

$$|\vec{N}_\eta\rangle_0 \quad \equiv \quad (C_1)^{N_1} (C_2)^{N_2} \ldots (C_M)^{N_M} |\vec{0}\rangle_0\,, \tag{14}$$

$$(C_\eta)^{N_\eta} \quad \equiv \quad \begin{cases} c^\dagger_{N_\eta\,\eta} c^\dagger_{(N_\eta - 1)\,\eta} \ldots c^\dagger_{1\,\eta} & \text{for} \quad N_\eta > 0\,, \\ 1 & \text{for} \quad N_\eta = 0\,, \\ c_{(N_\eta+1)\,\eta} c_{(N_\eta+2)\,\eta} \ldots c_{0\,\eta} & \text{for} \quad N_\eta < 0\,. \end{cases} \tag{15}$$

4.C Bosonic particle-hole operators $b^\dagger_{q\eta}$ and $b_{q\eta}$

Since all excited states within a given $\vec{N}$-particle Hilbert space have the same $\vec{N}$, they can all be regarded as particle-hole excitations built on the ground state $|\vec{N}\rangle_0$. We shall see below that for a systematic treatment

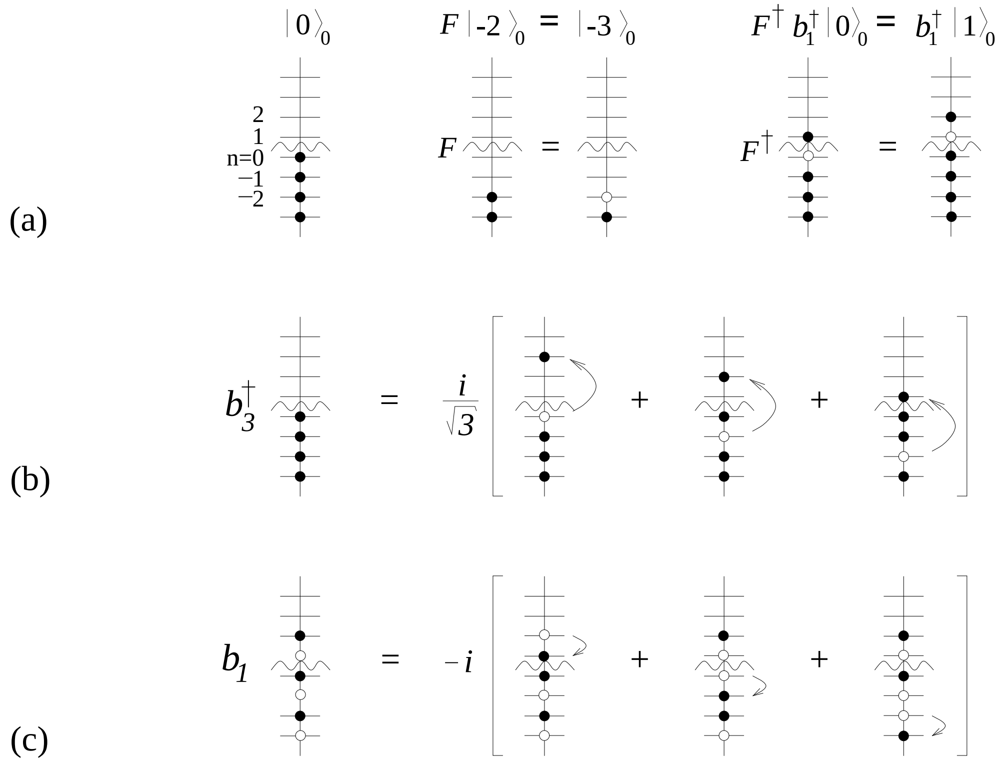

Figure 1: For the case $M = 1$ (i.e. the index η suppressed) we depict (a) the vacuum state $|0\rangle_0$ (the wiggly line indicates the "Fermi surface"), the action of F on the -2-particle ground state $|-2\rangle_0$, which yields $|-3\rangle_0$, and the action of $F^\dagger$ on the 0-particle excited state $ic_1^\dagger c_0|0\rangle_0 = b_1^\dagger|0\rangle_0$, which yields $b_1^\dagger|1\rangle_0$ [see Eqs. (25-26)]; (b) the action of $b_3^\dagger$ on $|0\rangle_0$ [see Eq. (16)]; (c) the action of b_1 on $c_2^\dagger c_0^\dagger c_{-2}^\dagger|-3\rangle_0$ [see Eq. (16)].

of all such operators it suffices to consider only the following *bosonic creation and annihilation* operators (both defined *only* for $q > 0$):

$$b^\dagger_{q\eta} \equiv \tfrac{i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k+q\,\eta} c_{k\eta} \;, \qquad b_{q\eta} \equiv \tfrac{-i}{\sqrt{n_q}} \sum_{k=-\infty}^{\infty} c^\dagger_{k-q\,\eta} c_{k\eta} \;, \tag{16}$$

with $q \equiv \frac{2\pi}{L} n_q > 0$, where $n_q \in \mathbb{Z}^+$ is a positive integer. For any $|\vec{N}\rangle$, the state $b^\dagger_{q\eta}|\vec{N}\rangle$ (or $b_{q\eta}|\vec{N}\rangle$) consists of a linear combination of particle-hole excitations relative to $|\vec{N}\rangle$, each term having q units of momentum more (or less) than $|\vec{N}\rangle$, as illustrated in Fig. 1(b-c). In this sense, $b^\dagger_{q\eta}$ and $b_{q\eta}$ can be viewed as momentum-raising or -lowering operators [they can also be identified with the Fourier-components of the electron density, see Eq. (36)]. Their normalization is purposefully chosen to produce harmonic oscillator commutation relations:

$$[b_{q\eta}, b_{q'\eta'}] = [b^\dagger_{q\eta}, b^\dagger_{q'\eta'}] = 0 \;, \qquad [N_{q\eta}, b_{q'\eta'}] = [N_{q\eta}, b^\dagger_{q'\eta'}] = 0 \;, \quad \text{for all} \quad q, q', \eta, \eta' \;; \tag{17}$$

$$\begin{aligned}
[b_{q\eta}, b^\dagger_{q'\eta'}] &= \delta_{\eta\eta'} \sum_{k=-\infty}^{\infty} \tfrac{1}{n_q} \left(c^\dagger_{k+q-q'\,\eta} c_{k\eta} - c^\dagger_{k+q\eta} c_{k+q'\eta} \right) \\
&= \delta_{\eta\eta'} \delta_{qq'} \sum_k \tfrac{1}{n_q} \Big\{ \Big[{}^*_* c^\dagger_{k\eta} c_{k\eta} {}^*_* - {}^*_* c^\dagger_{k+q\eta} c_{k+q\eta} {}^*_* \Big] + \Big({}_0\langle\vec{0}| c^\dagger_{k\eta} c_{k\eta} |\vec{0}\rangle_0 - {}_0\langle\vec{0}| c^\dagger_{k+q\eta} c_{k+q\eta} |\vec{0}\rangle_0 \Big) \Big\} \\
&= \delta_{\eta\eta'} \delta_{qq'} \;.
\end{aligned} \tag{18}$$

Eqs. (17) are easily checked, but the derivation of (18) requires some care, as first pointed out by Mattis and Lieb [6]: For $q \neq q'$ the two terms in the first line are both normal-ordered, and hence no subtleties can arise when subtracting them from each other to get zero (by shifting $k \to k - q'$ in the second term). However, for $q = q'$, before subtracting we first have to construct in the second line normal-ordered expressions (else we would be subtracting infinite expressions in an uncontrolled way). The normal-ordered terms in the second line cancel, as can be seen by writing $k + q = k'$ in the second term (this is allowed, since they are normal ordered, hence relabellings cannot produce problems). The definition of $|\vec{0}\rangle_0$ in Eqs. (10) and (11) implies that the remaining difference in expectation values in the second line gives

$$\frac{1}{n_q} \left(\sum_{n_k=-\infty}^{0} - \sum_{n_k=-\infty}^{-n_q} \right) = \frac{1}{n_q}\, n_q = 1 \;. \tag{19}$$

Note that the construction (16) of $b_{q\eta}$ and the derivation of its commutators (18) heavily rely on the set of k's being unbounded from below, which is why this property was stipulated in Section 2 to be a prerequisite for bosonization.

4.D Bosonic ground states $|\vec{N}\rangle_0$

Using Eq. (15), it is easy to verify that in each $\vec{N}$-particle Hilbert space $H_{\vec{N}}$, $|\vec{N}\rangle_0$ serves as vacuum state for the bosonic excitations:

$$b_{q\eta}|\vec{N}\rangle_0 = 0 \;, \qquad \text{for all} \quad q, \eta \;. \tag{20}$$

Intuitively, the reason is clear: Since $|\vec{N}\rangle_0$ is the $\vec{N}$-particle *ground* state, it does not contain any particle-hole excitations.

With respect to these boson vacuum states, one can define the operation of *boson-normal-ordering*, as follows: by definition, to boson-normal-order a function of b and $b^\dagger$'s, all $b_{q\eta}$'s are to be moved to the right of all $b^\dagger_{q\eta}$'s, so that

$$ {}^{*}_{*} ABC \dots {}^{*}_{*} = ABC \dots \; - \; {}_0\langle \vec{N} | ABC \dots | \vec{N} \rangle_0 \; ; , \qquad \text{for} \quad A, B, C, \dots \in \{ b_{q\eta}; b^\dagger_{q\eta} \} \; . \tag{21} $$

We use the same notation ${}^{*}_{*} \quad {}^{*}_{*}$ for boson as for fermion normal ordering, because a boson normal-ordered expression is automatically fermion normal ordered, as follows by taking $\vec{N} = \vec{0}$ in Eq. (21). Conversely, if a product purely of boson operators is fermion normal-ordered, i.e. if ${}_0\langle \vec{0} | ABC \dots | \vec{0} \rangle_0 = 0$, then automatically ${}_0\langle \vec{N} | ABC \dots | \vec{N} \rangle_0 = 0$ will hold for any $\vec{N}$, i.e. the product is also boson normal-ordered.

4.E Completeness of states in bosonic representation

It is obvious that every state $|\vec{N}\rangle$ in the $\vec{N}$-particle Hilbert space $H_{\vec{N}}$ can be obtained by acting on the corresponding ground state $|\vec{N}\rangle_0$ with some function of bilinear combinations of fermion operators: $|\vec{N}\rangle = \overline{f}(c^\dagger_{k\eta} c_{k'\eta}) |\vec{N}\rangle_0$. Remarkably, a much less obvious representation in terms of $b^\dagger_{q\eta}$'s also exists, namely:

For every $|\vec{N}\rangle$, there exists a function $f(b^\dagger)$ of $b^\dagger$s such that $|\vec{N}\rangle = f(b^\dagger)|\vec{N}\rangle_0$. (22)
i.e. the $b^\dagger$'s, acting on $|\vec{N}\rangle_0$, span the complete $\vec{N}$-particle Hilbert space $H_{\vec{N}}$.

(It is clearly not necessary to consider functions $f(b^\dagger, b)$ of b too, since $b|\vec{N}\rangle_0 = 0$.) This is a highly non-trivial statement, since the $b^\dagger$'s, being infinite sums, create complicated linear combinations of particle-hole excitations. For example, it is not at all obvious that even a state as simple as $c^\dagger_k c_{k'} |\vec{N}\rangle_0$ can be written in the form of Eq. (22).

To make plausible the validity of assertion (22), we offer here a (logically non-rigorous) "circular argument": By Eq. (4), we have

$$ c^\dagger_{k\eta} c_{k'\eta} = \frac{1}{2\pi L} \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dx' \, e^{i(k'x' - kx)} \, \psi^\dagger_\eta(x) \psi_\eta(x') \; . \tag{23} $$

Now, if one assumes that assertion (22) holds, the validity of the bosonization rules can readily be established (as shown below). As we shall see, they imply that $\psi^\dagger_\eta(x)\psi_\eta(x')$ can be expressed purely in terms of the $b^\dagger_{q\eta}$ and $b_{q\eta}$s. Therefore $\overline{f}(c^\dagger_{k\eta} c_{k'\eta}) |\vec{N}\rangle_0$ has the form (22) [using Eq. (23), rearranging its right-hand-side into boson-normal-ordered form and exploiting Eq. (20)], so that we have "proven our starting assumption".

Readers that find circular arguments unconvincing are referred to Appendix B for a rigorous proof of assertion (22).

4.F Klein factors $F^\dagger_\eta$ and F_η

As final bosonization ingredient, one has to define "ladder operators" that connect the various $\vec{N}$-particle Hilbert spaces, i.e. *raise or lower the total fermion number by one*, which no combination of bosonic operators can ever do. As a bonus, they also ensure that fermion fields of different species anticommute. Following the notation of Kotliar and Si [27], we shall call these ladder operators *Klein factors* and denote them by $F^\dagger_\eta$ and F_η. (In Haldane's paper, they are denoted by U and U^{-1}, see p. 2593 of Ref. [4]; for earlier discussions of such operators, see also [28, 29, 30].)

We define the Klein factors $F^\dagger_\eta$ and F_η to be operators with the following properties: Firstly, they commute with all bosonic operators:

$$ [b_{q\eta}, F^\dagger_{\eta'}] = [b^\dagger_{q\eta}, F^\dagger_{\eta'}] = [b_{q\eta}, F_{\eta'}] = [b^\dagger_{q\eta}, F_{\eta'}] = 0 \qquad \text{for all} \quad q, \eta, \eta' \; . \tag{24} $$

Secondly, their action on a general $\vec{N}$-particle state $|\vec{N}\rangle$, which can always be "factorized" as in (22) into a set of particle-hole excitations $f(b^\dagger)$ acting on the corresponding $\vec{N}$-particle ground state, $|\vec{N}\rangle = f(b^\dagger)|\vec{N}\rangle_0$, is thus defined:

$$F_\eta^\dagger|\vec{N}\rangle \equiv f(b^\dagger)c^\dagger_{N_\eta+1}|N_1,\dots,N_\eta,\dots N_M\rangle_0 \equiv f(b^\dagger)\widehat{T}_\eta|N_1,\dots,N_\eta+1,\dots N_M\rangle_0 \; ; \tag{25}$$

$$F_\eta|\vec{N}\rangle \equiv f(b^\dagger)c_{N_\eta}|N_1,\dots,N_\eta,\dots N_M\rangle_0 \equiv f(b^\dagger)\widehat{T}_\eta|N_1,\dots,N_\eta-1,\dots N_M\rangle_0 \; . \tag{26}$$

This is illustrated in Fig. 1(a) and can be visualized as follows: $F_\eta^\dagger$ (or F_η) commutes past $f(b^\dagger)$ and then adds (or removes) an η-electron to the lowest empty (from the highest occupied) η-level of $|\vec{N}\rangle_0$; this results in a new ground state, namely $c^\dagger_{N_\eta+1}|\vec{N}\rangle_0$ (or $c_{N_\eta}|\vec{N}\rangle_0$), on which the set of particle-hole excitations $f(b^\dagger)$ is then recreated. Thus the state $F_\eta^\dagger|\vec{N}\rangle$ (or $F_\eta|\vec{N}\rangle$) has the same set of bosonic excitations as the state $|\vec{N}\rangle$, but created on a ground state with one more (or less) η-electron.

The operator $\widehat{T}_\eta$ occuring in the last equalities in Eqs. (25) and (26), to be called the *phase-counting operator*, is defined by

$$\widehat{T}_\eta \equiv (-)^{\sum_{\bar\eta=1}^{\eta-1}\widehat{N}_{\bar\eta}} \; . \tag{27}$$

$\widehat{T}_\eta$ keeps track of the number of signs picked up when acting with a fermion operator $c_{k\eta}$ on $|\vec{N}\rangle_0$ to obtain a different $|\vec{N}'\rangle_0$:

$$c_{k\eta}(C_1)^{N_1}\dots(C_\eta)^{N_\eta}\dots(C_M)^{N_M}|\vec{0}\rangle_0 = \widehat{T}_\eta(C_1)^{N_1}\dots(C_{\eta-1})^{N_{\eta-1}}c_{k\eta}(C_\eta)^{N_\eta}\dots(C_M)^{N_M}|\vec{0}\rangle_0 \; . \tag{28}$$

The properties (24) to (26) completely specify $F_\eta^\dagger$ and F_η, and it is in principle not necessary to give a "more explicit" construction of operators with these properties. Nevertheless, such a construction is in fact easy to achieve: in Appendix D.3 we verify that, roughly speaking, the inverse of the bosonization identity, $F_\eta^\dagger \simeq a^{1/2}e^{-i\phi_\eta(0)}\psi_\eta^\dagger(0)$, does the job [see Eq. (D15)]. The most straightforward explicit representation of F, however, is in the bosonic representation [as first emphasized by Schönhammer[20], see his Eq.(B17)]: since the Fock space of all states is spanned by a set of orthornormal basis states of the form $|N;\{m_q\}\rangle = \prod_{q>0}^\infty \frac{b_q^{\dagger m_q}}{(m_q!)^{1/2}}|N\rangle_0$ [compare Eq. (B5)], the properties (24) to (26) immediately imply

$$F_\eta = \sum_{\vec{N}}\sum_{\{m_q\}} |N_1,\dots,N_\eta-1,\dots N_M;\{m_q\}\rangle\langle N_1,\dots,N_\eta,\dots N_M;\{m_q\}|\;\widehat{T}_\eta \; . \tag{29}$$

In fact, this equation can be viewed as a self-sufficient definition of F_η, alternative but equivalent to (24) to (26).

The defining properties (24) to (26) have the following consequences[5]: Firstly, since the spectrum of $\widehat{N}_\eta$ is unbounded from above or below, F_η is unitary: $F_\eta^{-1} = F_\eta^\dagger$. For this reason, it is often written as $F_\eta^\dagger \equiv e^{i\theta_\eta}$, with $\theta = \theta^\dagger$. We prefer not to use this notation, customary in the "field-theoretic" approach to bosonization, since it involves some (insufficiently well-known) subtleties and can lead to mistakes if used incorrectly, as discussed in Appendix D.2.

[5] Above, we took Eqs. (24) to (26) as the defining relations for F_η and $F_\eta^\dagger$, and derived (30) to (32) from them. Note that if instead Eqs. (24) and (30) to (32) were used as definitions, the action of F_η and $F_\eta^\dagger$ would be defined only up to a phase, since Eq. (32) implies that $F_\eta^\dagger|\vec{N}\rangle_0$ is equal to $|\dots,N_\eta+1,\dots\rangle_0$ modulo a phase. To fix this phase, additional definitions such as $F_\eta^\dagger|\vec{N}\rangle_0 \equiv \hat{T}_\eta|\dots,N_\eta+1,\dots\rangle_0$ are needed. Therefore, the approach chosen above is not only more explicit, but also more economical.

Secondly, the Klein factors can be checked to obey the following commution relations:

$$\{F_\eta^\dagger, F_{\eta'}\} = 2\delta_{\eta\eta'} \quad \text{for all} \quad \eta,\eta' \quad (\text{with } F_\eta F_\eta^\dagger = F_\eta^\dagger F_\eta = 1) \; ; \tag{30}$$

$$\{F_\eta^\dagger, F_{\eta'}^\dagger\} = \{F_\eta, F_{\eta'}\} = 0 \; , \quad \text{for} \quad \eta \neq \eta' \; ; \tag{31}$$

$$[\widehat{N}_\eta, F_{\eta'}^\dagger] = \delta_{\eta\eta'} F_\eta^\dagger \; , \qquad [\widehat{N}_\eta, F_{\eta'}] = -\delta_{\eta\eta'} F_\eta \; . \tag{32}$$

5 Boson fields – definition and properties

We define the boson fields $\phi_\eta(x)$ as Fourier sums over the $b_{q\eta}$ and $b_{q\eta}^\dagger$'s and derive some of their commonly-used properties, treating factors of $1/L$ with uncommon care.

When bosonizing $\psi_\eta(x)$ below, we shall find it useful to introduce the boson fields

$$\varphi_\eta(x) \equiv -\sum_{q>0} \tfrac{1}{\sqrt{n_q}} e^{-iqx} b_{q\eta} e^{-aq/2} \; , \qquad \varphi_\eta^\dagger(x) \equiv -\sum_{q>0} \tfrac{1}{\sqrt{n_q}} e^{iqx} b_{q\eta}^\dagger e^{-aq/2}, \tag{33}$$

and their Hermitian combination

$$\phi_\eta(x) \equiv \varphi_\eta(x) + \varphi_\eta^\dagger(x) = -\sum_{q>0} \tfrac{1}{\sqrt{n_q}} \left(e^{-iqx} b_{q\eta} + e^{iqx} b_{q\eta}^\dagger \right) e^{-aq/2} \; . \tag{34}$$

Here $a > 0$ is an infinitessimal mathematical regularization parameter needed to regularize ultraviolet ($q \to \infty$) divergent momentum sums that arise in certain non-normal-ordered expressions and commutators. Although a is often taken to be on the order of a lattice spacing, i.e. $a \simeq 1/k_F$, it was emphasized by Haldane [4] that it “in no way plays the role of a ‘cut-off’ length”. Nevertheless, $1/a$ can be interpreted as a kind of “effective band-width” (cf. the end of Section 10.C.2), in the sense that it represents the “maximum momentum difference” for the $c_{k\pm q}^\dagger c_k$-combinations occuring in ϕ. By construction, $\varphi_\eta(x)$ and $\phi_\eta(x)$ are periodic in x with period L. All properties of these fields follow directly from those [Eqs. (16) to (18)] of the $b_{q\eta}$, $b_{q\eta}^\dagger$ operators. Below we list some useful ones.

The normal-ordered electron density can be expressed in terms of the derivative field $\partial_x \phi_\eta(x)$, as follows:

$$\rho_\eta(x) \equiv {}^*_*\psi_\eta^\dagger(x)\psi_\eta(x){}^*_* = \tfrac{2\pi}{L} \sum_q e^{-iqx} \sum_k {}^*_* c_{k-q,\eta}^\dagger c_{k\eta} {}^*_* \tag{35}$$

$$= \tfrac{2\pi}{L} \sum_{q>0} i\sqrt{n_q} \left(e^{-iqx} b_{q\eta} - e^{iqx} b_{q\eta}^\dagger \right) + \tfrac{2\pi}{L} \sum_k {}^*_* c_{k\eta}^\dagger c_{k\eta} {}^*_* \tag{36}$$

$$= \partial_x \phi_\eta(x) + \tfrac{2\pi}{L} \widehat{N}_\eta \qquad (\text{for} \quad a \to 0) \; . \tag{37}$$

Eq. (36) shows that the $b_{q\eta}$ and $b_{q\eta}^\dagger$ are simply proportional to the Fourier-components of the electron density. (Note that the *actual* electron density is $\rho_\eta/(2\pi)$, since we normalized our ψ_η-fields to 2π instead of 1 in Eq. (8).)

The following commutators are often needed:

$$[\varphi_\eta(x), \varphi_{\eta'}(x')] = [\varphi_\eta^\dagger(x), \varphi_{\eta'}^\dagger(x')] = 0 \; , \tag{38}$$

$$[\varphi_\eta(x), \varphi_{\eta'}^\dagger(x')] = \delta_{\eta\eta'} \sum_{q>0} \tfrac{1}{n_q} e^{-q[i(x-x')+a]} \tag{39}$$

$$= -\delta_{\eta\eta'} \ln\left[1 - e^{-i\frac{2\pi}{L}(x-x'-ia)}\right] \tag{40}$$

$$\stackrel{L\to\infty}{\longrightarrow} -\delta_{\eta\eta'} \ln\left[i\tfrac{2\pi}{L}(x-x'-ia)\right] \; . \tag{41}$$

Eq. (40) was obtained using $\ln(1-y) = -\sum_{n=1}^{\infty} y^n/n$. Note that a cuts off the ultraviolet divergence at $x = x'$ that is typical of 1-D boson fields. The commutator (39) occurs, for example, when combining exponentials of boson fields as follows, using identity (C4) of Appendix C:

$$e^{i\varphi_\eta^\dagger(x)} e^{i\varphi_\eta(x)} = e^{i(\varphi_\eta^\dagger + \varphi_\eta)(x)} e^{[i\varphi_\eta^\dagger(x), i\varphi_\eta(x)]/2} = \left(\tfrac{L}{2\pi a}\right)^{1/2} e^{i\phi_\eta(x)} , \tag{42}$$

$$e^{-i\varphi_\eta(x)} e^{-i\varphi_\eta^\dagger(x)} = e^{-i(\varphi_\eta + \varphi_\eta^\dagger)(x)} e^{[-i\varphi_\eta(x), -i\varphi_\eta^\dagger(x)]/2} = \left(\tfrac{2\pi a}{L}\right)^{1/2} e^{-i\phi_\eta(x)} . \tag{43}$$

Note that the left-hand side of Eq. (42) is boson-normal-ordered, whereas the right-hand side is not. This is reflected in its prefactor factor $a^{-1/2}$, which would diverge in the limit $a \to 0$.

The commutator of $\phi_\eta(x)$ with its derivative[6] can be evaluated in two ways, depending on the order of limits for $L \to \infty$, $a \to 0$. If one takes the limit $L \to \infty$ first (but keeps terms of order $1/L$), one obtains, using Eq. (40):

$$[\phi_\eta(x), \partial_{x'}\phi_{\eta'}(x')] = \delta_{\eta\eta'}\; i\tfrac{2\pi}{L} \left[\frac{1}{e^{i\frac{2\pi}{L}(x-x'-ia)} - 1} + \frac{1}{e^{-i\frac{2\pi}{L}(x-x'+ia)} - 1} \right] \tag{44}$$

$$\stackrel{L\to\infty}{\longrightarrow} \quad \delta_{\eta\eta'} 2\pi i \left[\frac{a/\pi}{(x-x')^2 + a^2} - \tfrac{1}{L} \right] \quad \stackrel{a\to 0}{\longrightarrow} \quad 2\pi i \left[\delta(x-x') - \tfrac{1}{L} \right] . \tag{45}$$

Alternatively, if for some reason one wants to recover the periodic δ-function, the limit $L \to \infty$ can not be taken and instead one has to the limit $a \to 0$ first; using Eqs. (38), (39) and (6), one obtains

$$[\phi_\eta(x), \partial_{x'}\phi_{\eta'}(x')] = \delta_{\eta\eta'}\; i\tfrac{2\pi}{L} \sum_{q>0} e^{-qa} \left(e^{-iq(x-x')} + e^{iq(x-x')} \right) \tag{46}$$

$$\stackrel{a\to 0}{\longrightarrow} \quad \delta_{\eta\eta'}\; 2\pi i \left(\sum_{\bar{n}\in\mathbb{Z}} \delta(x - x' - \bar{n}L) - \tfrac{1}{L} \right) , \tag{47}$$

where the $1/L$ term in Eq. (47) is caused by the absence of a $q = 0$ term in Eq. (46). Note that retaining the leading $1/L$ term in Eqs. (45) and (47) ensures their consistency with

$$\int_{-L/2}^{L/2} dx' [\phi_\eta(x), \partial_{x'}\phi_{\eta'}(x')] = 0 , \tag{48}$$

which must hold since $\int_{-L/2}^{L/2} dx' \partial_{x'}\phi_{\eta'}(x') = \phi_{\eta'}(L/2) - \phi_{\eta'}(-L/2) = 0$.

Finally, the commutator of $\phi_\eta(x)$ with itself can be found by integrating Eq. (45) over x' over a region near x, and fixing the integration constant by requiring that the commutator be zero for $x = x'$:

$$[\phi_\eta(x), \phi_{\eta'}(x')] \stackrel{L\to\infty}{\longrightarrow} -\delta_{\eta\eta'}\; 2i \Big[\arctan\left[(x-x')/a\right] - \pi(x-x')/L \Big] \tag{49}$$

$$\stackrel{L=\infty, a\to 0}{\longrightarrow} \quad -\delta_{\eta\eta'}\; i\pi\, \epsilon(x-x') \qquad \text{where} \quad \epsilon(x) \equiv \begin{cases} \pm 1 & \text{for} \quad x \gtrless 0 , \\ 0 & \text{for} \quad x = 0 . \end{cases} \tag{50}$$

Eq. (50) is the form cited most often, but Eq. (49) shows that the step-function is actually smeared over a range a, and that there is a term of order $1/L$.

[6] In field-theoretical treatments, one often encounters the canonically conjugate field to $\phi_\eta(t,x)$, defined (in the Heisenberg picture) by $\Pi_\eta(t,x) \equiv \partial_t \phi_\eta(t,x)$. If (as is usual) a linear dispersion is assumed, with $v_F = 1$, then $\phi_\eta(t,x) = \phi_\eta(t+x)$ and $\Pi_\eta(t+x) = \partial_x \phi_\eta(t+x)$, so that Eq. (45) or (47) is the usual canonical commutation relation for boson fields.

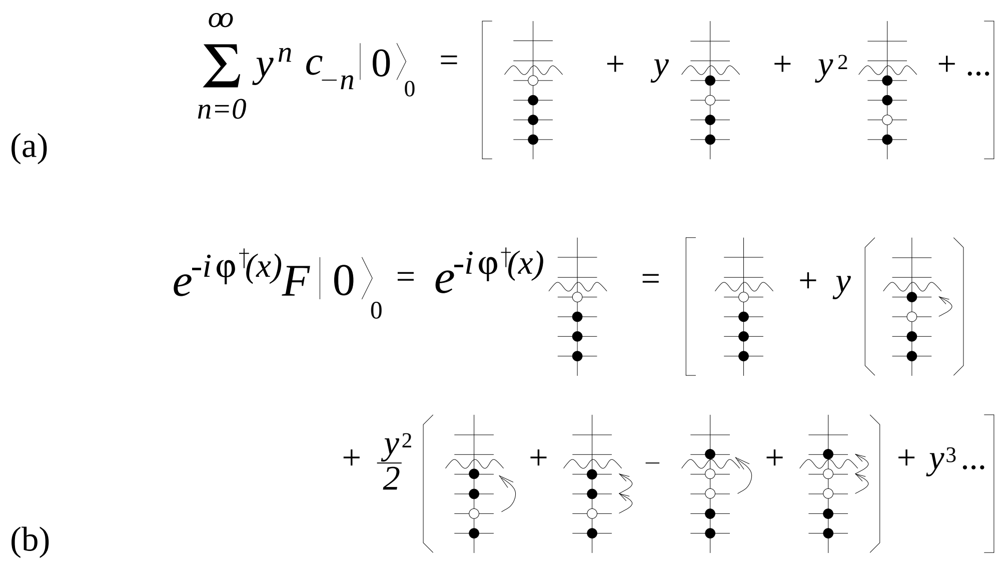

Figure 2: For the case $M = 1$ (i.e. the index η suppressed) we depict the action of $\psi(x)$ on $|0\rangle_0$ in two ways, using either (a) the Fourier expansion of Eq. (3), $\psi(x)|0\rangle_0 \sim \sum_{n=0}^{\infty} y^n c_{-n}|0\rangle_0$, where $y = e^{i2\pi x/L}$; or (b) the coherent state representation of Eq. (54), $\psi_\eta(x)|0\rangle_0 \sim e^{-i\varphi^\dagger(x)} F|0\rangle_0$. Although the second expansion appears to contain many more excited states than the first, remarkable cancellations of prefactors (some of which are investigated explicitly in Appendix E) guarantee that they are identical.

6 Derivation of the bosonization identity

We use the definitions of the preceding sections to give a novel derivation of the famous bosonization identities [(62) to (64)]. It is really quite straightforward, since we exploit some elementary properties of boson coherent states, which follow from standard operator identities stated and derived in Appendix C.

6.A $\psi_\eta|\vec{N}\rangle_0$ is a boson coherent state

We show that $\psi_\eta|\vec{N}\rangle_0$ is an eigenstate of $b_{q\eta}$ and hence has a coherent-state representation.

The definitions (3) of $\psi_\eta(x)$ and (16) of $b_{q\eta}$ imply that

$$[b_{q\eta'}, \psi_\eta(x)] \;=\; \delta_{\eta\eta'}\alpha_q(x)\psi_\eta(x) \;, \tag{51}$$

$$[b^\dagger_{q\eta'}, \psi_\eta(x)] \;=\; \delta_{\eta\eta'}\alpha^*_q(x)\psi_\eta(x) \;, \tag{52}$$

where $\alpha_q(x) = \frac{i}{\sqrt{n_q}}e^{iqx}$. Now, since $b_{q\eta}|\vec{N}\rangle_0 = 0$ [see Eq. (20)], Eq. (51) shows that $\psi_\eta(x)|\vec{N}\rangle_0$ is an *eigenstate of the boson annihilation operator* $b_{q\eta}$, with eigenvalue $\alpha_q(x)$:

$$b_{q\eta'}\psi_\eta(x)|\vec{N}\rangle_0 = \delta_{\eta\eta'}\alpha_q(x)\psi_\eta(x)|\vec{N}\rangle_0 \;. \tag{53}$$

Hence this state must have a coherent-state representation of the form (see e.g. [31])

$$\psi_\eta(x)|\vec{N}\rangle_0 \;=\; \exp\left[\sum_{q>0}\alpha_q(x)b^\dagger_{q\eta}\right]F_\eta\hat{\lambda}_\eta(x)|\vec{N}\rangle_0 \;=\; e^{-i\varphi^\dagger_\eta(x)}F_\eta\hat{\lambda}_\eta(x)|\vec{N}\rangle_0 \;, \tag{54}$$

where Eq. (33) was used for the second equality.[7] Here $\hat{\lambda}_\eta$ is a phase operator to be derived below, and F_η is needed because ψ_η removes exactly one η-particle, which the boson field $\varphi^\dagger_\eta(x)$ of course *cannot* accomplish. The representation (54) guarantees that Eq. (53) is satisfied, as can be seen by using identity (C3), with $A = b_{q\eta'}$, $B = -i\varphi^\dagger_\eta(x)$, $C = \delta_{\eta\eta'}\alpha_q(x)$.

Eq. (54) is a rather remarkable relation, since it shows that the action of $\psi_\eta(x)$ on $|N_\eta\rangle$ can be visualized in two different ways, illustrated in Fig. 2 (and discussed in more detail in Appendix E): when $\psi_\eta(x)$ is represented by its standard Fourier expansion (3), it creates an infinite linear combination of single-hole states, $\left(\frac{2\pi}{L}\right)^{1/2}\sum_k e^{-ikx}c_{n_k\eta}|\vec{N}\rangle_0$. The right-hand side of Eq. (54) states that the same result can be obtained in a different way: first F_η removes the highest η-electron in the $\vec{N}$-particle ground state $|\vec{N}\rangle_0$ to obtain a different ground state $c_{N_\eta\eta}|\vec{N}\rangle_0$, and then $e^{-i\varphi^\dagger_\eta(x)}$ creates on this a linear combination of hole states through the action of the raising operators $b^\dagger_{q\eta}$ which it contains. Eq. (54) states that both ways produce the *same* combination of single-hole states. This is a highly non-trivial statement, since intuitively one might have expected that $e^{-i\varphi^\dagger_\eta(x)}$, which is after all an exponential of an infinite sum of particle-hole operators of arbitrarily large momenta q, could produce much more complicated particle-hole excitations than the simple linear combination of *single-hole* states produced by $\psi_\eta(x)$. However, exploiting the remarkable properties of coherent states, Eq. (54) guarantees that of all the multitude of combinations of particle-hole excitations contained in $e^{-i\varphi^\dagger_\eta(x)}$, only those terms contribute, when acting on $c_{N_\eta\eta}|\vec{N}\rangle_0$, that fill its empty N_η-level

[7] In Eq. (54), strictly speaking $\sum_{q>0}\alpha_q(x)b^\dagger_{q\eta} = -i\varphi^\dagger_\eta(x)$ holds only if the regularization parameter in definition (33) of $\varphi^\dagger_\eta(x)$ equals zero, $a = 0$. Thus, all the manipulations below, up to and including Eq. (62), hold even if one strictly takes $a = 0$ (and regards $-i\varphi^\dagger_\eta(x)$ as shorthand for $\sum_{q>0}\alpha_q(x)b^\dagger_{q\eta}$). However, a $a \neq 0$ regularization parameter is required if one wants to un-normal-order the final result (62) to obtain (63) — but un-normal-ordering is usually done merely for notational convenience, and is never essential.

by moving to the latter a single η-electron from some lower filled state, leaving behind a single lower-lying hole; remarkably, *all* other particle-hole contributions (all of which would produce η-electrons *above* N_η) cancel out to zero. [In Appendix E we perform the instructive but cumbersome exercise of verifying this explicitly, for the highest few hole states, by expanding the exponential $e^{-i\varphi_\eta^\dagger(x)}$.]

To evaluate the operator $\hat\lambda_\eta(x)$, we calculate the following expectation value in two different ways: On the one hand,

$$ {}_0\langle \vec N | F_\eta^\dagger \psi_\eta(x) | \vec N\rangle_0 = \lambda_\eta(x) \; , \tag{55}$$

where we used Eq. (54) for $\psi_\eta(x)$, commuted $e^{-i\varphi_\eta^\dagger(x)}$ to the left past $F_\eta^\dagger$ [using Eq. (24)], and used ${}_0\langle\vec N| e^{-i\varphi_\eta^\dagger(x)} = {}_0\langle \vec N|$ [by Eq. (20)]. On the other hand, inserting the Fourier series (3) for $\psi_\eta(x)$ into Eq. (55), we note that since $|\vec N\rangle_0$ and ${}_0\langle\vec N|F_\eta^\dagger$ don't contain any particle-hole pairs, only the term in the sum with $n_k = N_\eta$ [i.e. $k = \frac{2\pi}{L}(N_\eta - \frac12\delta_b)$] can contribute:

$$ {}_0\langle \vec N | F_\eta^\dagger \psi_\eta(x) | \vec N\rangle_0 = \left(\tfrac{2\pi}{L}\right)^{1/2} e^{-i\frac{2\pi}{L}(N_\eta - \frac12\delta_b)x} \; . \tag{56}$$

Thus, we conclude that the operator $\hat\lambda_\eta(x)$ is given by:

$$ \hat\lambda_\eta(x) = \left(\tfrac{2\pi}{L}\right)^{1/2} e^{-i\frac{2\pi}{L}(\hat N_\eta - \frac12\delta_b)x} \; . \tag{57}$$

6.B Action of $\psi_\eta(x)$ on an arbitrary state $|\vec N\rangle$

We derive the bosonization identity by studying the action of $\psi_\eta(x)$ on an arbitrary state $|\vec N\rangle$.

Next we examine how $\psi_\eta(x)$ acts on an arbitrary state $|\vec N\rangle$ in Fock space, which by Eq. (22) we write as $|\vec N\rangle = f(\{b^\dagger_{q\eta'}\})|\vec N\rangle_0$. To this end, two identities are extremely useful:

$$\begin{aligned} \psi_\eta(x) f(\{b^\dagger_{q\eta'}\}) &= f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha_q^*(x)\})\,\psi_\eta(x) \; , &(58)\\ f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha_q^*(x)\}) &= e^{-i\varphi_\eta(x)}\, f(\{b^\dagger_{q\eta'}\})\, e^{i\varphi_\eta(x)} . &(59)\end{aligned}$$

The first follows from Eqs. (3) and (C8), with $A = b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha_q^*(x)$, $B = \psi_\eta(x)$ and $D = \delta_{\eta\eta'}\alpha_q^*(x)$; the second from Eqs. (33) and (C5), with $A = b^\dagger_{q\eta'}$, $B = i\varphi_\eta(x)$ and $C = -\delta_{\eta\eta'}\alpha_q^*(x)$.

Now evaluate $\psi_\eta|\vec N\rangle$ by commuting ψ_η past $f(\{b^\dagger_{q\eta'}\})$, then inserting Eq. (54) for $\psi_\eta|\vec N\rangle_0$, and finally rearranging:

$$\begin{aligned} \psi_\eta(x)|\vec N\rangle &= \psi_\eta(x)\, f(\{b^\dagger_{q\eta'}\})|\vec N\rangle_0 \\ &= f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha_q^*(x)\})\,\psi_\eta(x)|\vec N\rangle_0 && \text{[by Eq. (58)]}\\ &= f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha_q^*(x)\})\, e^{-i\varphi_\eta^\dagger(x)}\, F_\eta\hat\lambda_\eta(x)|\vec N\rangle_0 && \text{[by Eq. (54)]}\\ &= F_\eta\hat\lambda_\eta(x)\, e^{-i\varphi_\eta^\dagger(x)}\, f(\{b^\dagger_{q\eta'} - \delta_{\eta\eta'}\alpha_q^*(x)\})|\vec N\rangle_0 && \text{[by Eq. (24)]}\\ &= F_\eta\hat\lambda_\eta(x)\, e^{-i\varphi_\eta^\dagger(x)} \left[e^{-i\varphi_\eta(x)} f(\{b^\dagger_{q\eta'}\}) e^{i\varphi_\eta(x)}\right]|\vec N\rangle_0 && \text{[by Eq. (59)]}\\ &= F_\eta\hat\lambda_\eta(x) e^{-i\varphi_\eta^\dagger(x)}\, e^{-i\varphi_\eta(x)}\, f(\{b^\dagger_{q\eta'}\})|\vec N\rangle_0 && \text{[by Eq. (20)]}\\ &= F_\eta\hat\lambda_\eta(x)\, e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\eta(x)}|\vec N\rangle \; . && \text{[by Eq. (22)]}\end{aligned} \tag{60}$$

Since $|\vec{N}\rangle$ is an arbitrary state in the Fock space $\mathcal{F}$, (and *all* states in $\mathcal{F}$ have the form (22), see Section 4.E and appendix B), we conclude that the following so-called *bosonization formulas* for $\psi_\eta(x)$ hold as *operator identities* in Fock space,[8] valid for all L (i.e. all orders in an $1/L$ expansion):

$$\psi_\eta(x) = F_\eta \hat{\lambda}_\eta(x) e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\eta(x)} \tag{61}$$

$$= F_\eta \left(\tfrac{2\pi}{L}\right)^{1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\eta(x)} \qquad \text{[by Eq. (57)]} \tag{62}$$

$$= F_\eta\, a^{-1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\phi_\eta(x)} \qquad \text{[by Eq. (42)]} \tag{63}$$

$$= F_\eta\, a^{-1/2}\, e^{-i\Phi_\eta(x)} \qquad \text{with} \qquad \Phi_\eta(x) \equiv \phi_\eta(x) + \tfrac{2\pi}{L}(\widehat{N}_\eta - \tfrac{1}{2}\delta_b)x\ . \tag{64}$$

These forms are all equivalent. (Alternative notations used by Haldane [4], Kane & Fisher [16], Shankar [15] and others are discussed in Sections 10.A.4, 10.B.3, A.3 and D.2, respectively.) Eq. (62) is the "most rigorous", since it is normal ordered and hence valid even for $a = 0$ (compare footnote 7). Eq. (63) is the un-normal-ordered version of Eq. (62), obtained using (43), and evidently requires $a \neq 0$ [which is needed when unnormalordering to evaluate $[\varphi, \varphi^\dagger]$ in (43)]. The most common form is Eq. (64), which absorbs the factor $e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x}$ into the definition of a new Boson field $\Phi_\eta(x)$, following Haldane [4]. However, we prefer not to use this notation, for two reasons: firstly, $\phi_\eta(x)$ conveniently commutes with all Klein factors, whereas $[\Phi_\eta(x), F_{\eta'}^\dagger] = \delta_{\eta\eta'} \frac{2\pi}{L} x F_\eta^\dagger$ (by Eq. (32)); and secondly, the Klein factor F_η usually has a time-dependence $e^{-\frac{2\pi}{L}(\widehat{N}_\eta - \delta_b/2)\tau}$ (see Eq. (72)), so it is natural to view $e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x}$ as its x-dependence. If one is only interested in the limit $L \to \infty$, the factor $e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x}$ can be neglected.

This completes our derivation of the bosonization formulas. Since we deduced them explicitly and step by step from first principles, using only elementary operator identities, there is no need to "check" their validity by using them to calculate, for example, the anti-commutator $\{\psi_\eta, \psi_{\eta'}^\dagger\}$ or the correlator $\langle \psi_\eta \psi_{\eta'}^\dagger \rangle$, although these are instructive excercises, performed in Appendix F and Section 8, respectively [further such checks are performed in Appendix G.3, where the density ρ_η and free Hamilton of (66) below are bosonized using (62)]. This is one of the differences between the constructive approach to bosonization and the more formal field-theoretical one. In the latter, after fields $\psi_\eta(x)$ and $\phi_\eta(x)$ have been defined, the bosonization formula (63) is simply written down as a gift from the gods, whereupon its validity has to be established by calculating the anti-commutators and Green's functions of $e^{-i\phi_\eta(x)}$.

7 Hamiltonian with linear dispersion

We consider fermions with linear dispersion and bosonize the Hamiltonian in both the position and the momentum representation.

So far, no assumptions have been made about the Hamiltonian. Now assume a linear dispersion, $\varepsilon(k) = v_F \hbar k$, and measure all energies in units of $v_F \hbar$, i.e. set $v_F \hbar = 1$:

$$H_0 \equiv \sum_\eta H_{0\eta}, \qquad \text{with} \qquad H_{0\eta} \equiv \sum_{k=-\infty}^{\infty} k \,{}^*_* c_{k\eta}^\dagger c_{k\eta} {}^*_* \tag{65}$$

$$= \int_{-L/2}^{L/2} \frac{dx}{2\pi} \,{}^*_* \psi_\eta^\dagger(x) i\partial_x \psi_\eta(x) {}^*_*\ . \tag{66}$$

[8] It can readily be checked that Eq. (62) satisfies Eqs. (51) [and (52)], using identity (C3), with $A = b_{q\eta'}$, $B = -i\varphi_\eta^\dagger(x)$, $C = \delta_{\eta\eta'}\alpha_q(x)$ [or with $A = b_{q\eta'}^\dagger$, $B = -i\varphi_\eta(x)$, $C = \delta_{\eta\eta'}\alpha_q^*(x)$]. In fact, Emery uses this observation [3, Eq. (52)] to infer directly from Eqs. (51) and (52) that $\psi_\eta(x) \sim e^{-i\phi_\eta}$ (see also [10]). However, in order to also obtain the Klein factor F_η in Eq. (61) (which Emery did not), the more elaborate derivation given above is needed.

The second form is equivalent to the first, since inserting the Fourier expansions (3) for $\psi_\eta(x)$ into Eq. (66) reproduces (65).

Since $[H_{0\eta}, \widehat{N}_{\eta'}] = 0$ for all η, η', any $\vec{N}$-particle ground state is an eigenstate of $H_{0\eta}$, i.e. $H_{0\eta}|\vec{N}\rangle_0 = E^{\vec{N}}_{0\eta}|\vec{N}\rangle_0$. By inspection, its eigenvalue is

$$E^{\vec{N}}_{0\eta} = {}_0\langle\vec{N}|H_{0\eta}|\vec{N}\rangle_0 = \frac{2\pi}{L}\begin{cases}\displaystyle\sum_{n=1}^{N_\eta}(n-\delta_b/2) = \tfrac{1}{2}N_\eta^2 + \tfrac{1}{2}N_\eta(1-\delta_b) & \text{if } N_\eta \geq 0\ , \\ \displaystyle\sum_{n=N_\eta+1}^{0} -(n-\delta_b/2) = \tfrac{1}{2}N_\eta^2 + \tfrac{1}{2}|N_\eta|(1-\delta_b) & \text{if } N_\eta < 0\ , \end{cases}$$
$$= \frac{2\pi}{L}\frac{1}{2}N_\eta(N_\eta + 1 - \delta_b). \tag{67}$$

Furthermore, $b^\dagger_{q\eta}$ raises the energy of any eigenstate $|E\rangle$ by q units, as expected intuitively from $b^\dagger_{q\eta}$'s *fermionic* definition (16), which yields

$$[H_{0\eta}, b^\dagger_{q\eta'}] = q\, b^\dagger_{q\eta}\, \delta_{\eta\eta'}\ , \qquad \text{implying} \quad H_0 b^\dagger_{q\eta}|E\rangle = (E+q)b^\dagger_{q\eta}|E\rangle\ . \tag{68}$$

Now, the fact that the $b^\dagger$'s, acting on $|\vec{N}\rangle_0$, span the complete $\vec{N}$-particle Hilbert space $H_{\vec{N}}$ [recall Eq. (22)] implies that $H_{0\eta}$ must also have a representation *purely* in terms of bosonic variables. The only form that reproduces Eqs. (67) and (68) is:

$$H_{0\eta} = \sum_{q>0} q\, b^\dagger_{q\eta} b_{q\eta} + \frac{2\pi}{L}\frac{1}{2}\widehat{N}_\eta(\widehat{N}_\eta + 1 - \delta_b) \tag{69}$$
$$= \int_{-L/2}^{L/2} \frac{dx}{2\pi}\frac{1}{2}\, {}^*_*(\partial_x\phi_\eta(x))^2\, {}^*_* + (\tfrac{2\pi}{L})\tfrac{1}{2}\widehat{N}_\eta(\widehat{N}_\eta + 1 - \delta_b). \tag{70}$$

The second form is equivalent to the first, since inserting (34) for $\phi_\eta(x)$ into (70) reproduces (69). Both contain no F_η's, since $H_{0\eta}$ conserves particle number.

In field-theoretical treatments Eqs. (66) and (70) are often written using a so-called *point-splitting* prescription (denoted by : :) instead of the above normal-ordering prescription. Point-splitting, discussed in some detail in Appendix G, is another method of regularizing the product of fields at the same point in position space. It is in most cases equivalent to normal-ordering, in that it subtracts off diverging constants. In Appendix G we show that the point-split version of (70) can be obtained from the point-split version of (66) using the bosonization formula (63) [provided that regularization parameter used for point-splitting is the *same* a as that of the bosonic momentum cut-off $e^{-aq/2}$ in Eq. (33)].

For future reference, note that $H_{0\eta}$ transforms as follows under the unitary transformation $U \equiv e^{ic\phi_\eta(x)}$ [use Eq. (C5), with $[b_{q\eta}, \phi_\eta(x)] = -\frac{1}{\sqrt{n_q}}e^{iqx-aq/2}$, in Eq. (69)]:

$$UH_{0\eta}U^{-1} = \sum_{q>0} q\Big[b^\dagger_{q\eta} - ic\frac{1}{\sqrt{n_q}}e^{-iqx-aq/2}\Big]\Big[b_{q\eta} + ic\frac{1}{\sqrt{n_q}}e^{iqx-aq/2}\Big]$$
$$+\frac{2\pi}{L}\frac{1}{2}\widehat{N}_\eta(\widehat{N}_\eta + 1 - \delta_b)$$
$$= H_{0\eta} - c\partial_x\phi_\eta(x) + c^2(1/a - \pi/L) + O(a/L^2)\ . \tag{71}$$

The first two terms of (71) can also easily be derived from the position representation (70) for $H_{0\eta}$, using Eqs. (C5) and (45), which (for $L \to \infty$) imply $U\partial_{x'}\phi_\eta(x')U^{-1} = \partial_{x'}\phi_\eta(x') - 2\pi c\,\delta(x-x')$; however, this method is too crude to correctly reproduce the constants in Eq. (71), which is why we used the momentum representation here.

Also for future reference, note that the $\widehat{N}_\eta$-dependent terms in $H_{0\eta}$ imply that the Klein factors pick up an explicit time-dependence in the imaginary-time Heisenberg picture (with $\tau \in (-\beta, \beta]$ as time parameter):

$$F_\eta(\tau) \equiv e^{H_{0\eta}\tau} F_\eta e^{-H_{0\eta}\tau} = e^{-\frac{2\pi}{L}(\widehat{N}_\eta - \delta_b/2)\tau} F_\eta \, , \qquad F^\dagger_\eta(\tau) = e^{\frac{2\pi}{L}(\widehat{N}_\eta - \delta_b/2)\tau} F^\dagger_\eta \, . \tag{72}$$

Of course, $\frac{2\pi}{L}(\widehat{N}_\eta - \delta_b/2)$ is just the energy of the particle removed by F_η from the topmost occupied level of $|\vec{N}\rangle_0$. In the limit $L \to \infty$ in which this energy can be neglected, the time-ordered expectation value of Klein factors is simply ${}_0\langle \vec{N}| \mathcal{T} F_\eta(\tau) F^\dagger_{\eta'}(0) |\vec{N}\rangle_0 = \delta_{\eta\eta'} \operatorname{sgn}(\tau)$.

8 Relation between fermion and boson Green's functions

We show that the two-point Green's function for free fermions can be expressed in terms of that of free bosons [Eq. (78)], a fact which is sometimes used as the starting point for field-theoretical bosonization.

When discussing Green's functions, we shall always work in the imaginary-time Heisenberg picture (except in Section 10). Given the above Hamiltonian with linear dispersion, the free fields $\psi_\eta(\tau, x)$ and $\phi_\eta(\tau, x)$ only depend on the combination $z \equiv \tau + ix$ (and not on $\bar{z} \equiv \tau - ix$) (since $c_{k\eta}(\tau) = e^{-k\tau} c_{k\eta}$, etc., see Appendix H). Such fields are often called "chiral fields" (or "chiral left-movers", since after rotating back to real time, they depend only $t + x$). Therefore, we shall henceforth adopt the notation $\psi_\eta(z) \equiv \psi_\eta(\tau, x)$ and $\phi_\eta(z) \equiv \phi_\eta(\tau, x)$, $i\partial_z \phi_\eta(z) = \partial_x \phi_\eta(\tau, x)$. [This use of notation is somewhat sloppy: up to now we had used $\psi_\eta(0, x) = \psi_\eta(x)$, whereas in the new notation $\psi_\eta(0, x) = \psi_\eta(ix)$, but this should not cause confusion — if it does, the reader should imagine changing the old notation by replacing all preceding $\psi_\eta(x)$ by $\psi_\eta(ix)$, etc.] Real-time functions can obtained from imaginary-time ones by simply analytically continuing $\tau \to it$. It should remembered, though, that this works only for *free* fields: in the presence of interactions, the operators' time-development is more complicated, i.e. $c_{k\eta}(\tau) \neq e^{-k\tau} c_{k\eta}$.

8.A The limit $L \to \infty$ for $T \neq 0$

In the limit $L \to \infty$, the free imaginary-time ordered fermion and boson Green's functions (derived explicitly in Appendix H.2) have the following forms for $T \neq 0$:

$$\langle \mathcal{T} \psi_\eta(z) \psi^\dagger_{\eta'}(0) \rangle = \frac{\delta_{\eta\eta'}}{\frac{\beta}{\pi} \sin[\frac{\pi}{\beta}(z + \sigma a)]} \tag{73}$$

$$\langle \mathcal{T} \phi_\eta(z) \phi_{\eta'}(0) \rangle = - \, \delta_{\eta\eta'} \, \ln \left(\tfrac{2\beta}{L} \sin[\tfrac{\pi}{\beta}(\sigma z + a)] \right) \, . \tag{74}$$

$\langle \, \rangle$ is a thermal expectation value, $\mathcal{T}$ is the time-ordering operator for τ and $\sigma = \operatorname{sign}(\tau)$.

By observing that (up to a constant) Eq. (74) is the logarithm of Eq. (73), one might, even withouth prior knowledge of the bosonization formula, be led to conjecture that ψ_η must somehow be the exponential of ϕ_η; indeed, this is a common starting point for the field-theoretical treatment of bosonization. In the constructive approach to bosonization, however, the bosonization formula has already been established as an operator identity; hence, showing that fermion Green's functions can be calculated in terms of boson Green's functions merely has the status of a consistency check.

To perform this check, one exploits a remarkably identity [3] [see (C10), or more simply, (J6)], valid for any function $\hat{B} = \sum_{q>0} (\lambda_q b^\dagger_q + \tilde{\lambda}_q b_q)$ that is a linear combination of free boson operators governed by the boson Hamiltonian Eq. (69):

$$\langle e^{\lambda \hat{B}} \rangle = e^{\langle \hat{B}^2 \rangle \lambda^2/2} \, . \tag{75}$$

When combined with Eq. (C4), this implies:

$$\langle e^{\lambda_1 \hat{B}_1} e^{\lambda_2 \hat{B}_2} \rangle = e^{\langle \lambda_1 \hat{B}_1 \lambda_2 \hat{B}_2 + \frac{1}{2}(\lambda_1^2 \hat{B}_1^2 + \lambda_2^2 \hat{B}_2^2) \rangle} \tag{76}$$

Using the bosonization formula (63) [and (72) for the time-dependence of $F_\eta(\tau)$] we therefore have

$$\begin{aligned} \langle \mathcal{T} \psi_\eta(z) \psi^\dagger_{\eta'}(0) \rangle &= a^{-1} \Big[\theta(\tau) \langle F_\eta e^{-\frac{2\pi}{L}(\hat{N}_\eta - \frac{1}{2}\delta_b) z} e^{-i\phi_\eta(z)} e^{i\phi_{\eta'}(0)} F^\dagger_{\eta'} \rangle \\ &\qquad - \theta(-\tau) \langle e^{i\phi_{\eta'}(0)} F^\dagger_{\eta'} F_\eta e^{-\frac{2\pi}{L}(\hat{N}_\eta - \frac{1}{2}\delta_b) z} e^{-i\phi_\eta(z)} \rangle \Big] \qquad (77) \\ &= \delta_{\eta\eta'} \, \sigma a^{-1} e^{\langle \mathcal{T} \phi_\eta(z) \phi_\eta(0) - \phi_\eta(0) \phi_\eta(0) \rangle} \, . \qquad (78) \end{aligned}$$

Eq. (78) was obtained using (76) (and taking $e^{-\frac{2\pi}{L}(\hat{N}_\eta - \frac{1}{2}\delta_b) z} \simeq 1$ for $L \to \infty$); inserting Eq. (74) into Eq. (78) then readily reproduces Eq. (73).

8.B The limit $T = 0$ for $L \neq \infty$

For the case $T = 0$ but $L \neq \infty$, the fermion and boson Green's functions (derived explicitly in Appendix H.1) have the following forms:

$$\begin{aligned} \langle \mathcal{T} \psi_\eta(z) \psi^\dagger_{\eta'}(0) \rangle_{T=0} &= \frac{\delta_{\eta\eta'} e^{\frac{\pi}{L}(\delta_b + \sigma) z}}{\frac{L}{\pi} \sinh[\frac{\pi}{L}(z + \sigma a)]} \qquad (79) \\ \langle \mathcal{T} \phi_\eta(z) \phi_{\eta'}(0) \rangle_{T=0} &= - \, \delta_{\eta\eta'} \, \ln \left(1 - e^{-\frac{2\pi}{L}(\sigma z + a)} \right) \qquad (80) \end{aligned}$$

Here, too, Eq. (79) can be recovered from (80), by inserting the latter into the $T = 0$, $L \neq \infty$ version of (77-78) [with $\langle \mathcal{T} \phi_\eta(0) \phi_{\eta'}(0) \rangle_{T=0} = \delta_{\eta\eta'} \ln(L/2\pi a)$], namely

$$\langle \mathcal{T} \psi_\eta(z) \psi^\dagger_{\eta'}(0) \rangle_{T=0} = \delta_{\eta\eta'} \, \sigma a^{-1} e^{\frac{\pi}{L}\delta_b z} e^{\langle \mathcal{T} \phi_\eta(z) \phi_\eta(0) - \phi_\eta(0) \phi_\eta(0) \rangle_{T=0}} \, . \tag{81}$$

9 Vertex operators – some general properties

Exponentials of boson fields, $V^{(\eta)}_\lambda(\tau, x) \sim e^{i\lambda\phi_\eta(\tau,x)}$, are called *vertex operators* in the field theory literature and are the natural generalizations to $\lambda \neq \pm 1$ of the combinations $e^{\pm i\phi_\eta(\tau,x)}$ encountered so far. They occur in many applications of bosonization, e.g. Luttinger liquids (see Section 10) or the Kondo problem [17, 18]. Here we derive some of their general properties.

Since along the imaginary time axis all fields only depend on the combination $z \equiv \tau + ix$ (as is evident from Section 8), we henceforth use z as argument for all fields, writing e.g. $\phi_\eta(z)$. Moreover, all non-equal-time products of field-operators below will implicitly be assumed to be time-ordered (i.e. when we write $\hat{O}_1(z_1)\hat{O}_2(z_2)\ldots$, it is to be understood that $\tau_1 > \tau_2 \ldots$). This is important, since non-time-ordered products are ill-defined along the imaginary-time axis [31, p. 245].

9.A Definition of vertex operator

The boson normal-ordered form of the exponential $e^{i\lambda\phi_\eta(z)}$ is defined as follows:

$$\,^*_* e^{i\lambda\phi_\eta(z)} \,^*_* \equiv e^{i\lambda\varphi^\dagger_\eta(z)} e^{i\lambda\varphi_\eta(z)} = \left(\tfrac{L}{2\pi a}\right)^{\lambda^2/2} e^{i\lambda\phi_\eta(z)} = \frac{e^{i\lambda\phi_\eta(z)}}{\langle e^{i\lambda\phi_\eta(x)} \rangle} \tag{82}$$

The second equality is analogous to (42); the third follows from (75) and (74), and implies that normal-ordered exponentials indeed satisfy $\langle {}^*_* e^{i\lambda\phi_\eta(z)} {}^*_* \rangle = 1$, as they should.

To normal order the product of two normal-ordered exponentials like (82), proceed as follows:

$$ {}^*_* e^{i\lambda\phi_\eta(z)} {}^*_* \, {}^*_* e^{i\lambda'\phi_{\eta'}(z')} {}^*_* = e^{i(\lambda\varphi^\dagger_\eta(z)+\lambda'\varphi^\dagger_{\eta'}(z'))} \, e^{i(\lambda\varphi_\eta(z)+\lambda'\varphi_{\eta'}(z'))} \, e^{-\lambda\lambda'[\varphi_\eta(z),\varphi_{\eta'}(z')]} \tag{83} $$

$$ = {}^*_* e^{i(\lambda\phi_\eta(z)+\lambda'\phi_{\eta'}(z'))} {}^*_* \left[\tfrac{2\pi}{L}(z-z'+a)\right]^{\lambda\lambda'} \tag{84} $$

We used Eq. (C6) to commute $e^{i\lambda'\varphi^\dagger_{\eta'}(z')}$ to the left past $e^{i\lambda\varphi_\eta(z)}$, and used Eq. (41) to evaluate $[\varphi_\eta, \varphi^\dagger_{\eta'}]$ to leading order in L^{-1}. In this section we neglect finite-size corrections and hence always drop subleading order L^{-2} terms relative to L^{-1} terms. However, the former have to be kept if one is interested in finite-size corrections, see Appendix G.3.

A *vertex operator* is a normal-ordered exponential, characterized by a real number λ,

$$ V^{(\eta)}_\lambda(z) \equiv \left(\tfrac{L}{2\pi}\right)^{-\lambda^2/2} {}^*_* e^{i\lambda\phi_\eta(z)} {}^*_* = a^{-\lambda^2/2} e^{i\lambda\phi_\eta(z)} \, . \tag{85} $$

Its normalization (motivated below) is a generalization to the case $\lambda \neq 1$ of that of Eqs. (62) and (63). Evidently $\langle V^{(\eta)}_\lambda(z)\rangle = \delta_{\lambda,0}$ in the limit $L \to \infty$.

9.B Two-point correlator $\langle V^{(\eta)}_\lambda V^{(\eta')}_{\lambda'} \rangle$

The correlation function of two vertex operators can be derived precisely as in Section 8, using Eqs. (76) and (74) with the result [here $\sigma \equiv \mathrm{sgn}(\tau - \tau')$]:

$$ \langle \mathcal{T} V^{(\eta)}_\lambda(z) V^{(\eta')}_{\lambda'}(z')\rangle = \delta_{\eta\eta'} \left(a^{-\frac{1}{2}(\lambda_1^2+\lambda_2^2)}\right)\left(e^{\lambda\lambda' \ln\left[\frac{2\beta}{L}\sin[\frac{\pi}{\beta}(\sigma z - \sigma z' + a)]\right]}\right)\left(e^{\frac{1}{2}(\lambda_1^2+\lambda_2^2)\ln(\frac{2\pi a}{L})}\right) \tag{86} $$

$$ = \frac{\delta_{\eta\eta'} \, (L/2\pi)^{-(\lambda+\lambda')^2/2}}{\left(\frac{\beta}{\pi}\sin[\frac{\pi}{\beta}(\sigma z - \sigma z' + a)]\right)^{-\lambda\lambda'}} \overset{L\to\infty, T\to 0}{\longrightarrow} \frac{\delta_{-\lambda,\lambda'}}{(\sigma z - \sigma z' + a)^{\lambda^2}} \, . \tag{87} $$

The reason for including the factor $\left(\frac{L}{2\pi}\right)^{-\lambda^2/2}$ in definition (85) now becomes apparent: by producing the numerator in Eq. (87), it ensures that the above correlator is non-zero in the limit $L \to \infty$ only if $\lambda + \lambda' = 0$. This latter property is required on general grounds: since the boson Hamiltonian (70) is invariant under a shift $\phi_\eta(z) \to \phi_\eta(z) + const$, the same is expected for correlators of two properly normalized normal-ordered exponentials of boson fields. But for a correlator containing $\langle e^{i\lambda\phi(z)} e^{i\lambda'\phi(z')}\rangle$ this can clearly be true only if $\lambda + \lambda' = 0$, implying that such correlators must vanish otherwise.

The $T \to 0$ limit of Eq. (87) gives $\langle \mathcal{T} V^{(\eta)}_\lambda(z) V^{(\eta)}_{-\lambda}(0)\rangle_{T=0} = (\sigma z)^{-\lambda^2}$. Thus, the scaling dimension (as defined in Appendix G.1) of $V^{(\eta)}_\lambda$ and $V^{(\eta)\dagger}_\lambda = V^{(\eta)}_{-\lambda}$ is $\lambda^2/2$.

9.C OPEs involving vertex operators

The short-distance behavior of a product $V^{(\eta)}_\lambda V^{(\eta')}_{\lambda'}$ of vertex operators is summarized in its *operator product expansion* (OPE) (a concept reviewed in Appendix G.1). To derive its OPE, we simply have to normal order $V^{(\eta)}_\lambda V^{(\eta')}_{\lambda'}$. Since it is already normal-ordered if $\eta \neq \eta'$ (since then $[V^{(\eta)}_\lambda, V^{(\eta')}_{\lambda'}] = 0$), it suffices to consider $\eta = \eta'$:

$$ V^{(\eta)}_\lambda(z) \, V^{(\eta)}_{\lambda'}(z') = \left(\tfrac{L}{2\pi}\right)^{-(\lambda^2+\lambda'^2)/2} \left[\tfrac{2\pi}{L}(z-z'+a)\right]^{\lambda\lambda'} \left(1 + \lambda(z-z') i\partial_{z'}\varphi^\dagger_\eta(z')\right) \tag{88} $$

$$\times {}^*_*e^{i(\lambda\phi_\eta(z)+\lambda'\phi_{\eta'}(z'))}{}^*_* \big(1+\lambda(z-z')i\partial_{z'}\varphi_\eta(z')\big) \quad + \dots$$
$$= \frac{V^{(\eta)}_{\lambda+\lambda'}(z')}{(z-z'+a)^{-\lambda\lambda'}} + \frac{\lambda \, {}^*_*V^{(\eta)}_{\lambda+\lambda'}(z')\, i\partial_{z'}\phi_\eta(z')^*_*}{(z-z'+a)^{-\lambda\lambda'-1}} \quad + \dots \tag{89}$$

We used Eq. (84) to bring the left-hand side of Eq. (88) into normal-ordered form, and then took the limit $z \to z'$. Since all expressions within a normal-ordering symbol are well-defined (i.e. non-diverging), we Taylor-expanded inside the exponential, $\varphi_\eta(z) = \varphi_\eta(z') + (z-z')\partial_{z'}\varphi_\eta(z')$, but took care to maintain the normal order (which is why $\partial_{z'}\varphi^\dagger_\eta$ and $\partial_{z'}\varphi_\eta$ are right-most and left-most in Eq. (88), and why the second term of Eq. (89) explicitly needs the normal-ordering symbol).

Note that taking the expectation value of the OPE (89), namely $\langle V^{(\eta)}_\lambda(z)V^{(\eta)}_{\lambda'}(z')\rangle = (L/2\pi)^{-(\lambda+\lambda')^2/2}(z-z'+a)^{-\lambda^2}$, reproduces the $z/\beta \to 0$ limit of Eq. (86), i.e. its $T=0$ limit. This illustrates a rule of thumb, which can be proven quite generally:[9] $T \neq 0$ correlators of free (massless) fields can be obtained from $T=0$ ones by replacing $(z-z')$ by $\frac{\beta}{\pi}\sin[\frac{\pi}{\beta}(z-z')]$.

Another important OPE,

$$i\partial_z\phi_\eta(z)\, V^{(\eta')}_{\lambda'}(z') = \frac{\delta_{\eta\eta'}\,\lambda'}{z-z'+a}\, V^{(\eta')}_{\lambda'}(z') + {}^*_*V^{(\eta')}_{\lambda'}(z')\, i\partial_{z'}\phi_\eta(z')^*_* \,, \tag{90}$$

is obtained by commuting the $\partial_z\varphi_\eta(z)$ part of $\partial_z\phi_\eta$ past $V^{(\eta')}_{\lambda'}$ using Eq. (C3), and evaluating $[\partial_z\varphi_\eta, \varphi^\dagger_{\eta'}]$ using Eq. (41).

9.D Fermions as vertex operators

Comparing expressions (63) and (85), we see that *fermion* operators can be expressed in terms of vertex operators with $\lambda = \pm 1/2$:

$$\psi_\eta(z) = F_\eta e^{-\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)z} V^\eta_{-1/2}(z) \tag{91}$$

The factor $e^{-\frac{2\pi}{L}(\widehat{N}_\eta-\frac{1}{2}\delta_b)z}$ is a combination of the phase factor in Eq. (62) for $\psi_\eta(x)$ and the time dependence (72) of $F_\eta(\tau)$. Using Eq. (89), we thus find the following OPE for two fermion fields of the same species:

$$\psi^\dagger_\eta(z)\psi_\eta(z') \quad \stackrel{z\to z'}{\longrightarrow} \quad \frac{1}{(z-z'+a)} + i\partial_{z'}\phi(z') + \mathrm{Order}(\tfrac{1}{L}, a) \tag{92}$$

9.E General expectation values of vertex operators

It is possible to give the expectation value of a general time-ordered product of vertex operators in closed form [Eq. (96)]. To derive this result, we need some identities: Let $\hat{B}_i$ be linear in free boson variables, so that $[\hat{B}_i, \hat{B}_j] = c$-number. Then repeated application of Eq. (C4) gives:

$$e^{\hat{B}_1}e^{\hat{B}_2}\dots e^{\hat{B}_n} = e^{\sum_{j=1}^n \hat{B}_j} e^{\frac{1}{2}\sum_{i<j}[\hat{B}_i,\hat{B}_j]} \,. \tag{93}$$

By Eq. (75), we thus have:

$$\langle e^{\hat{B}_1}e^{\hat{B}_2}\dots e^{\hat{B}_n}\rangle = e^{\frac{1}{2}\langle(\sum_{j=1}^n \hat{B}_j)^2\rangle} e^{\frac{1}{2}\sum_{i<j}[\hat{B}_i,\hat{B}_j]} = e^{\frac{1}{2}\sum_{j=1}^n\langle\hat{B}_j^2\rangle} e^{\sum_{i<j}\langle\hat{B}_i\hat{B}_j\rangle} \,. \tag{94}$$

[9]The general proof of this rule exploits conformal invariance of correlation functions, by using a conformal mapping of the complex plane onto a cylinder with radius β (see e.g. Eq. (3.9) of Ref. [32]).

Now apply this identity to a product of vertex operators $V^{(\eta)}_{\lambda_j}(z_j) \equiv a^{-\lambda^2/2} e^{i\lambda_j \phi_\eta(z_j)}$, with $j = 1, \dots, n$. Using Eqs. (94) and (74) we readily obtain the following generalization of Eq. (87):

$$\langle \mathcal{T} V^{(\eta)}_{\lambda_1}(z_1) \dots V^{(\eta)}_{\lambda_n}(z_n) \rangle = \Big[a^{-\frac{1}{2}\sum_j \lambda_j^2} \Big] \Big[e^{\sum_{i<j} \lambda_i \lambda_j \ln\left[\frac{2\pi}{L} s(z_i, z_j)\right]} \Big] \Big[e^{\frac{1}{2}\sum_j \lambda_j^2 \ln\left(\frac{2\pi a}{L}\right)} \Big] \quad (95)$$

$$= \left(\tfrac{2\pi}{L}\right)^{\frac{1}{2}\left(\sum_{j=1}^n \lambda_j\right)^2} \prod_{i<j} [s(z_i, z_j)]^{\lambda_i \lambda_j} , \quad (96)$$

where $s(z_i, z_j) \equiv \frac{\beta}{\pi} \sin\left(\frac{\pi}{\beta}[(z_i - z_j)\mathrm{sgn}(\tau_i - \tau_j) + a]\right)$. Taking the limit $L \to \infty$, we see that this expectation value is non-zero only if $\sum_{j=1}^n \lambda_j = 0$. This is an important generalization of the corresponding result for two-point functions, discussed after (86), and again reflects invariance of the correlator under $\phi_\eta(z) \to \phi_\eta(z) + const$.

Suppose that all the λ_j are equal to $\pm\lambda$, with $\sum_{j=1}^n \lambda_j = 0$, and denote the corresponding arguments by $z^{(\pm)}$, i.e. use vertex operators $V^{(\eta)}_{+\lambda}(z_i^{(+)})$ and $V^{(\eta)}_{-\lambda}(z_j^{(-)})$. Then it can be shown by simple combinatorical algebra that the product in Eq. (96) can be rewritten as a sum:

$$\langle V^{(\eta)}_{+\lambda}(z_1^{(+)})\, V^{(\eta)}_{-\lambda}(z_1^{(-)})\, V^{(\eta)}_{+\lambda}(z_2^{(+)})\, V^{(\eta)}_{-\lambda}(z_2^{(-)})\, \dots \rangle \quad (97)$$

$$= \left[\sum_{\{z_i^{(+)}, z_j^{(-)}\}} \frac{1}{s(z_1^{(+)}, z_1^{(-)})\, s(z_2^{(+)}, z_2^{(-)}) \dots} + \frac{1}{s(z_1^{(+)}, z_2^{(-)})\, s(z_1^{(-)}, z_2^{(+)}) \dots} + \dots \right]^{\lambda^2} \quad (98)$$

where the sum goes over all combination of pairs $\{z_i^{(+)}, z_j^{(-)}\}$, with $i < j$. Actually, the algebra can be sidestepped via the following observation: Set $\lambda = 1$, so that (97) is an expectation value of free fermion fields. Then it has, on the one hand, a Wick-expansion in terms of two-point correlators $\langle \psi^\dagger(z_i^{(+)}) \psi(z_i^{(-)}) \rangle = 1/s(z_i^{(+)}, z_j^{(-)})$, which is simply (98) with $\lambda^2 = 1$; but on the other hand (97) is also equal to (96), with $\lambda_i \lambda_j = \pm 1$. Thus the product $\prod$ in (96) must equal the Wick-sum $\sum$ in (98), implying that also for $\lambda \neq 1$ one must have $(\prod)^{\lambda^2} = (\sum)^{\lambda^2}$. Hence (98) is, remarkably, a generalization of Wick's theorem to $\lambda \neq 1$!

10 Impurity in a Tomonaga-Luttinger liquid

To illustrate bosonization "in practice", we calculate the tunneling density of states, $\rho_{dos}(\omega)$, at the site of an impurity in a Tomonaga-Luttinger liquid. We resolve the recent controversy regarding $\rho_{dos}(\omega)$ by using a rigorous treatment of finite-size refermionization.

Though treating some elements of the theory of Tomonaga-Luttinger liquids [5, 7] in great detail, this section is by no means intended as a complete review of this subject (for such a review, see [20]); instead, it aims merely to illustrate at a detailed introductory level (and with more attention to subtleties than usual) the application of bosonization to a specific, non-trivial problem, namely the calculation of $\rho_{dos}(\omega)$. So non-trivial, in fact, that the exponent ν governing the low-energy behavior of the tunneling density of states, namely $\rho_{dos}(\omega) \sim \omega^{\nu-1}$ as $\omega \to 0$, has recently been subject to quite some controversy (summarized in Section 1.B of the introduction).

We begin in Section 10.A with a quantum wire of *free*, spinless L- and R-moving 1-D electrons and discuss the manipulations required to make the problem amenable to bosonization. In Section 10.B we switch on an electron-electron interaction (of dimensionless strength g) and diagonalize it in the boson basis. In Section 10.C we add a single impurity to the wire and, at a special value of the coupling constant ($g = 1/2$), refermionize and diagonalize the Hamiltonian and calculate a number of useful correlation functions. Section 10.D contains the culmination of the preceding developments: following a strategy due to Furusaki but implementing it more rigorously, we calculate $\rho_{dos}(\omega)$ at $g = 1/2$ and show that $\nu = 2$, confirming Fabrizio and Gogolin's [23] result and contradicting that of Oreg and Finkel'stein [21]. The calculation is appealingly straightforward — if it appears lengthy, this is only because for pedagogical reasons we show all details in full.

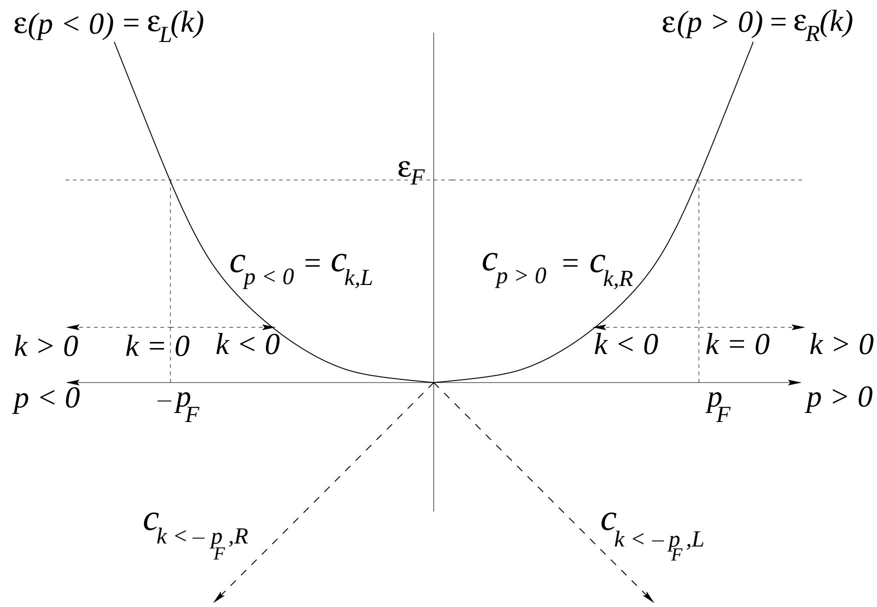

Figure 3: Schematic depiction of the dispersion relation $\varepsilon(p)$ of a 1-D wire containing R- and L-moving electrons with $p > 0$ and $p < 0$, respectively. From the original electron creation operators c_p, we construct $c_{k,L/R} \equiv c_{\mp(k+k_F)}$ for $k \in [-k_F, \infty]$, with corresponding dispersion $\varepsilon_{k,L/R} \equiv \varepsilon(\mp(k + k_F))$, see Eq. (100). Then we extend the Hilbert space by hand, by taking $k \in [-\infty, \infty]$, i.e. by adding "positron states" with $k < -k_F$, whose dispersion we took as $\varepsilon_{k,\nu} \equiv \varepsilon(0) + v_F(k + k_F)$.

10.A 1-Dimensional wire with free left- and right-moving electrons:

We introduce a quantum wire of **free,** *spinless L- and R-moving 1-D electrons, discuss the manipulations required to make the problem amenable to bosonization, and bosonize.*

10.A.1 Definition of $c_{k\eta}$ operators

Consider a 1-D conductor of length L containing free spinless left- and right-moving electrons, labelled by a momentum index $p \in (-\infty, \infty)$, with dispersion $\varepsilon(p)$ that is bounded from below, e.g. $\varepsilon(p) = (p^2 - p_F^2)/2m$.

The standard definition for the physical fermion field is

$$\Psi_{phys}(x) \equiv \left(\tfrac{2\pi}{L}\right)^{1/2} \sum_{p=-\infty}^{\infty} e^{ipx} c_p \;=\; \left(\tfrac{2\pi}{L}\right)^{1/2} \sum_{k=-k_F}^{\infty} \left(e^{-i(k_F+k)x} c_{-k_F-k} + e^{i(k_F+k)x} c_{k_F+k} \right) . \tag{99}$$

Here we wrote $p = \mp(k+k_F)$ with $k \in [-k_F, \infty)$, where $p \lessgtr 0$ corresponds to L and R-moving electrons. They can be viewed as two separate, independent "species", which we shall distinguish by an index $\nu = (L, R)$ (analogous to the η used hitherto), writing

$$c_{k\nu} \equiv c_{k,L/R} \equiv c_{\mp(k+k_F)} \qquad \text{with} \quad \varepsilon_{k,L/R} \equiv \varepsilon(\mp(k+k_F)) \, . \tag{100}$$

Our definition of k purposefully ensures that $\varepsilon_{k,\nu} \gtrless 0$ if $k \gtrless 0$ for both L- and R-movers.

10.A.2 Defining L- and R-moving fermion fields $\tilde{\psi}_{L/R}$

We now have to cast the problem in a form that meets the prerequisites for bosonization specified in Section 2. This is not yet the case, since $k \in [-k_F, \infty)$ is *bounded from below*, and *not discrete*. To remedy this, we proceed below in three steps: firstly, we extend the range of k to be unbounded, secondly introduce L- and R-moving fermion fields $\tilde{\psi}_{L/R}$ and thirdly impose boundary conditions on these to quantize k.

We begin by extending the single-particle Hilbert space by introducing (following Haldane [4]) additional unphysical "positron states" at the bottom of the Fermi sea: we simply extend the range of k to be *unbounded* by taking $k \in (-\infty, \infty)$, and define the corresponding energies in such a way that they all lie below $\varepsilon(p = 0)$, e.g. $\varepsilon_{k,\nu} \equiv \varepsilon(0) + v_F(k + k_F)$ for $k < -k_F$. The introduction of extra "unphysical" states does not change the low-energy physics of the system, since by construction they require very high energies ($> \varepsilon_F$) for their excitation. (However, they would be excited if a perturbation such as an electric field or impurity potential were sufficiently strong, so that strong perturbations cannot be dealt with using bosonization.)

Next, we factor out the rapidly fluctuating $e^{\mp i k_F x}$ phase factors and express $\Psi_{phys}(x)$ in terms of two fields $\tilde{\psi}_\nu(x)$ that vary *slowly* on the scale of $1/k_F$:

$$\Psi_{phys}(x) \quad \text{“=”} \quad e^{-ik_F x}\tilde{\psi}_L(x) + e^{+ik_F x}\tilde{\psi}_R(x), \tag{101}$$

$$\tilde{\psi}_\nu(x) \equiv \tilde{\psi}_{L/R}(x) \quad \equiv \quad \left(\tfrac{2\pi}{L}\right)^{1/2} \sum_{k=-\infty}^{\infty} e^{\mp ikx} c_{k,L/R} \, . \tag{102}$$

The "=" indicates that the r.h.s. of Eq. (102) differs from that of Eq. (99) by the inclusion of positron states. But since these do not change the low-energy physics, this difference does not matter and "for low-energy purposes" the first of Eq. (102) can effectively be regarded as a true equality. Our notation of using the index ν (instead of η) and putting a $\tilde{}$ on $\tilde{\psi}_\nu$ serves as a reminder that $\tilde{\psi}_L$ and $\tilde{\psi}_R$ are "mathematical L- and R-movers",[10] respectively, in contrast to the ψ_η's of earlier sections, which were *all* mathematical L-movers. (If one prefers to work purely with the latter, as is sometimes convenient, one can simply define purely L-moving fields by $\psi_{1,2}(x) \equiv \tilde{\psi}_{L,R}(\pm x)$, and similarly $\phi_{1,2}(x) \equiv \tilde{\phi}_{L,R}(\pm x)$ for the boson fields of Eq. (103) below.)

[10] A field is called a "mathematical L- or R- mover" if in the Heisenberg picture it depends on (real) time t and position x only via the combination $(t + x)$ or $(t - x)$, respectively. For example, the fields $\tilde{\psi}_{L/R}(t, x)$ defined in Eq. (102) are mathematical L- or R-movers if $c_{k,\pm}(t) = e^{-ikt} c_{k,\pm}$, which holds if (i) $\varepsilon_{k\pm} = v_F \hbar k$ (with $v_F \hbar = 1$) and (ii) no interactions are present. However, even if these two conditions do not hold, it is customary to refer to fields constructed as in Eq. (102) as L- or R-movers.

Finally, to quantize the allowed electron momenta k in units of $\frac{2\pi}{L}$, we impose boundary conditions on the fermion fields, choosing (for definiteness) anti-periodic ones: $\tilde{\psi}_\nu(L/2) = -\tilde{\psi}_\nu(-L/2)$ (i.e. $\delta_b = 1$ in Eqs. (2) and (5) — the specific choice of boundary condition becomes unimportant in the continuum limit $L \to \infty$).

10.A.3 Defining L- and R-moving boson fields $\tilde{\phi}_{L/R}$

The inclusion of "positron states" in the single-particle Hilbert space and the imposition of definite boundary conditions in the previous subsection should be viewed merely as formal tricks that make the problem amenable to bosonization. Now that the prerequisites of Section 2 are met, we can rigorously define number operators $\widehat{N}_{L/R}$, Klein factors $F_{L/R}$, and boson operators $b_{qL/R}$ in terms of the $c_{kL/R}$'s as in Sections 4.B, 4.F and 5, and procede to bosonize. Since the fields $\tilde{\psi}_L$ and $\tilde{\psi}_R$ formally differ from each other only by the factor $e^{\mp ikx}$ in Eq. (102), the only change needed for $\tilde{\psi}_R$ relative to $\tilde{\psi}_L$ is to replace x by $-x$ [and ∂_x by $-\partial_x$ and $\epsilon(x)$ by $-\epsilon(x)$), cf. Eqs. (33-34), (63), (37) and (50)]:

$$\tilde{\phi}_{L/R}(x) \equiv -\sum_{n_q \in \mathbb{Z}^+} \frac{1}{\sqrt{n_q}} e^{-aq/2} \left[e^{\mp iqx} b_{qL/R} + e^{\pm iqx} b^\dagger_{qL/R} \right] \quad (q = \tfrac{2\pi}{L} n_q > 0) , \tag{103}$$

$$\tilde{\psi}_{L/R}(x) = a^{-1/2} F_{L/R}\, e^{\mp i \frac{2\pi}{L} (\widehat{N}_{L/R} - \frac{1}{2}\delta_b) x} e^{-i\tilde{\phi}_{L/R}(x)} , \tag{104}$$

$$\tilde{\rho}_{L/R}(x) \equiv {}^*_* \tilde{\psi}^\dagger_{L/R}(x) \tilde{\psi}_{L/R}(x) {}^*_* = \pm \partial_x \tilde{\phi}_{L/R}(x) + \tfrac{2\pi}{L} \widehat{N}_{L/R} . \tag{105}$$

Note that $q = \frac{2\pi}{L} n_q$ implies that the boson fields and densities are periodic: $\tilde{\phi}_\nu(L/2) = \tilde{\phi}_\nu(-L/2)$ and $\tilde{\rho}_\nu(L/2) = \tilde{\rho}_\nu(-L/2)$.

The case of linear dispersion, $\varepsilon(p) = v_F(|p| - p_F)$, implying $\varepsilon_{kL/R} = v_F \hbar k$ for all k, is particularly simple. Then, in the imaginary-time Heisenberg picture used in Section 8 (with $v_F\hbar = 1$), the free L- (and R-) fields depend only on $z = \tau + ix$ (and $\bar{z} = \tau - ix$). Thus, we have (the comments just before Eq. (73) regarding the notation $\psi(z)$ apply here, too):

$$\begin{aligned} \tilde{\psi}_L(z) &\equiv \tilde{\psi}_L(\tau, x) \ \ [= \psi_1(\tau, x)], & \tilde{\psi}_R(\bar{z}) &\equiv \tilde{\psi}_R(\tau, x) \ \ [= \psi_2(\tau, -x)], \\ \tilde{\phi}_L(z) &\equiv \tilde{\phi}_L(\tau, x) \ \ [= \phi_1(\tau, x)], & \tilde{\phi}_R(\bar{z}) &\equiv \tilde{\phi}_R(\tau, x) \ \ [= \phi_2(\tau, -x)], \\ i\partial_z \tilde{\phi}_L(z) &\equiv \partial_x \tilde{\phi}_L(\tau, x) \ \ [= \partial_x \phi_1(\tau, x)], & i\partial_{\bar{z}} \tilde{\phi}_R(\bar{z}) &\equiv -\partial_x \tilde{\phi}_L(\tau, x) \ \ [= -\partial_x \phi_2(\tau, -x)]. \end{aligned} \tag{106}$$

These relations, given here for the sake of completeness, show that formulas involving relations between free R-movers can be obtained from ones involving relations between free L-movers by simply replacing L by R and z by $\bar{z}$ (and ∂_z by $\partial_{\bar{z}}$). This notation is used extensively, for example, by Affleck and Ludwig [32] in their confromal field theory solution of the Kondo problem.

10.A.4 Relation between our notation and that of Haldane

Readers interested in comparing our definitions with those used by Haldane in Ref. [4] (denoted by the subscript $_{Hal}$ below) should note that the normalization and particularly the phase factor in his definition (3.3), namely $\psi_{Hal}(x) \equiv L^{-1/2} \sum_{p=-\infty}^{\infty} e^{-ipx} c_{pHal}$, differ from ours in Eq. (99). Therefore he calls R/L-movers what we call L/R-movers. By identifying his index $p = (+,-) = (R, L)$ with our $\nu = (L, R)$ and making the identification $c_{\pm(k+k_F)Hal} = c_{k,L/R}$, one finds that his and our definitions are related as follows: for the fermion fields $\psi_{\pm Hal}(x) = e^{\mp ik_F x}(2\pi)^{-1/2} \tilde{\psi}_{L/R}(x)$, for the Klein factors $U_\pm = F^\dagger_{L/R}$ and $U^{-1}_\pm = F_{L/R}$, for the boson operators (which he defines for both $q > 0$ and < 0) one has $a^\dagger_{qHal} = -i[\theta(q) b^\dagger_{q,L} + \theta(-q) b^\dagger_{|q|,R}]$, and for the boson fields $\phi_{\pm Hal}(x) = \frac{\pm \pi x}{L} \widehat{N}_{L/R} + \tilde{\varphi}_{L/R}(x)$.

10.B Diagonalizing an electron-electron interaction by bosonizing

We consider a simple model with linear free-electron dispersion and a local electron-electron interaction. Diagonalizing it explicitly in the boson basis, we arrive at the "standard" bosonic form of the Tomonaga-Luttinger model, with dimensionless coupling g.

10.B.1 Turning on an electron-electron interaction

To illustrate the basic physics of a Tomonaga-Luttinger liquid, i.e. a system of interacting 1-D fermions, we shall consider the following simple model Hamiltonian:

$$H_{kin} = \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_*\Big[\psi_L^\dagger(x) i\partial_x \psi_L(x) \ + \ \psi_R^\dagger(x)(-i\partial_x)\psi_R(x)\Big]{}^*_* , \tag{107}$$

$$H_{int} = \int_{-L/2}^{L/2} \frac{dx}{2\pi} {}^*_*\big[g_2\, \tilde\rho_L(x)\tilde\rho_R(x) \ + \ \tfrac{1}{2} g_4 \left(\tilde\rho_L^2(x) + \tilde\rho_R^2(x)\right)\big]{}^*_* . \tag{108}$$

The kinetic term assumes a linear dispersion, $\varepsilon(p) \equiv \hbar v_F |p - p_F|$, i.e. $\varepsilon_{k,L/R} = \hbar v_F k$, with $\hbar v_F = 1$, cf. Eq. (66). H_{kin} describes a simple local (or point-like) electron-electron interaction,[11] parametrized by the dimensionless coupling strengths g_2 and g_4.

It is convenient to write the Hamiltonian as follows in terms of the densities $\tilde\rho_\nu$:

$$H_{kin} = \sum_{\nu=L,R}\left[\frac{2\pi}{L}\frac{1}{2}\widehat{N}_\nu^2 + \int_{-L/2}^{L/2}\frac{dx}{2\pi}{}^*_*\frac{1}{2}(\partial_x\phi_\nu(x))^2{}^*_*\right] = \int_{-L/2}^{L/2}\frac{dx}{2\pi}{}^*_*\frac{1}{2}\left[\tilde\rho_L^2 + \tilde\rho_R^2\right](x){}^*_* \tag{109}$$

$$H_0 = H_{kin} + H_{int} = \frac{v}{4}\int_{-L/2}^{L/2}\frac{dx}{2\pi}{}^*_*\Big[\frac{1}{g}(\tilde\rho_L + \tilde\rho_R)^2 \ + \ g\big(\tilde\rho_L - \tilde\rho_R\big)^2\Big](x){}^*_* , \tag{110}$$

$$\text{where} \qquad v \equiv \left[(1+g_4)^2 - g_2^2\right]^{1/2} , \qquad g \equiv \left[\frac{1+g_4-g_2}{1+g_4+g_2}\right]^{1/2} . \tag{111}$$

Eq. (109) follows from (107) via (70) and (37) (the $\hat N_\nu \partial_x \phi_\nu$ cross-terms in $\tilde\rho_\nu^2$ vanish when integrated, since $\tilde\phi_\nu$ is periodic); simple algebra then produces (110) for $H_{kin} + H_{int}$.

10.B.2 Diagonalizing H_0 in the boson basis

Now, the tremendous advantage of the bosonic representation is that $\tilde\rho_\nu$, though quadratic in the fermion field $\tilde\psi_\nu$, is linear in the boson field $\tilde\phi_\nu$ (see Eq. (105)). Thus H_0 is *quadratic* in bosonic variables and can be diagonalized straightforwardly by a Bogoljubov transformation of the $b_{q\nu}$'s:

$$H_0 = \frac{2\pi}{L}\frac{v}{2}\left\{\left(\frac{1}{g}+g\right)\sum_{\nu=L,R}\left[\frac{1}{2}\widehat{N}_\nu^2 + \sum_q n_q b_{q\nu}^\dagger b_{q\nu}\right] + \left(\frac{1}{g}-g\right)\left[\widehat{N}_L\widehat{N}_R - \sum_q n_q (b_{qR}b_{qL} + b_{qR}^\dagger b_{qL}^\dagger)\right]\right\}$$

[11] A more general local interaction such as ${}^*_*[\Psi^\dagger_{phys}(x)\Psi_{phys}(x)]^2{}^*_*$ also contains so-called *Umklapp* processes that do not conserve the number of L- or R-movers, e.g. $e^{i2k_Fx}\tilde\psi_L^\dagger\tilde\psi_R\tilde\rho_\nu$ or $e^{i4k_Fx}\tilde\psi_L^\dagger\tilde\psi_L^\dagger\tilde\psi_R\tilde\psi_R$. However, for the present spinless case they are zero, since Fermi statistics ensures that $\psi_\nu(x)\psi_\nu(x) = 0$. Note also that using a point-like interaction in conjunction with bosonization is strictly speaking somewhat sloppy: as emphasized in Section 10.A.2, the extention of Fock space to include infinitely many negative-energy "positron states" is strictly speaking justified only if these are not excited by a perturbation; however, a point-like interaction in position space is a constant in momentum space, i.e. it includes processes that couples states of arbitrarily large momentum differences, including physical with unphysical states (though their contribution to the low-energy physics is still negligible, because of the very large energy cost involved in exciting a positron state). A cleaner approach would thus be to explicitly use an interaction with a finite range, say $\tilde a$, corresponding to a finite cut-off $1/\tilde a$ in momentum space. For a more detailed discussion of such and other cut-off-related matters, see Ref. [20].

$$= \quad v\tfrac{2\pi}{L}\sum_{\nu=\pm}\left[g^{\nu}\widehat{\mathcal{N}}_{\nu}^{2}+\sum_{q}n_{q}B_{q\nu}^{\dagger}B_{q\nu}\right] \tag{112}$$

$$= \quad v\sum_{\nu=\pm}\left[\tfrac{2\pi}{L}g^{\nu}\widehat{N}_{\nu}^{2}+\int_{-L/2}^{L/2}\tfrac{dx}{2\pi}\,{}^{*}_{*}\tfrac{1}{2}\left(\partial_{x}\Phi_{\nu}(x)\right)^{2}\,{}^{*}_{*}\right] \;\equiv\; H_{0+}+H_{0-}\,, \tag{113}$$

where we have defined the following quantities:

$$B_{q\pm} \;=\; \tfrac{1}{\sqrt{8}}\left\{\left(\tfrac{1}{\sqrt{g}}+\sqrt{g}\right)\left(b_{qL}\mp b_{qR}\right)\pm\left(\tfrac{1}{\sqrt{g}}-\sqrt{g}\right)\left(b_{qL}^{\dagger}\mp b_{qR}^{\dagger}\right)\right\}\,, \tag{114}$$

$$\widehat{N}_{+} \;=\; \tfrac{1}{2}(\widehat{N}_{L}-\widehat{N}_{R})\,,\quad \widehat{N}_{-}=\tfrac{1}{2}(\widehat{N}_{L}+\widehat{N}_{R})\,, \tag{115}$$

$$\Phi_{\pm}(x) \;\equiv\; -\sum_{q>0}\tfrac{1}{\sqrt{n_{q}}}e^{-aq/2}\left[e^{-iqx}B_{q\pm}+e^{+iqx}B_{q\pm}^{\dagger}\right] \tag{116}$$

$$= \tfrac{1}{\sqrt{8}}\left\{\left(\tfrac{1}{\sqrt{g}}+\sqrt{g}\right)\left[\tilde{\phi}_{L}(x)\mp\tilde{\phi}_{R}(-x)\right]\pm\left(\tfrac{1}{\sqrt{g}}-\sqrt{g}\right)\left[\tilde{\phi}_{L}(-x)\mp\tilde{\phi}_{R}(x)\right]\right\}, \tag{117}$$

$$\rho_{\pm}(x) \;\equiv\; \partial_{x}\Phi_{\pm}(x)+\tfrac{2\pi}{L}\sqrt{2}g^{\pm1/2}\widehat{N}_{\pm} \tag{118}$$

$$= \tfrac{1}{\sqrt{8}}\left\{\left(\tfrac{1}{\sqrt{g}}+\sqrt{g}\right)\left[\tilde{\rho}_{L}(x)\mp\tilde{\rho}_{R}(-x)\right]\mp\left(\tfrac{1}{\sqrt{g}}-\sqrt{g}\right)\left[\tilde{\rho}_{L}(-x)\mp\tilde{\rho}_{R}(x)\right]\right\}\,. \tag{119}$$

The first line for H_0 follows by inserting (103) into (110). The diagonal form (112) was obtained by making the Bogoljubov transformation [12] (114) to a new set of orthonormal boson operators $B_{q\pm}$ (with $[B_{q\nu}, B^{\dagger}_{q'\nu'}] = \delta_{\nu\nu'}\delta_{qq'}$) and number operators $\mathcal{N}_{\pm}$. From these we constructed in Eqs. (116) and (118) two new boson fields $\Phi_{\pm}(x)$ and densities $\rho_{\pm}(x)$ [by analogy to Eqs. (103) and (105)]. By construction they *both* are manifestly L-moving (since $B_{q\pm}(t) = e^{-ivqt}B_{q\pm}$) and hence obey all the boson-field identies derived in Section 5 [we could equally well have defined $\Phi_{\pm}$ to be L/R-moving by replacing x by $\pm x$ in (116-119)]. Finally, (113) follows from (112) just as (70) follows from (69).

10.B.3 Relation between our notation and that of Kane and Fisher

For the sake of completeness, we briefly explain the notation used in the path-breaking papers of Kane and Fisher [16], denoting it by a subscript $_{kf}$. They use field-theoretic bosonization (another example of which is summarized in Appendix A) and write

$$\Psi_{phys}(x) \sim \sum_{n=odd} e^{-i\sqrt{\pi}\phi_{kf}(x)}e^{-in[\sqrt{\pi}\theta_{kf}(x)+k_F x]}\,, \tag{120}$$

where θ_{kf} and ϕ_{kf} are so-called "dual fields" that by definition satisfy

$$[\phi_{kf}(x), \theta_{kf}(x')] = -\tfrac{1}{2}i\epsilon(x-x'). \tag{121}$$

Kane and Fisher state that "the sum on n enforces the constraint that the particle density be discrete" [16]. Although this n-sum is also used in an early paper by Haldane [33], the present authors see no

[12] A simple way to derive the Bogoljubov transformation is to insert the general Ansatz $B_{q\nu} \equiv \sum_{q'\nu'}(\mathcal{A}^{\nu\nu'}_{qq'}b_{q'\nu'} + \bar{\mathcal{A}}^{\nu\nu'}_{qq'}b^{\dagger}_{q'\nu'})$ into the equation of motion implied by (112), namely $[B_{q\nu}, H] = vqB_{q\nu}$, and solve this for the $\mathcal{A}$'s, under the condition that $[B_{q\nu}, B^{\dagger}_{q'\nu'}] = \delta_{\nu\nu'}\delta_{qq'}$. Actually, the alternative combinations B_{q1} and $B_{q2} \equiv \frac{1}{\sqrt{2}}(B_{q+} \pm B_{q-})$ are more commonly used in the literature, since the corresponding fields Φ_1 and Φ_2, constructed as in (116), do not contain "non-local" combinations such as the $\tilde{\phi}_L(x) \mp \tilde{\phi}_R(-x)$ of (117). However, for the purpose of discussing scattering from a point-like impurity, as we do below, our "non-local" combinations cause no problems, and in fact are more convenient, since forward and backward scattering turn out to depend only on Φ_- and Φ_+, respectively, at $x = 0$.

need for it: using only $n = \pm 1$ and comparing Eq. (120) with our (101) and (104-105), we can make contact with the rigorous constructive approach (in the continuum limit $L = \infty$) through the identifications $\tilde{\phi}_{L/R,here} := \sqrt{\pi}(\phi_{kf} \pm \theta_{kf})$ i.e.

$$\theta_{kf}(x) := \tfrac{1}{2\sqrt{\pi}}\left[\tilde{\phi}_L(x) - \tilde{\phi}_R(x)\right]_{here}, \text{ thus } \partial_x\theta_{kf}(x) := \tfrac{1}{2\sqrt{\pi}}\left[\tilde{\rho}_L(x) + \tilde{\rho}_R(x)\right]_{here}, \tag{122}$$

$$\phi_{kf}(x) := \tfrac{1}{2\sqrt{\pi}}\left[\tilde{\phi}_L(x) + \tilde{\phi}_R(x)\right]_{here}, \text{ thus } \partial_x\phi_{kf}(x) := \tfrac{1}{2\sqrt{\pi}}\left[\tilde{\rho}_L(x) - \tilde{\rho}_R(x)\right]_{here}. \tag{123}$$

Eq. (121) is then consistent with our (50) (in which one must set $\epsilon(x) \to -\epsilon(x)$ for $\tilde{\phi}_{R,here}$). Since $\phi_{L/R,here} = \phi_{L/R,here}(x \pm t)$, it follows that $\partial_x\phi_{kf} = \partial_t\theta_{kf}$, thus $\partial_x\phi_{kf}$ is the canonically conjugate field to θ_{kf}. Translating our Hamiltonian H_0 as given by (110) into Kane and Fisher's notation, it takes a form often encountered in the literature, namely

$$H_0 = \frac{v}{2}\int_{-L/2}^{L/2} dx \, {}^*_*\left[\frac{1}{g}(\partial_x\theta_{kf}(x))^2 \; + \; g(\partial_x\phi_{kf}(x))^2\right]{}^*_* \, . \tag{124}$$

10.C Adding an impurity to a Tomonaga-Luttinger liquid

We add to the wire a single impurity at $x = 0$ that causes both forward and backward scattering and bosonize the Hamiltonian $H_0 + H_F + H_B$. We diagonalize H_F exactly for arbitrary g by a unitary transformation. We diagonalize H_B exactly for $g = \frac{1}{2}$ using refermionization, which we introduce in pedagogical detail, and calculate some useful correlation functions.

10.C.1 Adding an impurity

We turn on the impurity scattering term $H_F + H_B$, bosonize, show that $H_{F/B}$ depends only on $\Phi_\mp$, and diagonalize $H_{0-} + H_F$ for arbitrary g by a unitary transformation $U_- = e^{ic_-\Phi_-}$.

Assuming an impurity at $x = 0$ acts like a point scatterer causing both forward (L-L, R-R) and backward scattering (L-R, R-L), we consider the following perturbations (λ_F, λ_B and the phase θ_B are real, dimensionless constants, with $\lambda_B > 0$):

$$H_F = \sum_{\nu=L,R} \tfrac{v\lambda_F}{2\pi} \, {}^*_*\tilde{\psi}^\dagger_\nu(0)\tilde{\psi}_\nu(0){}^*_* = \tfrac{v\lambda_F}{2\pi}\left(\tilde{\rho}_L(0) + \tilde{\rho}_R(0)\right) = \tfrac{v\lambda_F}{2\pi}\sqrt{2g}\,\rho_- \, , \tag{125}$$

$$H_B = \tfrac{v\lambda_B}{2\pi}\left[e^{i\theta_B}\tilde{\psi}^\dagger_L(0)\tilde{\psi}_R(0) + e^{-i\theta_B}\tilde{\psi}^\dagger_R(0)\tilde{\psi}_L(0)\right] \tag{126}$$

$$= \tfrac{v\lambda_B}{2\pi a}\left[F^\dagger_L F_R e^{i(\tilde{\phi}_L(0) - \tilde{\phi}_R(0) + \theta_B)} + F^\dagger_R F_L e^{i(\tilde{\phi}_R(0) - \tilde{\phi}_L(0) - \theta_B)}\right] \tag{127}$$

$$= \tfrac{v\lambda_B}{2\pi a}\left[F^\dagger_L F_R e^{i(\sqrt{2g}\Phi_+ + \theta_B)} + F^\dagger_R F_L e^{-i(\sqrt{2g}\Phi_+ + \theta_B)}\right] , \tag{128}$$

We bosonized these using Eqs. (104-105) for the old and (117), (119) for the new boson fields, with

$$\Phi_\pm \equiv \Phi_\pm(0) = \tfrac{1}{\sqrt{2}} g^{\mp\frac{1}{2}}\left(\tilde{\phi}_L(0) \mp \tilde{\phi}_R(0)\right), \qquad \rho_\pm \equiv \rho_\pm(0) = \tfrac{1}{\sqrt{2}} g^{\pm\frac{1}{2}}\left(\tilde{\rho}_L(0) \mp \tilde{\rho}_R(0)\right). \tag{129}$$

Note that $[\Phi_+, \Phi_-] = [\rho_+, \rho_-] = 0$, since $[B_{q-}, B^\dagger_{q'+}] = 0$. The full Hamiltonian,

$$H \equiv H_0 + H_F + H_B = (H_{0+} + H_B) + (H_{0-} + H_F) \equiv H_+(\Phi_+) + H_-(\Phi_-) \, , \tag{130}$$

falls apart into two commuting parts, depending only on $\Phi_+(x)$ and $\Phi_-(x)$, respectively. The second of these can be written as

$$H_- = H_{0-} + H_F = v\left[\int_{-L/2}^{L/2} \tfrac{dx}{2\pi}\tfrac{1}{2}\,{}^*_*(\partial_x\Phi_-(x))^2\,{}^*_* \;+\; (\tfrac{2\pi}{L})\tfrac{1}{g}\widehat{N}_-^2 + c_-\left(\partial_x\Phi_- + \tfrac{2\pi}{L}\sqrt{\tfrac{2}{g}}\widehat{N}_-\right)\right], \tag{131}$$

with $c_- = \frac{\lambda_F}{2\pi}(2g)^{\frac{1}{2}}$, see Eqs. (113) and (125). It can be diagonalized using the unitary transformation $U_- = e^{ic_-\Phi_-}$, which maps it onto a Hamiltonian H'_- which is essentially free [we evaluate $U_-H_{0-}U_-^{-1}$ using (71) and $U_-H_FU_-^{-1}$ using (45)]:

$$H'_- \equiv U_-H_-U_-^{-1} = v\left[\int_{-L/2}^{L/2} \tfrac{dx}{2\pi}\tfrac{1}{2}\,{}^*_*(\partial_x\Phi_-(x))^2\,{}^*_* \;+\; \tfrac{2\pi}{L}\tfrac{1}{g}\widehat{N}_-^2 + c_-\tfrac{2\pi}{L}\sqrt{\tfrac{2}{g}}\widehat{N}_- - c_-^2\left(\tfrac{1}{a} - \tfrac{\pi}{L}\right)\right]. \tag{132}$$

10.C.2 Finite-size refermionization of H_+ at $g = \frac{1}{2}$

We give a rigorous introduction to the technique of finite-size refermionization, the "inverse" of bosonization. Then we refermionize $H_+ = H_{0+} + H_B$ for $g = \frac{1}{2}$ and make a unitary transformation $U_+ \sim e^{i\frac{\pi}{2}\hat{\mathcal{N}}^2}$ such that $H'_+ = U_+H_+U_+^{-1}$ is quadratic in refermionized operators.

The problem posed by the second term in Eq. (130), $H_+ = H_{0+} + H_B$, is not exactly solvable for general values of the coupling constant g, since H_B also involves Klein factors, i.e. is not expressed purely in bosonic language. However, H_+ *can* be diagonalized exactly for $g = \frac{1}{2}$, to which we henceforth restrict our attention.

$g = \frac{1}{2}$ is special, since then Φ_+ occurs in the backscattering term H_B only in the combination $e^{\pm i\Phi_+}$, which is precisely what occurs on the right-hand side of a bosonization identity![13] This can be exploited by *refermionizing:* we invert the line of reasoning of Section 4 to 6, where bosons and Klein factors were constructed from fermions, and here construct new fermions from bosons and Klein factors. We shall refermionize *at finite L,* since this allows us to discuss refermionization at the same level of rigor as bosonization, namely as an operator identity in Fock space. (Our treatment is an adaption of that invented by Zaránd for the 2-channel Kondo model [17, 18]; our way of defining the requisite new Klein factor F_+ is considerably more precise and natural than previous treatments in the literature.)

Besides the boson field Φ_+ and number operator $\widehat{N}_+$ occuring in H_+, we need new Klein factors F_+, $F_+^\dagger$ as ladder operators for $\widehat{N}_+$. Since Eq. (115) gives $\widehat{N}_+ = \frac{1}{2}(\widehat{N}_L - \widehat{N}_R)$, it is natural to simply define

$$F_+ \equiv F_R^\dagger F_L, \quad \text{implying} \quad \{F_+, F_+^\dagger\} = 2, \quad [\widehat{N}_+, F_+^\dagger] = F_+^\dagger, \quad [\widehat{N}_-, F_+^\dagger] = [\Phi_\pm(x), F_+^\dagger] = 0. \tag{133}$$

Thus F_+, $F_+^\dagger$ and $\widehat{N}_+$ satisfy the requisite standard relations (30) and (32). Next we define a new fermion field $\Psi_+(x)$ and its Fourier coefficients $c_{\bar{k}}$ via the "refermionization identity"

$$\sqrt{\tfrac{2\pi}{L}}\textstyle\sum_{\bar{k}} e^{-i\bar{k}x}c_{\bar{k}} \;\equiv\; \Psi_+(x) \;\equiv\; F_+\tfrac{1}{\sqrt{a}}e^{-i(\widehat{N}_+ - \frac{1}{2})\frac{2\pi x}{L}}e^{-i\Phi_+(x)}, \tag{134}$$

which should be read as follows: the combination $F_+\frac{1}{\sqrt{a}}e^{-i(\widehat{N}_+-\frac{1}{2})\frac{2\pi x}{L}}e^{-i\Phi_+(x)}$ is denoted by the short-hand notation $\Psi_+(x)$, since it is known (from Sections 6, 8 and Appendix F) *to behave precisely like a standard fermion field.* When Fourier-expanded, its Fourier coefficients $c_{\bar{k}}$ will thus be standard, easy-to-work-with

[13] In the language of Section 9.D, $e^{-i\Phi_+}$ is a vertex operator with scaling dimension $\frac{1}{2}$, which can be refermionized because a free fermion field has *also* has scaling dimension $\frac{1}{2}$.

fermion operators satisfying $\{c_{\bar k}, c^\dagger_{\bar k'}\} = \delta_{\bar k \bar k'}$. Formally, they can be defined by inverting the Fourier sum of Eq. (134) (cf. Eq. (4)):

$$c_{\bar k} \equiv \int_{-L/2}^{L/2} \tfrac{dx}{(2\pi L)^{1/2}}\, e^{ikx}\, \Psi_+(x) \;= \int_{-L/2}^{L/2} \tfrac{dx}{(2\pi L)^{1/2}}\, e^{ikx}\, F_+ \tfrac{1}{\sqrt{a}} e^{-i(\widehat N_+ - \frac12)\frac{2\pi x}{L}} e^{-i\Phi_+(x)} . \tag{135}$$

Since *all* operators on the right-hand side *were explicitly definded in terms of the original fermionic $c_{k\pm}$ operators,* Eq. (135) constitutes an extremely non-linear yet explicit and well-defined *construction of the new $c_{\bar k}$'s in terms of the old $c_{k\pm}$'s.* That such a direct, explicit construction is possible is one of the main advantages of constructive over field-theoretical bosonization.

What is the nature of the Fock space in which $\widehat N_+$, F_+ and the $c_{\bar k}$'s act? Since $N_L, N_R \in \mathbb{Z}$, we have $N_+ \in \mathbb{Z} + P/2$, where $P = 0$ or 1 if $(N_L - N_R)$ is even or odd. Since H_B contains only the combinations $F^\dagger_+$ and F_+, which leave $P = (2\widehat N_+) \mathrm{mod}\, 2$ invariant, the Fock space of states separates into two decoupled subspaces, labelled by $P = 0, 1$, with N_+ integer or half-integer in the $P = 0$ or 1 subspaces, respectively. The latter fact implies via Eq. (134) that the boundary condition on Ψ_+ is P-dependent, $\Psi_+(L/2) = e^{i\pi(1-P)}\Psi_+(-L/2)$, so that the $\bar k$-quantization must be too, with $\bar k = \frac{2\pi}{L}\left(n_{\bar k} - \frac{1-P}{2}\right)$, $n_{\bar k} \in \mathbb{Z}$ (cf. Eqs. (5) and (2), now with $\delta_b = 1 - P$)

The definition of a number operator $\widehat N_\eta$ and electron-hole operators $b_{q\eta}$ in Section 4 of course have analogues in the refermionized Fock space of $c_{\bar k}$'s. Firstly, we note that the following relation holds:

$$\widehat{\mathcal N} \equiv \sum_{\bar k} {}^*_* c^\dagger_{\bar k} c_{\bar k} {}^*_* = \widehat N_+ - P/2\;, \qquad (\text{with eigenvalues}\, \mathcal N \in \mathbb{Z}). \tag{136}$$

The left-hand side defines the number operator $\widehat{\mathcal N}$ of the new fermions, where now ${}^*_*\ {}^*_*$ denotes normal ordering of the $c_{\bar k}$'s with respect to a reference state, say $|0_+\rangle$, defined by $c_{\bar k}|0_+\rangle \equiv 0$ for $\bar k > 0$ and $c^\dagger_{\bar k}|0_+\rangle \equiv 0$ for $\bar k \le 0$ [cf. Eqs. (10-11)]. The right-hand side of (136) is an identity which can be proven by verifying that $\lim_{a\to 0} \int_{-L/2}^{L/2} \frac{dx}{2\pi}\left(\Psi^\dagger_+(x+a)\Psi_+(x) - \frac{1}{a}\right)$ yields the left- or right-hand sides of Eq. (136) when evaluated (to $\mathcal{O}(a, 1/L)$) using either the left- or the right-hand sides of Eq. (134) [see Eqs. (G3) or (G10) for details], respectively. More intuitively, since $\Psi_+ \sim F_+ \sim c_{\bar k}$ [by Eq. (134)], the action of Ψ_+ (or $\Psi^\dagger_+$) on any state decreases (or increases) *both* N_+ and $\mathcal N$ by one. These can thus differ only by a constant, which must be chosen such that $\mathcal N$ always is integer. Our definition of $|0_+\rangle$ above sets this constant equal to $P/2$, by setting $\mathcal N = 0$ for $N_+ = P/2$.

Secondly, the B_{q+}'s in terms of which $\Phi_+(x)$ was defined in Eq. (116) in fact can be expressed as particle-hole operators built from $c_{\bar k}$'s. To see this, we exploit the analogy between the refermionization identity (134) and the original bosonization identity (63) combined with its Fourier expansion (3), to conclude that

$$B^\dagger_{q+} = \tfrac{i}{\sqrt{n_q}} \sum_{\bar k} c^\dagger_{\bar k + q} c_{\bar k}\;, \quad B_{q+} = \tfrac{-i}{\sqrt{n_q}} \sum_{\bar k} c^\dagger_{\bar k - q} c_{\bar k}\;, \quad (q = \tfrac{2\pi}{L} n_q > 0), \tag{137}$$

in analogy to Eq. (16) for the $b_{q\eta}$ in the $\phi_\eta(x)$-fields of Eqs. (33-34).

We are now ready to refermionize H_+ by expressing it in terms of the $c_{\bar k}$'s:

$$H_{0+} \;=\; v\left[\tfrac{2\pi}{L}\left(\tfrac12 \widehat{\mathcal N}(\widehat{\mathcal N}+P) + \tfrac{P}{8}\right) + \int_{-L/2}^{L/2} \tfrac{dx}{2\pi}\tfrac12 {}^*_*(\partial_x \Phi_+(x))^2 {}^*_*\right] \tag{138}$$

$$\;=\; \sum_{\bar k} \varepsilon_{\bar k} {}^*_* c^\dagger_{\bar k} c_{\bar k} {}^*_* + \Delta_L \tfrac{P}{8}\;, \quad \text{where} \quad \Delta_L \equiv v\tfrac{2\pi}{L}\;, \quad \varepsilon_{\bar k} \equiv v\bar k = \Delta_L\left(n_{\bar k} - \tfrac{1-P}{2}\right); \tag{139}$$

$$H_B \;=\; \frac{v\lambda_B}{2\pi a}\left(F^\dagger_+ e^{i(\Phi_+ + \theta_B)} + F_+ e^{-i(\Phi_+ + \theta_B)}\right) = \frac{v\lambda_B}{2\pi\sqrt{a}}\left[\Psi^\dagger_+(0)e^{i\theta_B} + \Psi_+(0)e^{-i\theta_B}\right] \tag{140}$$

$$= \sqrt{\Delta_L \Gamma} \sum_{\bar{k}} \left(c^\dagger_{\bar{k}} e^{i\theta_B} + c_{\bar{k}} e^{-i\theta_B} \right) , \quad \text{where} \quad \Gamma \equiv \tfrac{v\lambda_B^2}{a(4\pi)^2} \; . \tag{141}$$

For H_{0+} in (138) we started from Eq. (113) and wrote $\frac{1}{2}\widehat{N}_+^2 = \frac{1}{2}\widehat{\mathcal{N}}(\widehat{\mathcal{N}} + P) + \frac{P}{8}$ [using Eq. (136)]; to obtain (139) we inverted the line of reasoning that lead from Eq. (65) to (70) [or from (G12) to (G16)]. For H_B we simply inserted (134) into (128). Eq. (141) manifestly shows that $\widehat{N}_+$ is *not* conserved, as expected for a back-scattering term that converts L- into R-movers and vice versa. (In contrast, the total number of fermions, $2\widehat{N}_- = \widehat{N}_L + \widehat{N}_R$, obviously *is* conserved, since by Eq. (133) $[F_+, \widehat{N}_-] = 0$.)

It is easiest to diagonalize H_+ if it is nominally *quadratic* in fermionic operators. It can be made so (and the phase $e^{i\theta_B}$ absorbed) by a trivial unitary phase transformation:

$$U_+ \equiv e^{i(\frac{\pi}{2}\widehat{\mathcal{N}}^2 - \theta_B \widehat{\mathcal{N}})} \qquad \text{gives} \qquad U_+ \left(F_+^\dagger e^{i\theta_B} \right) U_+^{-1} = F_+^\dagger \, e^{i\pi(\widehat{\mathcal{N}} + \frac{1}{2})} = F_+^\dagger (i\sqrt{2}\, \alpha_d) \; . \tag{142}$$

Here $\alpha_d \equiv \frac{1}{\sqrt{2}} e^{i\pi\widehat{\mathcal{N}}}$ is a "local Majorana fermion", since its definition implies the following properties [the first two follow from the fact that $\mathcal{N} \in \mathbb{Z}$, the last four from Eq. (133)]:

$$\{\alpha_d, \alpha_d\} = 1 \; , \qquad \alpha_d^\dagger = \alpha_d \; , \qquad \{F_+, \alpha_d\} = \{F_+^\dagger, \alpha_d\} = \{c_{\bar{k}}, \alpha_d\} = \{c_{\bar{k}}^\dagger, \alpha_d\} = 0 \; . \tag{143}$$

Since Eqs. (142) and (135) imply $U_+ \left(c^\dagger_{\bar{k}} e^{i\theta_B} \right) U_+^{-1} = c^\dagger_{\bar{k}} (i\sqrt{2}\, \alpha_d)$, the transformed version of H_+ is quadratic in fermions, as desired:

$$H'_+ \equiv U_+ H_+ U_+^{-1} = \Delta_L \tfrac{P}{8} + \sum_{\bar{k}} \left[\varepsilon_{\bar{k}^*} {}^* c^\dagger_{\bar{k}} c_{\bar{k}^*} {}^* + \sqrt{\Delta_L \Gamma} \left(c^\dagger_{\bar{k}} + c_{\bar{k}} \right) \left(i\sqrt{2}\, \alpha_d \right) \right] \; . \tag{144}$$

The trick of converting a term linear in fermions to a quadratic form using a Majorana fermion was also used by Matveev [34]. In contrast to his work, however, our use of constructive bosonization, in which electron counting operators such as $\widehat{\mathcal{N}}$ play a fundamental role, allows us to precisely formulate the unitary transformation that causes this Majorana fermion to appear naturally.

The fact that $\Gamma \propto 1/a$ is consistent with the well-known fact that for $g < 1$, an impurity in a Tomonoga-Luttinger liquid "scales into the strong-coupling regime". By this statement one means that under a renormalization group transformation designed to focus on the low-energy regime of the model, the effective strength of the impurity scattering *increases.* To see this explicitly, one can adopt for example Anderson's poor man's scaling approach, in which the RG is generated by reducing (at fixed L, usually $= \infty$) the bandwidth while adjusting the couplings to keep the dynamical properties invariant. Since the cut-off used when bosonizing is $1/a \, (\sim p_F)$, reducing the bandwidth means changing a to a larger value a', which must be accompanied by a change in coupling constant from its initial value $\lambda_B(a)$ to a new value $\lambda'_B(a')$. Since a occurs in H'_+ *only* through Γ, one immediately concludes that $\lambda'_B(a') = \lambda_B \sqrt{a'/a}$, which implies that λ_B grows under rescaling. (This is completely analogous to what happens for the 2-channel Kondo model [17, 18].)

10.C.3 Finite-size diagonalization of H'_+ at $g = \frac{1}{2}$

We give the linear transformation (derived in Appendix I) that diagonalizes the refermionized H'_+ (for finite L) by expressing the $c_{\bar{k}}$'s in terms of new fermions $\tilde{\alpha}_\varepsilon$ and $\beta_{\bar{k}}$.

As pointed out by Oreg and Finkel'stein [25] and Furusaki [22], the form (144) for H'_+ is related to that arising after bosonizing and refermionizing the 2-channel Kondo model. Following the latter's solution in

Refs. [17, 18], this fact can be exploited to explicitly diagonalize H'_+ for finite L in the $P = 0$ sector[14] [i.e. with $\bar{k} = \frac{2\pi}{L}(n_{\bar{k}} - \frac{1}{2})$]. This elementary excercise is performed in Appendix I, with the following results:

$$c_{\bar{k}}(t) = \tfrac{1}{\sqrt{2}}\left(\alpha_{\bar{k}}(t) + i\beta_{\bar{k}}(t)\right) , \tag{145}$$

$$\alpha_{-\bar{k}} \equiv \alpha^\dagger_{\bar{k}} , \qquad \beta_{-\bar{k}} \equiv \beta^\dagger_{\bar{k}} , \qquad \{\alpha_{\bar{k}}, \alpha^\dagger_{\bar{k}'}\} = \{\beta_{\bar{k}}, \beta^\dagger_{\bar{k}'}\} = \delta_{\bar{k}\bar{k}'} , \qquad \{\alpha, \beta\} = 0 ; \tag{146}$$

$$\alpha_{\bar{k}}(t) = \sum_\varepsilon A_{\bar{k},\varepsilon} e^{-i\varepsilon t}\tilde{\alpha}_\varepsilon , \qquad \alpha_d(t) = \sum_\varepsilon A_{d,\varepsilon} e^{-i\varepsilon t}\tilde{\alpha}_\varepsilon , \qquad \beta_{\bar{k}}(t) = e^{-i\varepsilon_{\bar{k}} t}\beta_{\bar{k}} , \tag{147}$$

$$\tilde{\alpha}_{-\varepsilon} = \tilde{\alpha}^\dagger_\varepsilon , \qquad \{\tilde{\alpha}_\varepsilon, \tilde{\alpha}^\dagger_{\varepsilon'}\} = \delta_{\varepsilon\varepsilon'} , \qquad \{\tilde{\alpha}, \beta\} = 0 ; \tag{148}$$

$$A_{\bar{k},\varepsilon} = \frac{i2\sqrt{\Delta_L\Gamma}A_{d,\varepsilon}}{\varepsilon - \varepsilon_{\bar{k}}} , \quad A_{d,\varepsilon} = -i\,\mathrm{sign}(\varepsilon)\left[\frac{4\Delta_L\Gamma}{4\Delta_L\Gamma + \varepsilon^2 + (4\pi\Gamma)^2}\right]^{1/2} , \quad [\mathrm{sign}(0) \equiv i] \tag{149}$$

$$H'_+ = \sum_{\varepsilon>0} \varepsilon\left(\tilde{\alpha}^\dagger_\varepsilon\tilde{\alpha}_\varepsilon - \tfrac{1}{2}\right) + \sum_{\bar{k}>0}\varepsilon_{\bar{k}}\left(\beta^\dagger_{\bar{k}}\beta_{\bar{k}} + \tfrac{1}{2}\right) ; \tag{150}$$

$$\frac{\varepsilon}{4\Gamma} = \Delta_L \sum_{\bar{k}=-\infty}^{\infty}\frac{1}{\varepsilon - \varepsilon_{\bar{k}}} ; \tag{151}$$

$$\langle G'_B|\beta_{\bar{k}}\beta^\dagger_{\bar{k}'}|G'_B\rangle = \delta_{\bar{k}\bar{k}'}\theta(\varepsilon_{\bar{k}}), \qquad \langle G'_B|\tilde{\alpha}_\varepsilon\tilde{\alpha}^\dagger_{\varepsilon'}|G'_B\rangle = \delta_{\varepsilon\varepsilon'}\theta(\varepsilon), \qquad (\text{with } \theta(0) \equiv \tfrac{1}{2}) . \tag{152}$$

Eqs. (145) and (147) express $c_{\bar{k}}$ and α_d in terms of two sets of fermions, $\{\beta_{\bar{k}}\}$ and $\{\tilde{\alpha}_\varepsilon\}$, that diagonalize H'_+ [cf. (150)]. The sums $\sum_\varepsilon$ in (147) run over all real solutions ε of the eigenvalue equation (151). Analyzing (151) graphically (cf. [18]) shows that each ε lies within $\Delta_L/2$ of some $\varepsilon_{\bar{k}}$, to which it reduces as $\Gamma \to 0$, with the exception of one solution, namely $\varepsilon = 0$. The latter is associated with the Majorana fermion $\tilde{\alpha}_0 = \tilde{\alpha}^\dagger_0$, which reduces to α_d as $\Gamma \to 0$ (and whose contribution to $c_{\bar{k}}$ is negligible for $L \to \infty$). For each $\varepsilon > 0$ that solves (151), $-\varepsilon$ does too; however, by (146) and (148) the negative-energy operators $\beta^\dagger_{-|\bar{k}|}$ and $\tilde{\alpha}^\dagger_{-|\varepsilon|}$ are not independent, but should be viewed as shorthand (making some equations more compact) for $\beta_{|\bar{k}|}$ and $\tilde{\alpha}_{|\varepsilon|}$. The latter annihilate the ground state $|G'_+\rangle$ of H'_+ [cf. (152)].

Correlation functions with respect to H'_+ and $|G'_B\rangle$, which we denote by $\langle\quad\rangle'$, i.e.

$$\langle O_1(t)O_2(0)\rangle' \equiv \langle G'_B|e^{iH'_+t}O_1e^{-iH'_+t}O_2|G'_B\rangle , \tag{153}$$

are straightforward to calculate using the above results, provided that O_1 and O_2 can be expressed in terms of the $c_{\bar{k}}$'s and α_d. In the process it is often convenient to take the continuum limit $L \to \infty$, in which the spectrum of ε's and $\varepsilon_{\bar{k}}$'s becomes continuous:

$$\Delta_L\sum_{\bar{k}} \stackrel{L\to\infty}{\longrightarrow} \int d\varepsilon_{\bar{k}}, \qquad \Delta_L\sum_\varepsilon \stackrel{L\to\infty}{\longrightarrow} \int d\varepsilon, \qquad \frac{1}{\varepsilon - \varepsilon_{\bar{k}}} \stackrel{L\to\infty}{\longrightarrow} P\frac{1}{\varepsilon - \varepsilon_{\bar{k}}} + \frac{\varepsilon}{4\Gamma}\delta(\varepsilon - \varepsilon_{\bar{k}}) . \tag{154}$$

In the third relation, P denotes principle value and the δ-function is needed to ensure consistency with Eq. (151). Where necessary, divergent integrals will be regularized by inserting a factor $e^{-|\bar{k}|a} = e^{-|\varepsilon_{\bar{k}}|a/v}$ or $e^{-|\varepsilon|a/v}$ [just as one does for free Green's functions, cf. Appendix H.1.a]. For example (see also Appendix J)

$$D_\beta(t) \equiv \Delta_L\sum_{\bar{k}\bar{k}'}\langle\beta_{\bar{k}}(t)\beta^\dagger_{\bar{k}'}(0)\rangle' \stackrel{L\to\infty}{\longrightarrow} \int_0^\infty d\varepsilon_{\bar{k}}\, e^{-\bar{k}(it+a/v)} = \frac{1}{it + a/v} ; \tag{155}$$

$$D_{\alpha_d}(t) \equiv \langle\alpha_d(t)\alpha_d(0)\rangle' = \sum_{\varepsilon\varepsilon'} e^{-i\varepsilon t}A_{d,\varepsilon}A^*_{d,\varepsilon'}\langle\alpha_\varepsilon\alpha^\dagger_{\varepsilon'}\rangle' \tag{156}$$

[14] The differences between the $P = 0$ and 1 sectors disappear in the continuum limit $L \to 0$ that we are ultimately interested in.

$$\stackrel{L\to\infty}{\longrightarrow} \quad \int_0^\infty d\varepsilon \, \frac{e^{-i\varepsilon t}\, 4\Gamma}{\varepsilon^2 + (4\pi\Gamma)^2} \;=\; \begin{cases} \frac{1}{2} & (t=0), \\ \frac{1}{4\pi^2\Gamma i t}\left[1 + \mathcal{O}\left(\frac{1}{\Gamma t}\right)\right] & (\Gamma t \gg 1). \end{cases} \tag{157}$$

We used (147) and (149) for α_d in (156), and (154) when taking the limit $L \to \infty$. The asymptotic $\Gamma t \gg 1$ behavior of (157) was obtained using the general result

$$\int_0^\infty d\varepsilon \, \frac{e^{-\varepsilon(it+a)}\, \varepsilon^n}{(\varepsilon^2 + c^2)^m (\varepsilon + \bar{c})^{\bar{m}}} \;\sim\; \frac{n!}{c^{2m}\, \bar{c}^{\bar{m}}\, (it)^{1+n}} \qquad \text{for} \quad ct, \bar{c}t \gg 1, \tag{158}$$

(with $n, m, \bar{m} \geq 0$ integer, $c, \bar{c} > 0$ real), which follows by noting that for $ct, \bar{c}t \gg 1$ the integrals are dominated by the regime $\varepsilon \ll c, \bar{c}$, in which $(\varepsilon^2 + c^2)^m (\varepsilon + \bar{c})^{\bar{m}} \simeq c^{2m} \bar{c}^{\bar{m}}$.

10.C.4 Bosonic correlation functions at $g = 1/2$

We express $\Phi_+(t,x)$ in terms of the fermions $\beta_{\bar{k}}$ and $\tilde{\alpha}_\varepsilon$, which enables us to express **arbitrary** *bosonic correlation functions in terms of fermionic ones. We then show explicitly that for $\Gamma t \gg 1$, $\langle \Phi_+(t,0)\Phi_+(0,0)\rangle' \sim t^{-2}$ and $\langle e^{i\lambda\Phi_+(t,0)} e^{-i\lambda\Phi_+(0,0)}\rangle' \sim const.$*

The key to this endeavour is that the Fourier coefficients B_{q+} of $\Phi_+(t,x)$ in (116) have a refermionized representation, namely (137), a fact that has to our knowledge not been exploited before. Using (145) and (146), B_{q+} can be rewritten as

$$B_{q+} = \frac{-i}{\sqrt{n_q}} \sum_{\bar{k}} (\alpha_{-\bar{k}+q} - i\beta_{-\bar{k}+q})(\alpha_{\bar{k}} + i\beta_{\bar{k}}) = -\frac{1}{\sqrt{n_q}} \sum_{\bar{k}} \beta_{\bar{k}} \alpha_{-\bar{k}+q} \, . \tag{159}$$

(Since $\sum_{\bar{k}} = \sum_{\bar{k}-q}$, we have $\sum_{\bar{k}} \alpha_{-\bar{k}+q}\alpha_{\bar{k}} = \frac{1}{2}\sum_{\bar{k}}\{\alpha_{-\bar{k}+q}, \alpha_{\bar{k}}\} = 0$, etc.) Inserting (159) into (116) for $\Phi_+(t,x)$, (147) for $\alpha_{-\bar{k}+q}$, and (149) for $A_{-\bar{k}+q,\varepsilon}$ yields

$$\Phi_+(t,x) \;=\; \sum_{q\neq 0} \frac{e^{-a|q|/2} e^{-iqx}}{n_q} \sum_{\bar{k}} \beta_{\bar{k}}(t) \alpha_{-\bar{k}+q}(t) = \sum_{\bar{k}\varepsilon} \Phi_{\bar{k},\varepsilon}(x)\, \beta_{\bar{k}}(t) \tilde{\alpha}_\varepsilon(t) \, , \tag{160}$$

$$\Phi_{\bar{k},\varepsilon}(x) \;\equiv\; \sum_{q\neq 0} \frac{e^{-a|q|/2} e^{-iqx}}{n_q} A_{-\bar{k}+q,\varepsilon} \;=\; \frac{\operatorname{sgn}(\varepsilon)\, 4\Delta_L \Gamma}{[4\Delta_L\Gamma + \varepsilon^2 + (4\pi\Gamma)^2]^{1/2}} \sum_{q\neq 0} \frac{e^{-a|q|/2} e^{-iqx}}{n_q(\varepsilon + \varepsilon_{\bar{k}} - \varepsilon_q)} \, . \tag{161}$$

Here $\varepsilon_q = \Delta_L n_q$, and $\sum_{q\neq 0}$ means a sum over all $n_q \in \mathbb{Z}$ (positive and negative) except $q = 0$. Using (154) to perform this sum in the continuum limit, we obtain for $x = 0$:

$$\sum_{q\neq 0} \frac{e^{-a|q|/2}}{n_q(\varepsilon + \varepsilon_{\bar{k}} - \varepsilon_q)} \;=\; \frac{\Delta_L}{\varepsilon + \varepsilon_{\bar{k}}} \sum_{q\neq 0} \Big[\frac{e^{-a|\varepsilon_q|/2v}}{\varepsilon + \varepsilon_{\bar{k}} - \varepsilon_q} - \frac{e^{-a|\varepsilon_q|/2v}}{\varepsilon_q} \Big] \stackrel{L\to\infty}{\longrightarrow} \frac{\varepsilon\, e^{-|\varepsilon+\varepsilon_{\bar{k}}|a/2v}}{4\Gamma(\varepsilon + \varepsilon_{\bar{k}})} \tag{162}$$

(we kept only terms that do not vanish when $a \to 0$). Thus (161) yields [again using (154)]:

$$\Phi_{\bar{k},\varepsilon} \equiv \Phi_{\bar{k},\varepsilon}(0) \stackrel{L\to\infty}{\longrightarrow} \frac{\Delta_L |\varepsilon|}{[\varepsilon^2 + (4\pi\Gamma)^2]^{1/2}} \left[e^{-|\varepsilon+\varepsilon_{\bar{k}}|a/2v} \frac{P}{\varepsilon + \varepsilon_{\bar{k}}} + \frac{\varepsilon}{4\Gamma}\delta(\varepsilon + \varepsilon_{\bar{k}}) \right] . \tag{163}$$

Eqs. (160) and (163) for $\Phi_+(t,0)$ and $\Phi_{\bar{k}\varepsilon}(0)$ instructively show that the infrared divergence inherent in a free boson field (due to the $1/q$ in $\Phi_+(0) = \sum_{q\neq 0} \frac{i}{q} \sum_k c^\dagger_{\bar{k}-q} c_{\bar{k}})$ is cut off by backscattering at a scale Γ.

This has dramatic consequences for the 2-point correlator:

$$D_{\Phi_+}(t) \equiv \langle \Phi_+(t,0)\Phi_+(0,0)\rangle' \tag{164}$$

$$= \sum_{\bar k \bar k' \varepsilon\varepsilon'} \Phi_{\bar k,\varepsilon}\Phi^*_{\bar k',\varepsilon'} \langle \beta_{\bar k}(t)\tilde\alpha_\varepsilon(t)\tilde\alpha^\dagger_{\varepsilon'}(0)\beta^\dagger_{\bar k'}(0)\rangle' = \sum_{\bar k>0}\sum_{\varepsilon\geq 0} e^{-i(\varepsilon_{\bar k}+\varepsilon)t}\theta(\varepsilon)|\Phi_{\bar k,\varepsilon}|^2 \tag{165}$$

$$= \mathcal{P}\!\!\int_0^\infty d\varepsilon\, d\varepsilon_{\bar k}\, \frac{e^{-(\varepsilon_{\bar k}+\varepsilon)(it+a/v)}\,\varepsilon^2}{[\varepsilon^2+(4\pi\Gamma)^2](\varepsilon+\varepsilon_{\bar k})^2} = \begin{cases} -\ln\left(e^\gamma 4\pi\Gamma a/v\right) & \text{for } t=0,\ \Gamma a \ll 1; \\ \frac{1}{(4\pi\Gamma i t)^2}\left[1+\mathcal{O}\left(\frac{1}{\Gamma t}\right)\right] & \text{for } \Gamma t \gg 1\,. \end{cases} \tag{166}$$

[The $\Gamma t \gg 1$ result was obtained by doing first the $\varepsilon_{\bar k}$, then the ε integral, using Eq. (158).] Compare the results (166) with (80) for a free boson correlator, namely $-\ln[\frac{2\pi}{L}(it+a)]$: For $t=0$, the infrared divergence is now cut off by Γ instead of $1/L$ ($\gamma = 0.577\ldots$ is Euler's constant); and for $\Gamma t \gg 1$ the t^{-2} decay of the correlator is much faster than the free logarithmic behavior. This strong suppression of the long-time fluctuations of $\Phi_+(t,x=0)$ has been paraphrased [21, 23] by saying that *at $x=0$ the backscattering impurity "pins" the field $\Phi_+(t,0)$ to its average value $\langle\Phi_+(t,0)\rangle'$, the fluctuations around which are "massive"*[15]. In more physical terms, the *current fluctuations* at the impurity site, which are governed by $\partial_x\Phi_+(x)|_{x=0}$, are suppressed by the backscattering impurity. This eminently plausible result was first found by Kane and Fisher [16], who showed via an RG analysis that the conductance past such an impurity is 0 at $T=0$ whenever $g<1$. Note, however, that density fluctuations, governed by $\partial_x\Phi_-(x)|_{x=0}$, are *not* suppressed by backscattering (since $[H_B,\Phi_-]=0$).

The pinning of Φ_+ has important consequences; for example, it immediately implies that the correlator of two vertex functions is asymptotically constant:

$$D_{V_\lambda}(t) \equiv a^{-\lambda^2}\langle e^{i\lambda\Phi_+(t,0)}e^{-i\lambda\Phi_+(0,0)}\rangle' \tag{167}$$

$$= a^{-\lambda^2}\sum_{n=0}^\infty\sum_{n'=0}^\infty \frac{(i\lambda)^n(-i\lambda)^{n'}}{n!\,n'!}\langle\Phi_+^n(t,0)\,\Phi_+^{n'}(0,0)\rangle' \tag{168}$$

$$= a^{-\lambda^2}\langle e^{i\lambda\Phi_+(t,0)}\rangle'\langle e^{-i\lambda\Phi_+(0,0)}\rangle' + \mathcal{O}(\Gamma t)^{-2} \qquad \text{for} \quad \Gamma t \gg 1\,. \tag{169}$$

To obtain (169), we invoked the fermionic expression (160) for Φ_+ and Wick's theorem for fermions to envisage each correlator in (168) as a sum of products of contractions of the kind $\langle\tilde\alpha\tilde\alpha^\dagger\rangle'$ and $\langle\beta\beta^\dagger\rangle'$. All terms involving no contractions at all between an operator at time t and one at 0, to be called "disconnected", can be reorganized to yield the first term of (169), a t-independent constant; all other, "connected", terms decay at least as t^{-2}, since $D_{\Phi_+}(t)$ of (164) is the leading such term. Note that the value of the leading constant in (169) is *not* simply equal to $a^{-\lambda^2}e^{-\lambda^2\langle\Phi_+(0,0)^2\rangle'} = (e^\gamma 4\pi\Gamma/v)^{\lambda^2}$ [by (166)], since the identity (75) which would yield this result holds only for *free* boson fields (cf. the end of Appendix J.1).

10.D Tunneling density of states at the impurity site

For $g=\frac12$, we calculate the low-energy ($\omega\to 0$) asymptotic behavior of the tunneling density of states at the impurity site, $\rho_{dos}(\omega)\sim\omega^{\nu-1}$, following Furusaki. Our result $\nu=2$ resolves the controversy between

[15] By Eq. (160), this average value is $\langle\Phi_+(t,0)\rangle'=0$, but in general it depends on one's choice of gauge: if we had not "gauged away" the phase factor $e^{i\theta_B}$ in H_+ of (128) by including the factor $e^{-i\theta_B\mathcal{N}_+}$ in the unitary transformation U_+ of (142), we would have obtained $\langle\Phi_+\rangle'=\theta_B$ here [23]. The fluctuations are called "massive" because similar behavior ($\langle\Phi_+(t)\Phi_+(0)\rangle\sim t^{-2}$) occurs in models involving only bosonic fields, but with a mass term, e.g. with Hamiltonian $H_M = \int_{-L/2}^{L/2}\frac{dx}{2\pi}\frac12 {}^*_*(\partial_x\Phi_+(x))^2{}^*_* + \frac12 M[\Phi_+(x=0)]^2$. In fact, such effective models can be used to approximately treat the general case $g\neq\frac12$, which cannot be refermionized [35, 22].

Fabrizio & Gogolin and Furusaki vs. and Oreg & Finkel'stein in favor of the former authors.

The tunneling electron density of states at the impurity site, $\rho_{dos}(\omega)$, is defined by

$$\rho_{dos}(\omega) \equiv \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{i\omega t} \langle G|\Psi_{phys}(t)\Psi^\dagger_{phys}(0) + \Psi^\dagger_{phys}(0)\Psi_{phys}(t)|G\rangle , \tag{170}$$

where $\Psi_{phys}(t) \equiv e^{iHt}\Psi_{phys}(x=0)e^{-iHt}$ and $|G\rangle$ is the ground state of $H = H_0 + H_F + H_B$. Both terms in (170) are real, and $\rho_{dos}(\omega) = \rho_{dos}(-\omega)$ by particle-hole symmetry ($c_{k,L/R} \to c^\dagger_{-k,L/R}$ maps $H(\lambda_B)$ to $H(-\lambda_B)$, both of which have the same spectrum). Since at $T = 0$ the second term in (170) does not contribute to the $\omega > 0$ part of $\rho_{dos}(\omega)$, its asymptotic $\omega \to 0^+$ behavior is determined by the asymptotic $t \to \infty$ behavior of

$$D_{phys}(t) \equiv \langle G|\Psi_{phys}(t)\Psi^\dagger_{phys}(0)|G\rangle \sim (it)^{-\nu} \quad \text{for } t \to \infty , \tag{171}$$

which implies $\rho_{dos}(\omega) \sim \omega^{\nu-1}$ for $\omega \to 0^+$. We shall throughout take the continuum limit $L \to \infty$ and neglect all $1/L$ terms. The reason why $D_{phys}(t)$ asymptotically does not contain a fluctuating factor $e^{-i\Delta t}$ for $t \to \infty$ is that H has gapless excitations, implying that Δ must be zero [22]. Our goal is to calculate the exponent ν, which has recently been subject to quite some controversy, as mentioned in the introduction to Section 10.

We start by bosonizing the physical fermion field Ψ_{phys} occuring in (171), using (101), (104) and (129):

$$\begin{aligned} \Psi_{phys}(x=0) &= \tilde\psi_L(x=0) + \tilde\psi_R(x=0) = a^{-1}\left(F_L e^{-i\tilde\phi_L} + F_R e^{-i\tilde\phi_R}\right) \\ &= a^{-1} e^{-\frac{i}{\sqrt{2g}}\Phi_-}\left(F_L e^{-i\sqrt{\frac{g}{2}}\Phi_+} + F_R e^{i\sqrt{\frac{g}{2}}\Phi_+}\right) . \end{aligned} \tag{172}$$

Since $H = H_+ + H_-$ and $[H_-, H_+] = 0$ [see Eq. (130)], $D_{phys}(t)$ can be factorized as $D_{phys}(t) \equiv D_F(t)D_B(t)$, where

$$D_F(t) \equiv \langle G_F|e^{iH_- t}\left(e^{-\frac{i}{\sqrt{2g}}\Phi_-}\right)e^{-iH_- t}\left(e^{\frac{i}{\sqrt{2g}}\Phi_-}\right)e^{i\hat E t}|G_F\rangle , \tag{173}$$

$$\begin{aligned} D_B(t) &\equiv a^{-1}\langle G_B|e^{iH_+ t}\left(F_L e^{-i\sqrt{\frac{g}{2}}\Phi_+} + F_R e^{i\sqrt{\frac{g}{2}}\Phi_+}\right)e^{-iH_+ t}\left(F_L^\dagger e^{i\sqrt{\frac{g}{2}}\Phi_+} + F_R^\dagger e^{-i\sqrt{\frac{g}{2}}\Phi_+}\right)|G_B\rangle \\ &\equiv D_{LL}(t) + D_{RR}(t) + D_{LR}(t) + D_{RL}(t) . \end{aligned} \tag{174}$$

Here $|G_F\rangle$ and $|G_B\rangle$ are the ground states of H_- and H_+, respectively; $\hat E$ is defined by $e^{iH_- t}F_{L/R}e^{-iH_- t} \equiv e^{i\hat E t}$ and, being of order $\frac{2\pi}{L}$, will be neglected henceforth.

10.D.1 Free tunneling density of states

In the absence of an impurity the calculation of $D_F(t)$ and $D_B(t)$ is straightforward, since $H_{0\pm}$ are free boson Hamiltonians, so that the free-boson relations (74) and (76) [or equivalently Eq. (87)] can be used:

$$D_F(t) = \langle G_F|e^{iH_- t}\left(e^{-\frac{i}{\sqrt{2g}}\Phi_-(0)}\right)e^{-iH_- t}\left(e^{\frac{i}{\sqrt{2g}}\Phi_-(0)}\right)|G_F\rangle \tag{175}$$

$$\simeq e^{\frac{1}{2g}\langle G_F|\Phi_-(t)\Phi_-(0) - \Phi_-(0)\Phi_-(0)|G_F\rangle} = (1 + ivt/a)^{-\frac{1}{2g}} \tag{176}$$

$$D_B(t) = a^{-1}\langle G_B|e^{iH_+ t}\left(e^{-i\sqrt{\frac{g}{2}}\Phi_+(0)}\right)e^{-iH_+ t}\left(e^{i\sqrt{\frac{g}{2}}\Phi_+(0)}\right)|G_B\rangle + \text{h.c.} \tag{177}$$

$$\simeq a^{-1}e^{\frac{g}{2}\langle G_B|\Phi_+(t)\Phi_+(0) - \Phi_+(0)\Phi_+(0)|G_B\rangle} + \text{c.c.} = a^{-1}(1 + ivt/a)^{-\frac{g}{2}} + \text{c.c.} \tag{178}$$

It follows that $\nu = \frac{1}{2}(g + \frac{1}{g})$. For the free-fermion case $g = 1$ we have $\nu = 1$, i.e. for $\omega \to 0$ we recover the standard "Fermi-liquid" property $\rho_{dos}(\omega = 0) \neq 0$. However, for any $g \neq 1$ we have $\nu > 1$, i.e. $\rho_{dos}(\omega) \to 0$ for $\omega \to 0$. Thus, the *interactions in 1-D cause the density of states to vanish at the Fermi energy*. This property is one of the most spectacular differences between a Tomonaga-Luttinger liquid (without impurities) and a Fermi liquid.

10.D.2 Effect of an impurity on $\rho_{dos}(\omega)$

Let us now consider the effect of turning on H_F and H_B. The calculation of $D_F(t)$ is again simple, since we can use the unitary transformation $U_- = e^{-ic_-\Phi_-}$ of Section 10.C.1 to map H_- onto a free Hamiltonian $H'_- \equiv U_- H_- U_-^{-1}$, see Eq. (132). Denoting the corresponding ground state by $|G'_F\rangle = U_-|G_F\rangle$, the function $D_F(t)$ can be evaluated by first making this transformation in Eq. (175), and proceeding as before:

$$\begin{aligned} D_F(t) &= \langle G'_F| e^{iH'_- t} \left(e^{-\frac{i}{\sqrt{2g}}\Phi_-(0)} \right) e^{-iH'_- t} \left(e^{\frac{i}{\sqrt{2g}}\Phi_-(0)} \right) |G'_F\rangle && (179) \\ &\simeq e^{\frac{1}{2g}\langle G'_F|\Phi_-(t)\Phi_-(0) - \Phi_-(0)\Phi_-(0)|G'_F\rangle} = (1 + ivt/a)^{-\frac{1}{2g}} && (180) \end{aligned}$$

Now $\Phi_-(t) \equiv \Phi_-(t, x = 0)$ denotes time-development w.r.t. H'_-.

To calculate $D_B(t)$, we restrict ourselves to the case $g = \frac{1}{2}$ for which we have refermionized H_+ above. First, using a trick due to Furusaki[16] [22] and Fabrizio and Gogolin [23], we note that the following relations hold:

$$\begin{aligned} 2\,\alpha_d H'_+(\lambda_B)\alpha_d &= H'_+(-\lambda_B) \equiv \bar{H}'_+(\lambda_B)\,, && (181) \\ F_+ H'_+(\lambda_B) F_+^\dagger &= \bar{H}'_+(\lambda_B) + v\tfrac{2\pi}{L}(\hat{\mathcal{N}} + \tfrac{1}{2}P + \tfrac{1}{2})\,, && (182) \\ F_{L/R} H_+(\lambda_B) F_{L/R}^\dagger &= H_+(-\lambda_B) + O(\tfrac{2\pi}{L}) \equiv \bar{H}_+(\lambda), && (183) \end{aligned}$$

The first two follow from (144) for H'_B and (143), and the third from using $\{F_L, F_R\} = 0$, etc., in (128) for H_B. Thus, commuting α_d or F_+ past $H'_+(\lambda_B)$ yields a similar Hamiltonian with backscattering term of opposite sign, $\bar{H}'_+(\lambda_B) \equiv H'_+(-\lambda_B)$, and similarly for commuting $F_{L/R}$ past $H_+(\lambda_B)$. The extra $\mathcal{O}(\frac{2\pi}{L})$ term in (182) results from the $\frac{1}{2}\hat{\mathcal{N}}(\hat{\mathcal{N}} + P)$ in $H'_{0+} = H_{0+}$ [see (138)] and can be neglected in the continuum limit, and similarly for that in (183).

To calculate the LL and RR contributions in Eq. (174), we now proceed as follows:

$$\begin{aligned} D_{LL/RR}(t) &\equiv a^{-1}\langle G_B| e^{iH_+ t} \left(F_{L/R} e^{\mp\frac{i}{2}\Phi_+} \right) e^{-iH_+ t} \left(F_{L/R}^\dagger e^{\pm\frac{i}{2}\Phi_+} \right) |G_B\rangle && (184) \\ &= a^{-1}\langle G'_B| e^{iH'_+ t} e^{\mp\frac{i}{2}\Phi_+} e^{-i\bar{H}'_+ t} e^{\pm\frac{i}{2}\Phi_+} |G'_B\rangle && (185) \\ &\sim 2a^{-1}\langle e^{iH'_+ t}\alpha_d e^{\mp\frac{i}{2}\Phi_+} e^{-iH'_+ t} e^{\pm\frac{i}{2}\Phi_+}\alpha_d\rangle' \equiv 2a^{-3/4} D_{\alpha_d V_{\mp 1/2}}(t)\,. && (186) \end{aligned}$$

[16] Since Furusaki uses *field-theoretical* bosonization, he somewhat nonrigorously treats Klein factors (which he denotes by η) as though they were Majorana fermions (which they are not, since $F^2 \neq 1$), and hence uses the same η for what here are three distinct operators, F_+, $F_+^\dagger$ and α_d. Moreover, he uses a different argument [when discussing *his* Eqs. (25-28)] to evaluate $D_{LL/RR}(t)$ than our Eqs. (184-190): He points out that in Eq. (185) one can use (71) to write $e^{\pm\frac{i}{2}\Phi_+} e^{-i\bar{H}'_+ t} e^{\mp\frac{i}{2}\Phi_+} = e^{-i(\bar{H}'_+ \mp \frac{v}{2}\partial_x\Phi_+ + \frac{v}{4a})t}$, then argues that $\partial_x\Phi_+ \equiv \partial_x\Phi_+(x=0)$ is an irrelevant operator (since, as can be readily confirmed using our methods, $\langle \partial_x\Phi_+(t)\partial_x\Phi_+(0)\rangle \sim t^{-4}$), which hence does not affect the asymptotic behavior of $D_{LL/RR}$. This argument leads to $D_{LL/RR}(t) \sim \langle G'_B| e^{iH'_+ t} e^{-i(\bar{H}'_+ + \frac{v}{4a})t} |G'_+\rangle \sim t^{-1} e^{-i\Delta t}$, which produces the desired asymptotic t^{-1} decay. However, it also produces an oscillatory factor $\Delta = \frac{v}{4a}$, which Furusaki seems to have overlooked but which cannot be correct: since H is gapless, we know that Δ must be zero, as Furusaki points out himself earlier in his paper [after his Eq. (11)]. Thus, his argumentation subtly contradicts itself — the resolution is probably [37] that infinite-order perturbation theory in the irrelevant operator $\partial_x\Phi_+$ will produce an oscillatory factor $e^{i\Delta t}$ that exactly cancels the above $e^{-i\Delta t}$ (without affecting the asymptotic t^{-1} decay), but showing this explicitly seems like a rather non-trivial task.

To obtain (185), we first commuted $F^\dagger_{L/R}$ to the front (changing H_+ into $\bar{H}_+$), where it drops out via $F_{L/R}F^\dagger_{L/R} = 1$, and then performed the unitary transformation $U_+ = e^{i(\frac{\pi}{2}\hat{\mathcal{N}}^2 - \theta_B \hat{\mathcal{N}})}$ of Eq. (142) to change $H_+(\pm\lambda_B)$ to $H'_+(\pm\lambda_B)$ and $|G_B\rangle$ to $|G'_B\rangle$. To obtain (186), we used (181) to change $\bar{H}'_+$ back to H'_+. The resulting correlator, or its generalization to arbitrary λ, can be asymptotically evaluated in a way analogous to D_{V_λ} of (167):

$$\begin{aligned} D_{\alpha_d V_\lambda}(t) &\equiv a^{-\lambda^2} \langle \alpha_d(t) e^{i\lambda\Phi_+(t)} e^{-i\lambda\Phi_+(0)} \alpha_d(0)\rangle' && (187)\\ &= a^{-\lambda^2} \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} \frac{(i\lambda)^n(-i\lambda)^{n'}}{n!\,n'!} \langle \alpha_d(t)\, \Phi_+^n(t)\, \Phi_+^{n'}(0)\, \alpha_d(0)\rangle' && (188)\\ &= a^{-\lambda^2} \langle \alpha_d(t)\alpha_d(0)\rangle' \langle e^{i\lambda\Phi_+(t)} e^{-i\lambda\Phi_+(0)}\rangle' + \text{connected terms} && (189)\\ &= \frac{C_\lambda}{(it)}\left[1 + \mathcal{O}\left(\frac{1}{\Gamma t}\right)\right] \qquad \text{for} \quad \Gamma t \gg 1\,. && (190) \end{aligned}$$

First note that the factors of α_d in $D_{\alpha_d V_\lambda}(t)$ guarantee that it can not approach a non-zero constant for $\Gamma t \to \infty$; if it did, then $\langle \alpha_d e^{i\lambda\Phi_+}\rangle'$ would itself have to be non-zero, which it trivially is not (since $\langle\alpha_d\rangle' = 0$ and $[\alpha_d, \Phi_+] = 0$). That $D_{\alpha_d V_\lambda}(t) \sim t^{-1}$ follows from the observation that each correlator in (188), when expressed [via (160)] in terms of $\tilde{\alpha}$'s and β's and evaluated using Wick's theorem, contains *at least one* contraction of the kind $\Delta_L \sum_{\bar{k}} C_{\bar{k}} \langle \beta_{\bar{k}}(t)\beta^\dagger_{\bar{k}}(0)\rangle'$ or $\Delta_L \sum_\varepsilon C_\varepsilon \langle \tilde{\alpha}_\varepsilon(t)\tilde{\alpha}^\dagger_\varepsilon(0)\rangle'$, i.e. between fermions at times t and 0. This necessarily gives rise to a factor of at least t^{-1}, as illustrated by (155) for $D_\beta(t) \sim t^{-1}$ [where $C_{\bar{k}} = 1$] or (157) for $D_{\alpha_d}(t) \sim t^{-1}$ [where $C_\varepsilon = 4\Gamma/(\varepsilon^2 + (4\pi\Gamma)^2)$], or (164) for $D_{\Phi_+}(t) \sim t^{-2}$ [which features a product of two such contractions, with $C_{\bar{k}}C_\varepsilon = \varepsilon^2([\varepsilon^2 + (4\pi\Gamma)^2][\varepsilon + \varepsilon_{\bar{k}}]^2)^{-1}$]. (In general, the coefficients $C_{\bar{k}}$ and C_ε that occur in the Wick expansion are of the form occuring under the integral in (158), and hence cause a $t^{-(1+n)}$ decay, with $n \geq 0$, see Appendix J.3 for examples.)

In (189), we gathered in the first term all "disconnected" terms in which $\alpha_d(t)$ and $\alpha_d(0)$ are contracted only with *each other* (and not with any $\tilde{\alpha}$'s from Φ_+'s); that this contribution goes like $(it)^{-1}$ (as first pointed out by Fabrizio and Gogolin [23]) follows from (157) and (169). The remaining "connected" terms are all those in which $\alpha_d(t)$ and $\alpha_d(0)$ are contracted with some $\tilde{\alpha}$'s arising from the Φ_+'s; of these terms, the prefactors of those going like $\sim t^{-1}$ contribute to the constant C_λ in (190), though most contain more than one t-to-0 contractions and hence decay faster. In Appendix J.3 this is illustrated explicitly for several such connected terms. In Appendix J.2 we also check the result $D_{\alpha_d V_\lambda}(t) \sim t^{-1}$ explicitly for the case $\lambda = -1$, for which $D_{\alpha_d V_{-1}}(t)$ can be related to the correlator $D_\Psi(t) \equiv \langle \Psi_+(t)\Psi^\dagger_+(0)\rangle' \sim (2vit)^{-1}$, which we calculate exactly there.

From the result $D_{\alpha_d V_\lambda}(t) \sim t^{-1}$ we conclude from (186) that also $D_{LL/RR} \sim t^{-1}$.

The $D_{LR}(t) = D^*_{RL}(-t)$ contributions to (174) for $D_B(t)$ can be evaluated analogously:

$$\begin{aligned} D_{LR}(t) &\equiv a^{-1}\langle G_B| e^{iH_+t}\left(F_L e^{-\frac{i}{2}\Phi_+}\right) e^{-iH_+t}\left(F^\dagger_R e^{-\frac{i}{2}\Phi_+}\right)|G_B\rangle && (191)\\ &= a^{-1}\langle G'_B| e^{iH'_+t}\left(-F_+ i\sqrt{2}\alpha_d e^{i\theta_B} e^{-\frac{i}{2}\Phi_+}\right) e^{-i\bar{H}'_+t} e^{-\frac{i}{2}\Phi_+}|G'_B\rangle && (192)\\ &= -\sqrt{2/a}\, e^{i\theta_B}\langle e^{iH'_+t}\Psi_+ e^{\frac{i}{2}\Phi_+} e^{-iH'_+t} e^{-\frac{i}{2}\Phi_+} i\alpha_d\rangle' \,\sim (it)^{-1}. && (193) \end{aligned}$$

For (192), we first commuted $F^\dagger_R$ to the front (changing H_+ to $\bar{H}_+$), where it combines with F_L to give $F_L F^\dagger_R = -F_+$, and then made the unitary transformation U_+, which changes F_+ to $F_+ i\sqrt{2}\alpha_d e^{i\theta_B}$, see (142); for (193) we factorized $F_+ e^{-\frac{i}{2}\Phi_+}$ as $\sqrt{a}\Psi_+ e^{\frac{i}{2}\Phi_+}$, and commuted $i\alpha_d$ to the right using (181). The resulting correlator (193) decays as t^{-1} for the same reasons as $D_{\alpha_d V_\lambda}$ of (187); the leading term, $\langle \Psi_+(t)[\Phi(t) - \Phi_+(0)]\alpha_d(0)\rangle \sim t^{-1}$ is calculated explicitly in Appendix J.4.

Putting everything together, we conclude that for $g = \frac{1}{2}$, $D_B(t) \sim t^{-1}$ and hence $D_{phys}(t) = D_F(t)D_B(t) \sim t^{-2}$. Thus, the sought-after exponent in (171) is $\nu = 2$, hence (170) implies $\rho_{dos}(\omega) \sim \omega$.

10.D.3 Discussion of the controversy regarding $\rho_{dos}(\omega)$.

Our result $\nu = 2$ agrees with that of Fabrizio and Gogolin (FG) [23] and Furusaki [22], who found $\nu = \frac{1}{g}$ [21] for general g, but not with that of Oreg and Finkel'stein (OF) [21], who found $\nu = \frac{1}{2g}$, using an exact mapping to a Coulomb gas problem. In this mapping, the correlation function $D_B(t)$ of (174) is represented as $D_B(t) = Z_e(t) - Z_o(t)$, where Z_e and Z_o are two Coulomb gas partition functions whose asymptotic $t \to \infty$ behavior, $Z_{e,o}(t) \sim C_{e,o} + D_{e,o}/t^{\mu_{e,o}}$, needs to be determined ($C_{e,o}$, $D_{e,o}$ and $\mu_{e,o} > 0$ are constants). Using a seemingly rather natural mean-field approximation, OF concluded that $C_e > C_o$, implying that $Z_e(t) - Z_o(t)$ asymptotically approaches a *constant* $\sim (C_e - C_o) \neq 0$]. This disagrees, however, with our exact result that for $g = 1/2$, $D_B(t)$ asymptotically *decays* as $1/t$ [see (186), (190)]. Now, Fabrizio and Gogolin (FG) were the first to point out explicitly, in a Comment [23] on OF's work, that this decay stems from a correlator of Klein factors namely

$$\langle e^{iH_+ t} F_{L/R} e^{-iH_+ t} F^\dagger_{L/R} \rangle = \langle e^{iH'_+ t} e^{-i\bar{H}'_+ t} \rangle' = 2 \langle e^{iH'_+ t} \alpha_d e^{-iH'_+ t} \alpha_d \rangle' = 2 D_{\alpha_d}(t) \tag{194}$$

[by (183), (181)]. This correlator occurs in (189) (in which FG somewhat cavalierly ignored the connected terms), and is at the heart of the dispute — FG concluded that for general g it decays as $t^{-1/2g}$ (their argument is explained in Appendix J.1), in agreement with our $g = 1/2$ result (157), $D_{\alpha_d} \sim (it)^{-1}$. Hence FG concluded that "the neglect of Klein factors is the likely origin" of the missing $1/t^{\mu_{e,o}}$ in OF's result.

In our opinion, however, this conclusion of FG is somewhat premature and the matter is more subtle: OF *did* treat Klein factors correctly – they lead to the minus sign in $Z_e - Z_o$, as emphasized by OF in their Reply [24] to FG's Comment, and as we confirm explicitly in Appendix K. On the other hand, our exact $g = 1/2$ result that $D_B(t) = Z_e(t) - Z_o(t) \sim 1/t$ unambiguously implies that $C_e = C_o$, i.e. that for $g = 1/2$ the leading constants in fact *cancel* precisely. We must therefore conclude that a "sign problem", i.e. the cancellation of two large quantities, occurs in OF's Coulomb gas, and that *the mean field approximation which they used is not sufficiently accurate to correctly deal with this sign problem.* Since the latter arose *because of Klein factors*, Fabrizio and Gogolin evidently *were* correct in emphasizing their importance.

In replying to FG's Comment, OF rejected their result for D_{α_d}, arguing that the effect of density fluctuations, which, as OF correctly point out, are *not* suppressed by a backscattering impurity, were not properly taken into account by FG. We disagree with this criticism and believe that FG's calculation is sound. The technical details of FG's calculation and OF's objection to it are described in Appendix J.1.

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Appendices

A Relation between field-theoretical and constructive bosonization

We make explicit the connection between the constructive and field-theoretical approaches to bosonization, by showing how the operators used in the latter can be constructed in terms of the former. To be specific, we shall transcribe into the L/R-language of Section 10.A the excellent treatment of Shankar [15] (which we summarize here without giving details, since these are well explained in Ref. [15]), denoting his notation by the subscript $_{Sha}$.

A.1 Definition of boson fields

Shankar starts from a set of boson operators $\phi_{Sha}(p)$ satisfying $[\phi_{Sha}(p), \phi^\dagger_{Sha}(p')] = 2\pi\delta(p-p')$, where $p \in (-\infty, \infty)$ is a 1-D continuous momentum index. His Hamiltonian is

$$H \equiv \int_{-\infty}^{\infty} \tfrac{dp}{2\pi} |p| \phi^\dagger_{Sha}(t,p)\phi_{Sha}(t,p) = \tfrac{1}{2}\int_{-\infty}^{\infty} dx \left[(\partial_x \phi_{Sha}(t,x))^2 + \Pi^2_{Sha}(t,x) \right] , \tag{A1}$$

where the 1-D boson field $\phi_{Sha}(t,x)$ is defined for $x \in (-\infty, \infty)$ as

$$\phi_{Sha}(t,x) \equiv \int_{-\infty}^{\infty} \tfrac{dp}{2\pi\sqrt{2|p|}} \left[\phi_{Sha}(p) e^{ipx} + \phi^\dagger_{Sha}(p) e^{-ipx} \right] e^{-a|p|/2} \tag{A2}$$

($a > 0$ being an infinitessimal regularization parameter), and $\Pi_{Sha}(t,x) \equiv \partial_t \phi_{Sha}(t,x)$ is the canonically conjugate field. One can check that $[\phi_{Sha}(t,x), \Pi_{Sha}(t,x')] = i\delta(x-x')$ in the limit $a \to 0$. From these fields, the combinations

$$\phi_{\pm Sha}(t,x) \equiv \lim_{x_0 \to -\infty} \tfrac{1}{2} \left[\phi_{Sha}(t,x) \mp \int_{x_0}^{x} dx' \, \Pi_{Sha}(t,x') \right] \tag{A3}$$

are constructed, whose commutations relations can be checked to be

$$[\phi_{\pm Sha}(t,x), \phi_{\pm Sha}(t,x')] = \pm \tfrac{i}{4}\, \epsilon(x-x') , \quad \text{where} \quad \epsilon(x) \equiv \begin{cases} \pm 1 & \text{for} \quad x \gtrless 0 \\ 0 & \text{for} \quad x = 0 \end{cases} ; \tag{A4}$$

$$[\phi_{+Sha}(t,x), \phi_{-Sha}(t,x')] = \pm \tfrac{i}{4} . \tag{A5}$$

(The absence in Shankar's Eq. (3.12) of the factor $\pm i$ occuring in Eq. (A4) is a typo.) The fields $\phi_{\pm Sha}(t,x)$ can be checked to depend only on $(t \mp x)$ and hence are called R- and L-moving.

A.2 Bosonization postulate

The bosonization formula for R- and L-moving fermion fields is postulated to be

$$\psi_{\pm Sha}(t,x) = (2\pi a)^{-1/2} e^{\pm i\sqrt{4\pi}\phi_{\pm Sha}(t,x)} , \tag{A6}$$

and its correctness is verified by checking explicitly that the correlation functions and anti-commutators of $\psi_{\pm Sha(x)}$ are correctly reproduced.

Note that Eq. (A6) lacks Klein factors $F_{R/L}$ that lower the number of R- or L-moving electrons by one. Therefore (A6) does not have the status of an operator identity in Fock space, but has meaning only within expectation values containing precisely one $\psi^\dagger_{+Sha}$ (or $\psi^\dagger_{-Sha}$) for every ψ_{+Sha} (or ψ_{-Sha}), i.e. in which the product of all Klein factors would equal one anyway. Note furthermore that in the absence of (anti-commuting) Klein factors, which in the constructive approach guarantee that $\{\tilde\psi_R(x), \tilde\psi^\dagger_L(x')\} = 0$, "special tricks" are required to ensure that this relation is correctly reproduced. In the above construction, the trick is that ϕ_{+Sha} and ϕ_{-Sha} do *not* commute, so that $\{\psi_{+Sha}(x), \psi^\dagger_{-Sha}(x')\} = 0$ follows from Eqs. (A5) and (C6).

A.3 Relation between our notation and that of Shankar

Let us now transcribe the above field-theoretical approach into our notation, as used in Section 10.A. Because the lack of Klein factors in the former, such a transcription can never be completely one-to-one. Our aim is therefore merely to find the relation between Shankar's $\phi_{\pm Sha}(x)$ and our $\tilde\phi_{R/L}(x)$ fields. There is some freedom in choice of signs, etc., which we shall exploit to ensure that Shankar's bosonization formula Eq. (A6) for $\psi_{\pm Sha}$ is consistent with our Eq. (104) for $\tilde\psi_{R/L}$. To this end, we identify Shankar's $\phi_{Sha}(p)$-operators (defined for positive and negative p) with our two species of boson operators $b_{q,L/R}$ (defined only for positive q), as follows:

$$\phi_{Sha}(p) := L^{1/2}\left(\theta(-p) b_{|p|,L} - \theta(p) b_{p,R}\right) \tag{A7}$$

The above Hamiltonian (A1) then equals our H_{kin} of (107), in the limit $L \to \infty$. The $\phi_{\pm Sha}(x)$ of Eq. (A3) can be expressed in terms of the $\tilde\phi_{R/L}(x)$ fields defined in Eq. (103) by using Eq. (A7) to translate $\phi_{Sha}(p)$'s into $b_{|p|L/R}$'s:

$$\phi_{\pm Sha}(t,x) \;:=\; \mp \tfrac{1}{2\sqrt{\pi}} \left(\tilde\phi_{R/L}(t,x) - \tilde\phi_0(t)\right) , \tag{A8}$$

$$\tilde\phi_0(t) \;=\; \tfrac{1}{2}\left[\tilde\phi_L(t,x_0) + \tilde\phi_R(t,x_0)\right] , \tag{A9}$$

$$\psi_{\pm Sha}(t,x) \;=\; (2\pi a)^{-1/2} e^{-i(\tilde\phi_{R/L}(t,x) - \tilde\phi_0(t))} := (2\pi)^{-1/2}\tilde\psi_{R/L}(t,x)] , \tag{A10}$$

$$:\psi^\dagger_{\pm Sha}(t,x)\psi_{\pm Sha}(t,x): \;=\; \tfrac{1}{\sqrt{\pi}}\partial_x\tilde\phi_{\pm Sha}(t,x) := \mp\tfrac{1}{2\pi}\partial_x\tilde\phi_{R/L}(t,x) = \tfrac{1}{2\pi} :\psi^\dagger_{R/L}(t,x)\psi_{R/L}(t,x):$$

Eqs. (A8) and (A9) (in the limit $x_0 \to -\infty$) allow one to readily reproduce the commutators of Eqs. (A4) and (A5), and in doing so to pin-point the reason why the latter is non-zero: it is the presence of the "zero-mode term" $\tilde\phi_0(t)$ in Eq. (A8) [which corresponds to the terms that Shankar calls "oscillating pieces at spatial infinity" after his Eq. (3.11)]. Thus the ability of Eq. (A6) to represent two different species of *anti-commuting* fermion fields $\psi_{\pm Sha}$ purely in terms of bosonic operators and without using Klein factors comes at a price: in Eq. (A8) one effectively adds to the $\tilde\phi_{R/L}$, which are built purely from operators of the *same* R/L species ($b_{|p|,R/L}$, $b^\dagger_{|p|,R/L}$), a zero mode term $\tilde\phi_0$, which contains a *mixture* of operators of *opposite* species (both $b_{|p|,R/L}$, $b^\dagger_{|p|,R/L}$ *and* $b_{|p|,L/R}$, $b^\dagger_{|p|,L/R}$). This price is rather high from a conceptual point of view, because as soon as $\psi_{\pm Sha}$ contains such mixtures, it no longer makes sense to try to construct, as in Eq. (16), the $b_{qR/L}$ in terms of the "original" fermion operators $c_{kR/L}$ in terms of which the fermion fields were "originally" defined: each $\psi_{\mp Sha}$ would then depend on both c_{kL} and c_{kR}, which clearly does not make sense. Thus, in the above way of formulating the field-theoretic approach the $\phi_{Sha}(p)$ are introduced for purely formal reasons, and the very concrete relation between boson and fermion operators $b_{qR/L}$ and $c_{kR/L}$ that is the hallmark of the constructive approach to bosonization is lost. Nevertheless, the field-theoretical approach is perfectly well-defined and produces correct answers if used with sufficient care (though this is sometimes easier said than done, and several consequential mistakes have been made in the past (Ref. [27] discusses an example).

B Completeness of boson representation

We prove, following Haldane [4], the following theorem for a single species of 1-D fermions (i.e. $M = 1$, hence the fixed index η is not shown explicitly below):

Theorem: *The Fock space (F_c) spanned by arbitrary combinations of the fermion operators c_k and $c_k^\dagger$'s acting on the "vacuum state" $|N = 0\rangle_0$, is identical to the Fock space (F_b) spanned by arbitrary combinations of the bosonic operators $b_q^\dagger$'s acting on the set of all N-particle ground states $\{|N\rangle_0, N \in \mathbb{Z}\}$.*

This is equivalent to proving Eq. (22): $F_c = F_b$ implies that any $|N\rangle \in F_c$ can be written as $|N\rangle = \sum_{N' \in \mathbb{Z}} f_{N'}(b_q^\dagger)|N'\rangle_0$, where $f_{N'}(b_q^\dagger)$ is some function, labelled by N', of $b_q^\dagger$'s; but since $[b_q^\dagger, \widehat{N}] = 0$, only the $N' = N$ term can be non-zero. Since the generalization to several species is trivial (all operators with $\eta \neq \eta'$ commute), Eq. (22) immediatly follows.

Proof: It is evident that F_b is a subspace of F_c, since the $b^\dagger$'s are functions of the $c^\dagger$ and c's. To prove that in fact $F_b = F_c$, we are thus confronted by the state-counting problem of showing that every state in F_c also occurs in F_b. This can be done elegantly by calculating the corresponding grand-canonical partition functions: since both are sums over positive definite quantities, one would find $F_c > F_b$ if F_c had more states than F_b; conversely, if one found $Z_c = Z_b$, this would imply $F_c = F_b$. Since this argument is independent of the form of the Hamiltonian, we are free to choose H such that the calculation of partition functions becomes tractable. To this end, we choose the linear dispersion $\varepsilon_k = k$ of Eq. (65), with $\delta_b = 1$, so that

$$H_0 \equiv \tfrac{2\pi}{L} \sum_k (n_k - \tfrac{1}{2}) \,{}^*_* c_k^\dagger c_k {}^*_* \,. \tag{B1}$$

The calculation of Z_c is an elementary text-book excercise: Summing over all $n_k \in \mathbb{Z}$, with the corresponding fermion state $c_k^\dagger|N = 0\rangle$ either empty or occupied, yields

$$\begin{aligned} Z_c &= \mathrm{Tr}[e^{-\beta(H_0 - \mu\widehat{N})}] & \text{(B2)}\\ &= \prod_{n_k=1}^{\infty} (1 + e^{-\beta 2\pi/L(n_k - 1/2)} e^{\beta\mu n_k}) \prod_{n_k=-\infty}^{-1} (1 + e^{-\beta 2\pi/L|n_k - 1/2|} e^{\beta\mu n_k}) & \text{(B3)}\\ &= \prod_{n=1}^{\infty} (1 + w^{2n-1} v^{n})(1 + w^{2n-1} v^{-n}) \qquad \text{where} \quad w = e^{-\beta\pi/L} \quad \text{and} \quad v = e^{\beta\mu}. & \text{(B4)} \end{aligned}$$

Next we calculate Z_b: We note that F_b is spanned by the set of all states of the form

$$|N; \{m_q\}\rangle = \prod_{q>0}^{\infty} \frac{b_q^{\dagger m_q}}{(m_q!)^{1/2}} |N\rangle_0 \tag{B5}$$

where $N \in \mathbb{Z}$, and for each $q = \frac{2\pi}{L} n_q > 0$ (with $n_q \in \mathbb{Z}^+$) the $m_q \geq 0$ are integers specifying how many bosonic quanta of momentum q have been excited. By Eqs. (67) and (68), each $|N; \{m_q\}\rangle$ is an eigenstate of H_0, with eigenvalue $\frac{2\pi}{L}\frac{1}{2}N^2 + \sum_{q>0} q\, m_q$. Therefore

$$\begin{aligned} Z_b &= \sum_{N=-\infty}^{\infty} \sum_{\{m_q\}} \langle N; \{m_q\}| e^{-\beta(H_0 - \mu\widehat{N})} |N; \{m_q\}\rangle & \text{(B6)}\\ &= \sum_{N=-\infty}^{\infty} \sum_{\{m_q\}} e^{-\beta 2\pi/L(N^2/2 + \sum_{q>0} n_q m_q)} e^{\mu\beta N} & \text{(B7)} \end{aligned}$$

$$= \left(\sum_{N=-\infty}^{\infty} w^{N^2} v^N \right) \left(\sum_{M=0}^{\infty} P(M) w^{2M} \right) = \frac{\sum_{N=-\infty}^{\infty} w^{N^2} v^N}{\prod_{n=1}^{\infty}(1-w^{2n})} = Z_c \; . \tag{B8}$$

In the first of (B8), we denoted by $P(M)$ the number of states $|N; \{m_q\}\rangle$ (for fixed N) satisfying $\sum_{n_q=1}^{\infty} n_q m_q = M$, and, since this number is independent of N, factorized the sum. For the second of (B8), we noted that $P(M)$ is just the number of partitions of M, and hence (by simple combinatorics) also occurs in the series expansion of the function

$$\frac{1}{\prod_{n=1}^{\infty}(1-y^n)} = \prod_{n=1}^{\infty} \Big[\sum_{m=0}^{\infty} (y^n)^m \Big] = \sum_{M=0}^{\infty} P(M) y^M \; . \tag{B9}$$

Finally, the last of (B8) follows by inserting Jacobi' triple product identity,[17]

$$\sum_{N=-\infty}^{\infty} w^{N^2} v^N = \prod_{n=1}^{\infty} (1 + w^{2n-1} v)(1 + w^{2n-1} v^{-1})(1 - w^{2n}) \; , \tag{B10}$$

and comparing with Eq. (B4) for Z_c. The rather remarkable result $Z_b = Z_c$ immediately implies Eq. (22), as argued above, which completes the proof.

Incidentally, Eq. (22) can also proven more formally: Since Eq. (B2) has the form $Z_c = \sum_{N \in \mathbb{Z}} Z_{cN} v^N$, where Z_{cN} is the *canonical* N-particle partition function, and since $Z_c = Z_b$, Eq. (B7) shows that

$$Z_{cN} = \sum_{\{m_q\}} \langle N; \{m_q\}| e^{-\beta H_0} |N; \{m_q\}\rangle \; . \tag{B11}$$

This proves that the bosonic representation of fermion states is complete also within any of the *fixed-N* Hilbert spaces H_N.

C Useful identities

We derive the operator identities needed for bosonization, all of which are standard [39, 3].

Below, A and B are operators; we define an operator-valued function $f(A)$ through its Taylor expansion, i.e. $f(A) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) A^n$.

Theorem 1 (Baker-Hausdorff): Define $[A,B]_{n+1} \equiv [[A,B]_n, B]$ and $[A,B]_0 \equiv A$. Then

$$e^{-B} A e^{B} \;=\; \sum_{n=0}^{\infty} \frac{1}{n!} [A,B]_n \;=\; A + [A,B] + \tfrac{1}{2!}[[A,B],B] + \ldots \; . \tag{C1}$$

Proof: Consider the operator-valued function $\mathcal{A}(s) \equiv e^{-sB} A \, e^{sB}$, where $s \in \mathbb{R}$. Since $\frac{d^n \mathcal{A}(s)}{ds^n} = e^{-sB}[A,B]_n \, e^{sB}$, as can be shown inductively, the Taylor series about $s = 0$ gives

$$A(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \left(\frac{d^n \mathcal{A}(s)}{ds^n} \right)_{s=0} = \sum_{n=0}^{\infty} \frac{s^n}{n!} [A,B]_n \; . \tag{C2}$$

[17] Jacobi's triple product identity can be proven using both the series and product representations of the elliptic theta function $\theta(v;w)$: equating Gradshteyn and Ryzhik's [38] Eqs. (8.192.3) and (8.181.2) for $\theta(-iv/2;w)$ yields (B10).

Taking $s = 1$ in Eq. (C2) yields the Baker-Hausdorff theorem (C1). □

Theorem 2: If $C \equiv [A, B]$ satisfies $[A, C] = [B, C] = 0$, then

$$\begin{aligned}
&\text{(i)} & e^{-B} A e^{B} &= A + C \quad \text{or} \quad [A, e^{B}] = C e^{B}\ ; & \text{(C3)}\\
&\text{(ii)} & e^{A} e^{B} &= e^{A+B} e^{C/2} = e^{A+B+C/2}\ ; & \text{(C4)}\\
&\text{(iii)} & e^{-B} f(A) e^{B} &= f(A + C)\ ; & \text{(C5)}\\
&\text{(iv)} & e^{A} e^{B} &= e^{B} e^{A} e^{C}\ . & \text{(C6)}
\end{aligned}$$

Proof: (i) Inserting the stated condition into the Baker-Hausdorff theorem (C1) yields (C3).
(ii) The operator-valued function $\mathcal{T}(s) \equiv e^{sA} e^{sB}$ $(s \in \mathbb{R})$ satisfies the differential equation

$$\tfrac{d\mathcal{T}(s)}{ds} = e^{sA} A e^{sB} + e^{sA} e^{sB} B = \mathcal{T}(s)(A + sC + B) \tag{C7}$$

[the second equality follows from (C3)], with boundary condition $\mathcal{T}(0) = 1$. Since $[A + B, C] = 0$, its solution by inspection is $\mathcal{T}(s) = e^{s(A+B)} e^{s^2 C/2}$. Thus $\mathcal{T}(1)$ yields (C4).
(iii) By induction, (C3) implies $e^{-B} A^n e^{B} = (A + C)^n$, which yields (C5) when inserted into the Taylor expansion for $f(A)$. (iv) is a special cases of (iii), with $f(A) = e^{A}$. □

Theorem 3: If $[A, B] = DB$ and $[A, D] = [B, D] = 0$, then $f(A)B = Bf(A + D)$. (C8)

Proof: Use $AB = B(A + D)$ to show inductively that $A^n B = B(A + D)^n$. This yields (C8) when inserted into the Taylor expansion for $f(A)$. □ — Using (C8), one readily finds:

$$e^{A} B = B e^{A+D}\ , \qquad e^{A} B^n = B^n e^{A+nD} = (B e^{D})^n e^{A}\ , \qquad e^{A} e^{B} = e^{B e^{D}} e^{A}\ . \tag{C9}$$

Theorem 4 (free bosons): For a free boson Hamilton $H = \sum_j \hbar\omega_j (b_j^\dagger b_j + \frac{1}{2})$ with $[b_j, b_{j'}^\dagger] = \delta_{jj'}$, the thermal average of $e^{\hat{B}}$, where $\hat{B} = \sum_j (\lambda_j b_j^\dagger + \tilde{\lambda}_j b_j)$ is linear in bosons, is

$$\langle e^{\hat{B}} \rangle = e^{\frac{1}{2}\langle \hat{B}^2 \rangle}, \qquad \text{where} \quad \langle \hat{O} \rangle = \mathrm{Tr}\left(e^{-\beta H} \hat{O} \right) / \mathrm{Tr} e^{-\beta H}\ . \tag{C10}$$

Proof [3] (see also (J6) for a simpler proof): Since the bosons are independent, the thermal averages separate into independent parts, $\langle e^{\hat{B}} \rangle = \prod_j C_j$ and $\langle \hat{B}^2 \rangle / 2 = \sum_j \lambda_j \tilde{\lambda}_j \langle b_j^\dagger b_j + 1/2 \rangle$, thus it suffices to show that

$$C_j \equiv \langle e^{\lambda_j b_j^\dagger + \tilde{\lambda}_j b_j} \rangle = e^{\lambda_j \tilde{\lambda}_j \langle b_j^\dagger b_j + 1/2 \rangle}\ . \tag{C11}$$

We denote the j-th mode partition function by Z_j and drop the index j henceforth. Then

$$\begin{aligned}
Z &= \sum_{m=0}^{\infty} \langle m | e^{-\beta\hbar\omega(b^\dagger b + 1/2)} | m \rangle = \sum_{m=0}^{\infty} x^{m+1/2} = \frac{x^{1/2}}{1 - x}, \quad \text{where } x \equiv e^{-\beta\hbar\omega}; & \text{(C12)}\\
\langle b^\dagger b \rangle &= Z^{-1} \sum_{m=0}^{\infty} \langle m | e^{-\beta\hbar\omega(b^\dagger b + 1/2)} b^\dagger b | m \rangle = Z^{-1} \sum_{m=0}^{\infty} x^{m+1/2} m = (x^{-1} - 1)^{-1}\ ; & \text{(C13)}\\
C &= Z^{-1} \sum_{m=0}^{\infty} \langle m | e^{-\beta\hbar\omega(b^\dagger b + 1/2)} e^{\lambda\tilde{\lambda}/2} e^{\lambda b^\dagger} e^{\tilde{\lambda} b} | m \rangle & \text{(C14)}\\
&= Z^{-1} e^{\lambda\tilde{\lambda}/2} \sum_{m=0}^{\infty} x^{m+1/2} \sum_{r=0}^{m} \frac{\lambda^r \tilde{\lambda}^r}{(r!)^2} \frac{m!}{(m-r)!} & \text{(C15)}
\end{aligned}$$

$$= Z^{-1} e^{\lambda\tilde{\lambda}/2} \sum_{r=0}^{\infty} \frac{(\lambda\tilde{\lambda})^r}{r!} S_r \; , \tag{C16}$$

$$\text{where} \quad S_r \;\equiv\; \sum_{m=r}^{\infty} x^{m+1/2} \binom{m}{r} = x^{r+1/2} \sum_{\bar{m}=0}^{\infty} x^{\bar{m}} \binom{r+\bar{m}}{r} = \frac{x^{r+1/2}}{(1-x)^{r+1}} = Z \langle b^\dagger b \rangle^r . \tag{C17}$$

(C12) and (C13) are standard. In (C14) we normal-ordered the $e^{\lambda_j b_j^\dagger + \tilde{\lambda}_j b_j}$ of (C11) using (C4). For (C15) we expanded the last two exponentials of (C14) and evaluated the matrix elements $\langle m | b^{\dagger r} b^{r'} | m \rangle$ using $b^r |m\rangle = \sqrt{m}\, b^{r-1} |m-1\rangle = [m(m-1)\dots(m-r+1)]^{1/2} |m-r\rangle$ for $r \ge m$, and 0 for $r < m$. For (C16) we reordered the double sum, using $\sum_{m=0}^{\infty} \sum_{r=0}^{m} = \sum_{r=0}^{\infty} \sum_{m=r}^{\infty}$. To evaluate the sum S_r defined in (C17), we wrote $\bar{m} = m - r$ in the second equality, for the third evaluated the sum $\sum_{\bar{m}=0}^{\infty}$ using the identity $\binom{r+\bar{m}}{r} = (-1)^{\bar{m}} \binom{-r-1}{\bar{m}}$ and the binomial theorem, and for the fourth simplified using (C12) and (C13). Inserting the last of (C17) into (C16) yields (C11). □

D More on Klein factors

We explain why Klein factors are often ignored, discuss the pitfalls behind the notation $F_\eta = e^{-i\widehat{\theta}_\eta}$, and give an explicit construction of F_η in terms of $c_{k\eta}^\dagger$ and $c_{k\eta}$ operators.

D.1 Why Klein factors are often ignored

In very many papers, the Klein factors are tacitly assumed but not written explicitly, or simply ignored. This is usually OK if one calculates correlation functions of the form

$$G = \langle \psi_1 \psi_2 \dots \psi_n \psi_n^\dagger \psi_{n-1}^\dagger \dots \psi_1^\dagger \rangle \tag{D1}$$

for a Hamiltonian that conserves each separate $\widehat{N}_\eta$, because such functions are only non-zero if they contain an *equal* number of $\psi_\eta^\dagger$ and ψ_η and thus of $F_\eta^\dagger$ and F_η, so that the latter can be commuted past each other and all bosonic operators and combined to give 1. Of course, one has to be careful to keep track of the minus signs that arise in this procedure, but various authors have their own way of doing this (some of which are discussed in the next subsection, e.g. using a set of anti-commuting Majorana fermions, or adding so-called "zero modes" with appropriate properties to the boson fields). Moreover, for free bosons the bosonic fields themselves see to it that only correlation functions containing an *equal* number of $\psi_\eta \propto e^{-i\phi_\eta}$ and $\psi_\eta^\dagger \propto e^{i\phi_\eta}$ are non-zero, because invariance of the bosonic Hamiltonian (70) under $\phi_\eta(x) \to \phi_\eta(x) + const$ implies that $\langle e^{i(\lambda_1 \phi_\eta(x_1) + \dots + \lambda_n \phi_\eta(x_n))} \rangle = 0$ unless $\sum_n \lambda_n = 0$ [see Eq. (86) for details, and Appendix K for an example].

However, if the Hamiltonian does *not* conserve each separate $\widehat{N}_\eta$, the above arguments are no longer applicable. Neglecting Klein factors is then very dangerous and has been known to lead to mistakes, as discussed, for example, in Refs. [27].

D.2 The Notation $F_\eta^\dagger = e^{i\hat{\theta}_\eta}$

In the literature, the Klein factors are often written as $F_\eta^\dagger \equiv e^{i\widehat{\theta}_\eta}$ and $F_\eta \equiv e^{-i\widehat{\theta}_\eta}$, complemented by the *mnemonical* relation

$$[\widehat{N}_\eta, i\widehat{\theta}_{\eta'}] \equiv \delta_{\eta\eta'} \; , \tag{D2}$$

which suggests that the "phase operator" $\widehat{\theta}$ is conjugate to $\widehat{N}$. (For an explicit construction of $\hat{\theta}$ and an enlightening discussion thereof, see Eq. (B.20) of Ref. [20], where it is denoted by $\hat{k}$) The motivation for this notation is that these relations can be used to "derive" the crucial relation $[\widehat{N}_\eta, F^\dagger_\eta] = \delta_{\eta\eta'} F^\dagger_\eta$ [see Eq. (32)] using identity (C3), which states that $[A, e^B] = Ce^B$ if $C = [A, B]$ commutes with A and B.

Furthermore, to ensure that the F_η's anti-commute[18] for different η's [Eqs. (30) and (31)] one takes, for example [*again as mnemonic only,* to be used with (C6)]

$$[\widehat{\theta}_\eta, \widehat{\theta}_{\eta'}] \equiv \begin{cases} i\pi & \text{if} \quad \eta > \eta' \\ -i\pi & \text{if} \quad \eta < \eta' \\ 0 & \text{if} \quad \eta = \eta' \end{cases} , \qquad \text{thus} \qquad \{e^{i\widehat{\theta}_\eta}, e^{\pm i\widehat{\theta}_{\eta'}}\} = 2\delta_{\eta\eta'} e^{i(\widehat{\theta}_\eta \pm \widehat{\theta}_{\eta'})} . \tag{D3}$$

However, the reader should be warned that Eq. (D2) merely has the status of a *mnemonic*, to be used *only* in conjunction with $[A, e^B] = ce^B$ in order to evaluate $[\widehat{N}_\eta, F^\dagger_\eta]$, as described above. The reason for this warning is the following: If Eq. (D2) is viewed as an operator identity and $\widehat{N}_\eta$ is treated as an Hermitian operator, a contradiction arises (which unfortunately is not always appreciated, although this is discussed at length by Carruthers and Nieto in Ref. [40]). To see this, take the diagonal expectation value of Eq. (D2): on the one hand, Eq. (D2) gives $\langle N_\eta | [\widehat{N}_\eta, i\widehat{\theta}_\eta] | N_\eta \rangle = \langle N_\eta | 1 | N_\eta \rangle = 1$, and on the other

$$\langle N_\eta | \widehat{N}_\eta \, i\widehat{\theta}_\eta - i\widehat{\theta}_\eta \widehat{N}_\eta | N_\eta \rangle = (N_\eta - N_\eta) \langle N_\eta | i\widehat{\theta}_\eta | N_\eta \rangle = 0. \tag{D4}$$

To understand the origin of this contradiction, recall the following general result from quantum mechanics: If $\hat{X}$ and $\hat{Y}$ are conjugate operators (i.e. $[\hat{X}, i\hat{Y}] = 1$) and the spectrum of $\hat{X}$ are the discrete integers, then $\hat{X}$ is Hermitian only in the space of states "periodic in $\hat{Y}$", i.e. obtained by acting on a reference state (say $|0\rangle$) with *periodic* functions of $\hat{Y}$, i.e. functions depending *only* on the exponentials $e^{\pm i\hat{Y}}$. In the present case where $\hat{X} = \widehat{N}_\eta$ and $\hat{Y} = \widehat{\theta}_\eta$, the above contradiction thus arose since in Eq. (D4) in fact $\langle N_\eta | \widehat{N}_\eta \widehat{\theta}_\eta \neq N_\eta \langle N_\eta | \widehat{\theta}_\eta$. (A formal way of keeping track of the non-Hermiticity of $\widehat{N}_\eta$ is discussed in the appendix of Ref. [41].)

To avoid contradictions, $\hat{\theta}$ should be defined not through Eq. (D2), but by writing

$$[\widehat{N}_\eta, e^{\pm i\widehat{\theta}_{\eta'}}] = \pm \delta_{\eta\eta'} e^{\pm i\widehat{\theta}_{\eta'}} , \tag{D5}$$

[which is just Eq. (32)], which evidently identifies the exponentials $e^{\pm i\widehat{\theta}_\eta}$ as raising and lowering operators. Since these *do not* have any diagonal matrix elements, no contradiction occurs: one the one hand $\langle N_\eta | [\widehat{N}_\eta, e^{\pm i\widehat{\theta}_\eta}] | N_\eta \rangle = (N_\eta - N_\eta) \langle N_\eta | e^{\pm i\widehat{\theta}_\eta} | N_\eta \rangle = 0$, and on the other, using Eq. (D5), $\langle N_\eta | \pm e^{\pm i\theta_\eta} | N_\eta \rangle = \pm \langle N_\eta | N_\eta \pm 1 \rangle = 0$.

Note that the above discussion also implies that all non-integer powers of raising or lowering operators, i.e. expressions of the form $\left(e^{\pm i\widehat{\theta}_\eta}\right)^g$ with $g \notin \mathbb{Z}$, are ill-defined, since they would be plagued by the same kind of inconsistencies as Eq. (D2). Unfortunately, many authors unwittingly do use such objects: it is quite common to "absorb" $\widehat{\theta}_\eta$ (often called a "zero mode" in this context) into the boson field $\Phi_\eta(s)$ of (64) by writing $\tilde{\Phi}_\eta(x) \equiv \widehat{\theta}_\eta + \Phi_\eta(x)$, and to write the bosonization relation (64) simply as $\psi_\eta(x) = a^{-1/2} e^{-i\tilde{\Phi}_\eta(x)}$. This procedure is formally perfectly valid. However, it is also quite common to subsequently use expressions like $e^{-ig\tilde{\Phi}_\eta(x)}$ with $g \notin \mathbb{Z}$, which arise, e.g., when making a linear transformation of the form $\tilde{\Phi}'_\eta(x) = A_{\eta\eta'} \tilde{\Phi}_\eta(x)$ [as one does for the Kondo problem[17, 18] or a Tomonaga-Luttinger, see Section 10.C, Eq. (117)]. *Strictly speaking, this procedure is formally meaningless, since the factors* $e^{\pm ig\widehat{\theta}_\eta}$ *contained in such expressions are ill-defined.*

[18] Some authors (e.g. Fabrizio and Gogolin [35]) instead ensure this by writing $F_\eta = \widehat{\chi}_\eta e^{-i\widehat{\theta}_\eta}$ with $[\widehat{\theta}_\eta, \widehat{\theta}_{\eta'}] = 0$, where the $\widehat{\chi}_\eta$ are a set of Majorana fermions satisfying $\{\widehat{\chi}_\eta, \widehat{\chi}_{\eta'}\} = 2\delta_{\eta\eta'}$.

D.3 Fermionic representation of $F^\dagger_\eta$

In this section we give an explicit construction for $F^\dagger_\eta$ in terms of nothing but fermionic $c^\dagger_{k\eta}$ and $c_{k\eta}$'s, and check explicitly that it satisfies the defining properties Eqs. (24) to (26). Though in principle such a construction is not necessary, since Eqs. (24) to (26) fully define $F^\dagger_\eta$, its existence serves as a useful consistency check for the formalism.

The construction is in fact almost trivial: by inverting Eq. (62), $F^\dagger_\eta$ is immediately expressed in terms of ψ_η and ϕ_η, both of which are functions only of $c^\dagger_{k\eta}$'s:

$$F^\dagger_\eta = \left(\tfrac{L}{2\pi}\right)^{1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\varphi_\eta(x)} e^{-i\varphi^\dagger_\eta(x)} \psi^\dagger_\eta(x) \,. \tag{D6}$$

Upon inserting Eq. (3) for $\psi^\dagger_\eta(x)$ and Eqs. (33) and (33) for $\varphi^\dagger_\eta(x)$ and $\varphi_\eta(x)$, this equation expresses $F^\dagger_\eta$ purely in terms of electron operators. This expression seems to be x-dependent, but is not, since all its matrix-elements between (x-independent) states are x-independent, as we shall see below.

Eq. (D6) can be used to check whether $F^\dagger_\eta$ does have all required properties (see Section 4.F): E.g., to check that $[F^\dagger, b] = 0$, note that $b_{q\eta}$ commutes with all operators in Eq. (D6) except $\varphi^\dagger_{\eta'}(x)$ and $\psi^\dagger_{\eta'}(x)$, but that the two contributions from

$$[b_{q\eta}, e^{-i\varphi^\dagger_{\eta'}(x)}] = \delta_{\eta\eta'}\alpha_q(x) e^{-i\varphi^\dagger_{\eta'}(x)} \tag{D7}$$

$$[b_{q\eta}, \psi^\dagger_{\eta'}(x)] = -\delta_{\eta\eta'}\alpha_q(x)\psi^\dagger_{\eta'}(x) \tag{D8}$$

exactly cancel (an observation due to Haldane [4]). The other commutators of Eq. (24) can be similarly verified.

Next, one has to verify that

$$_0\langle \vec{N}'|\widehat{T}_\eta F^\dagger_\eta|\vec{N}\rangle_0 = \delta_{N_1',N_1}\ldots\delta_{N_\eta',N_\eta+1}\ldots\delta_{N_M',N_M}\,, \tag{D9}$$

where $\widehat{T}_\eta$ is the phase-counting operator of Eq. (27). Insert Eq. (D6) into the left-hand side of Eq. (D9), and use the identities

$$e^{-i\varphi_\eta(x)} e^{-i\varphi^\dagger_\eta(x)} = e^{-i\varphi^\dagger_\eta(x)} e^{-i\varphi_\eta(x)} e^{-[\sum_{q>0}\frac{1}{n_q}e^{-qa}]} \quad \text{[using Eq. (C6) and (39)]} \tag{D10}$$

$$e^{-i\varphi_\eta(x)}\psi^\dagger_\eta(x) = \psi^\dagger_\eta(x) e^{-i\varphi_\eta(x)} e^{[\sum_{q>0}\frac{1}{n_q}e^{-qa}]} \quad \text{[using Eq. (C9)]} \tag{D11}$$

to commute $e^{-i\varphi^\dagger_\eta(x)}$ to the very left and $e^{-i\varphi_\eta(x)}$ to the very right, where they are equal to unity when acting on $_0\langle\vec{N}'|$ and $|\vec{N}\rangle_0$ respectively. Since the two c-number exponentials thus produced cancel ($e^{(-1+1)[\sum_{q>0}\frac{1}{n_q}e^{-qa}]} = 1$), we get

$$_0\langle \vec{N}'|\widehat{T}_\eta F^\dagger_\eta|\vec{N}\rangle_0 = {}_0\langle \vec{N}'|\widehat{T}_\eta \left(\tfrac{L}{2\pi}\right)^{1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta-\frac{1}{2}\delta_b)x} e^{-i\varphi_\eta(x)} e^{-i\varphi^\dagger_\eta(x)}\psi^\dagger_\eta(x)|\vec{N}\rangle_0 \tag{D12}$$

$$= {}_0\langle \vec{N}'|\widehat{T}_\eta \left(\tfrac{L}{2\pi}\right)^{1/2} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta-\frac{1}{2}\delta_b)x} \left[\left(\tfrac{2\pi}{L}\right)^{1/2}\sum_k e^{ikx}c^\dagger_{k\eta}\right]|\vec{N}\rangle_0\,, \tag{D13}$$

where we have inserted Eq. (3) for $\psi^\dagger_\eta(x)$. Now the argument is just like that given in the determination of $\widehat{\lambda}_\eta$ in Section 7: Commuting $c^\dagger_{k\eta}$ past $|N_1\rangle\otimes\ldots\otimes|N_{\eta-1}\rangle$ produces a factor T_η which exactly cancels the phase contributed by $\widehat{T}_\eta$. Since neither $_0\langle\vec{N}'|$ nor $|\vec{N}\rangle_0$ contain any particle-hole-excitations, Eq. (D13) is non-zero only if $c^\dagger_{k\eta}$ adds the $(N_\eta+1)$-th particle [which has momentum $k = \frac{2\pi}{L}(N_\eta + 1 - \frac{1}{2}\delta_b)$] to $|N_\eta\rangle$ to produce $|N_\eta + 1\rangle$, and if at the same time $N'_\eta = N_\eta + 1$. Hence all c-number-exponentials cancel, showing that indeed Eq. (D13) is equal to Eq. (D9).

Finally, it immediately follows from $[F^\dagger, b] = 0$ that $F_\eta^\dagger$ creates no particle-hole excitations, i.e. that

$$
{}_0\langle \vec{N}' | f(\{b_{q\bar\eta}\}) F_\eta^\dagger | \vec{N}\rangle_0 = 0 \qquad \text{for all } f(\{b_{q\bar\eta}\}),\ \vec{N} \text{ and } \vec{N}'. \tag{D14}
$$

Thus, we conclude that all matrix elements (between x-independent states) of $F_\eta^\dagger$ are indeed x-independent, as stated earlier. Hence the x-independence of $F_\eta^\dagger$ in Eq. (D6) can be made explicit by either setting $x = 0$, or by including a dummy integration $L^{-1}\int_{-L/2}^{L/2} dx$ (we also unnormal-ordering the exponentials):

$$
F_\eta^\dagger = a^{1/2} e^{-i\phi_\eta(0)} \psi_\eta^\dagger(0) = \frac{a^{1/2}}{L} \int_{-L/2}^{L/2} dx\, e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)x} e^{-i\phi_\eta(x)} \psi_\eta^\dagger(x)\ . \tag{D15}
$$

E Remarkable cancellations involving bosonization

To gain intuition into the remarkable way in which the bosonization identity works, we compare the expansions of $\psi(x)|0\rangle_0$ and $a^{-1/2}Fe^{-i\phi(x)}|0\rangle_0$.

Consider the state $\psi(x)|0\rangle_0$ for a single species of fermions (i.e. $M = 1$, and we drop the index η). As illustrated in Fig. 2, we can obtain two equivalent representations for this state by either Fourier-expanding ψ using (3), or bosonizing it using (62):

$$
\left(\tfrac{2\pi}{L}\right)^{1/2} \sum_{n\in\mathbb{Z}} e^{-i(n-\frac{1}{2}\delta_b)2\pi x/L} c_n |0\rangle_0 = F_\eta \left(\tfrac{2\pi}{L}\right)^{1/2} e^{-i(\widehat{N}_\eta - \frac{1}{2}\delta_b)2\pi x/L} e^{-i\varphi_\eta^\dagger(x)} e^{-i\varphi_\eta(x)} |0\rangle_0 \tag{E1}
$$

$$
\sum_{n=0}^{\infty} y^n c_{-n}|0\rangle_0 \quad = e^{-\left(\sum_{n=1}^{\infty}\frac{1}{n} y^n \rho_n\right)} c_0 |0\rangle_0\ , \quad \text{where} \quad y \equiv e^{i2\pi x/L}, \quad \rho_n \equiv \sum_{\bar n\in\mathbb{Z}} c^\dagger_{\bar n + n} c_{\bar n}, \tag{E2}
$$

$$
= \left[1 - y\rho_1 + y^2(-\tfrac{1}{2}\rho_2 + \tfrac{1}{2}\rho_1^2) + y^3(\tfrac{1}{3}\rho_3 + \tfrac{1}{2}\rho_1\rho_2 - \tfrac{1}{6}\rho_1^3) + \ldots\right] c_0|0\rangle_0 \tag{E3}
$$

$$
= \sum_{n=0}^{\infty} \left[A_n y^n c_{-n} + B_n y^{n+2}(c^\dagger_{n+1}c_{-1})c_0 + C_n y^{n+3}(c^\dagger_{n+1}c_{-2})c_0 + \ldots\right] |0\rangle_0. \tag{E4}
$$

Eqs. (26), (33), (16) and (20) were used to obtain from the right-hand side of (E1) that of (E2), and some of the terms arising in the latter's expansion are indicated in (E3-E4), where the dots represent infinitely many further contributions. Eq. (E4) implies that the only non-zero coefficients in (E4) are $A_n = 1$, whereas *all others* are zero, $B_n = C_n = \ldots = 0$. This is quite astonishing, since it implies that when $e^{-i\varphi^\dagger(x)}$ acts on $c_0|0\rangle_0$, all the many ways in which excited states such as $c_n^\dagger c_{-1} c_0|0\rangle_0$ can arise must somehow cancel each other, with only terms of the form $y^n c_{-n}|0\rangle_0$ surviving. To get a feeling for how this can possibly be, we consider the lowest few terms in (E3) (by inserting the sums from ρ_n) that contribute to the A_n and B_n series explicitly (more general terms become intractable):

$$
\sum_{n=0}^{\infty} A_n y^n c_{-n}|0\rangle_0 = \Big\{ 1 - y(c_0^\dagger c_{-1}) + y^2\left[-\tfrac{1}{2}(c_0^\dagger c_{-2}) + \tfrac{1}{2}(c^\dagger_{-1}c_{-2})(c_0^\dagger c_{-1})\right] \tag{E5}
$$

$$
+ y^3\left[-\tfrac{1}{3}(c_0^\dagger c_{-3}) + \tfrac{1}{2}(c^\dagger_{-2}c_{-3})(c_0^\dagger c_{-2}) - \tfrac{1}{6}(c^\dagger_{-2}c_{-3})(c^\dagger_{-1}c_{-2})(c_0^\dagger c_{-1})\right] + \ldots \Big\} c_0|0\rangle_0
$$

$$
= \quad \left\{ 1 + yc_{-1} + y^2\left[\tfrac{1}{2} + \tfrac{1}{2}\right] c_{-2} + y^3\left[\tfrac{1}{3} + \tfrac{1}{2} + \tfrac{1}{6}\right] c_{-3} + \ldots \right\} |0\rangle_0 \tag{E6}
$$

$$\sum_{n=0}^{\infty} B_n y^{n+2} (c_n^\dagger c_{-1}) c_0 |0\rangle_0 = \Big\{ y^2 \Big[-\tfrac{1}{2}(c_1^\dagger c_{-1}) + \tfrac{1}{2}(c_1^\dagger c_0)(c_0^\dagger c_{-1}) \Big] \tag{E7}$$

$$+ y^3 \Big[-\tfrac{1}{3}(c_2^\dagger c_{-1}) + \tfrac{1}{2}(c_2^\dagger c_1)(c_1^\dagger c_{-1}) - \tfrac{1}{6}(c_2^\dagger c_1)(c_1^\dagger c_0)(c_0^\dagger c_{-1}) \Big] + \dots \Big\} c_0 |0\rangle_0$$

$$= \Big\{ y^2 \left[-\tfrac{1}{2} + \tfrac{1}{2} \right] (c_1^\dagger c_{-1}) + y^3 \left[-\tfrac{1}{3} + \tfrac{1}{2} - \tfrac{1}{6} \right] (c_2^\dagger c_{-1}) + \dots \Big\} c_0 |0\rangle_0 \tag{E8}$$

This shows that the first few terms of these series (illustrated in Fig. 2) do give $A_n = 1$ and $B_n = 0$, and illustrates how the remarkable cancellations of excited states occur. To confirm that this happens for all $n \geq 0$, we note that the systematics according to which (E6) and (E8) arose imply that the A_n and B_n can be found by summing the following two series, corresponding to taking $\rho_n = -1$ or 1 in (E2) (the sign difference arises because c_0 is commuted past the c_{-n} in the factor $(c_0^\dagger c_{-n})$ in the A series, but not in the B series):

$$\sum_{n=0}^{\infty} A_n y^n \equiv e^{-\left(\sum_{n=1}^{\infty} \frac{-1}{n} y^n\right)} = e^{-\ln(1-y)} = (1-y)^{-1} = \sum_{n=0}^{\infty} y^n \,, \quad \text{implying } A_n = 1; \tag{E9}$$

$$\sum_{n=-2}^{\infty} B_n y^{n+2} \equiv e^{-\left(\sum_{n=1}^{\infty} \frac{1}{n} y^n\right)} = e^{\ln(1-y)} = 1 - y \,, \quad \text{implying } B_{n\geq 0} = 0 \,. \tag{E10}$$

The B_{-2} and B_{-1} terms correspond to the same terms as A_0 and $-A_1$, namely $c_0|0\rangle_0$ and $c_{-1}|-\rangle_0$.

Doing such checks explicitly for more general terms than the A_n and B_n series becomes intractably complicated. But we know that the seemingly miraculous cancellations needed to make them vanish will indeed occur and are not really miraculous at all, since the remarkable properties of coherent states allowed us to rigorously derive the bosonization identity as an operator identity in Section 6.

F Checking anti-commutators

We check explicitly (following Haldane [4]) that the bosonized versions of $\psi_\eta(x)$ correctly reproduce the anti-commutation relations (8).

Note first that the anti-commutation relations of the Klein factors [Eq. (31)] trivially guarantee $\{\psi_\eta, \psi_{\eta'}\} = \{\psi_\eta, \psi_{\eta'}^\dagger\} = 0$ for $\eta \neq \eta'$. That this works is of course no miracle, since Eq. (31) is a consequence of the anti-commutation relations of the original $c_{\eta k}$-operators.

For $\eta = \eta'$, start from Eq. (62), and use Eqs. (C6) and (32) to commute the exponentials into the order that occurs in the operators O_1 and O_2 defined below. One finds:

$$\psi_\eta(x)\psi_\eta(x') = O_1(x,x')\, e^{i\frac{2\pi}{L}x}\, e^{[-i\varphi_\eta(x), -i\varphi_\eta^\dagger(x')]} \tag{F1}$$

$$= O_1(x,x')\, e^{i\frac{2\pi}{L}x} \left(1 - y e^{-2\pi a/L}\right) ; \tag{F2}$$

$$\psi_\eta(x)\psi_\eta^\dagger(x') = O_2(x,x')\, e^{-i\frac{2\pi}{L}(x-x')}\, e^{[-i\varphi_\eta(x), i\varphi_\eta^\dagger(x')]} \tag{F3}$$

$$= O_2(x,x')\, y \left(1 - y e^{-2\pi a/L}\right)^{-1} ; \tag{F4}$$

$$\psi_\eta^\dagger(x')\psi_\eta(x) = O_2(x,x')\, e^{[i\varphi_\eta(x'), -i\varphi_\eta^\dagger(x)]} \tag{F5}$$

$$= O_2(x,x') \left(1 - y^{-1} e^{-2\pi a/L}\right)^{-1} . \tag{F6}$$

Here we have defined $y \equiv e^{-i\frac{2\pi}{L}(x-x')}$, and the operators O_1 and O_2 are given by

$$O_1(x,x') = \tfrac{2\pi}{L} F_\eta F_\eta e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)(x+x')} e^{-i\left(\varphi^\dagger_\eta(x)+\varphi^\dagger_\eta(x')\right)} e^{-i\left(\varphi_\eta(x)+\varphi_\eta(x')\right)} \quad \text{(F7)}$$

$$O_2(x,x') = \tfrac{2\pi}{L} e^{-i\frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)(x-x')} e^{-i\left(\varphi^\dagger_\eta(x)-\varphi^\dagger_\eta(x')\right)} e^{-i\left(\varphi_\eta(x)-\varphi_\eta(x')\right)} \, . \quad \text{(F8)}$$

It follows immediately that

$$\{\psi_\eta(x), \psi_\eta(x')\} = O_1(x,x')\, e^{i\frac{\pi}{L}(x+x')} \left[y^{-\frac{1}{2}}(1 - y e^{-2\pi a/L}) + y^{\frac{1}{2}}(1 - y^{-1} e^{-2\pi a/L}) \right] \quad \text{(F9)}$$

$$\stackrel{a\to 0}{\longrightarrow} \quad 0\,; \quad \text{(F10)}$$

$$\{\psi_\eta(x), \psi^\dagger_\eta(x')\} = O_2(x,x')\, y^{\frac{1}{2}} \left[y^{\frac{1}{2}}(1 - y e^{-2\pi a/L})^{-1} + y^{-\frac{1}{2}}(1 - y^{-1} e^{-2\pi a/L})^{-1} \right] \quad \text{(F11)}$$

$$\stackrel{a\to 0}{\longrightarrow} \quad O_2(x,x') \sum_{\bar n \in \mathbb{Z}} y^{\bar n} \; = \; O_2(x,x')\, L \sum_{\bar n \in \mathbb{Z}} \delta(x - x' - \bar n L) \quad \text{(F12)}$$

$$= \quad 2\pi \sum_{\bar n \in \mathbb{Z}} \delta(x - x' - \bar n L) e^{i\pi \bar n \delta_b} . \quad \text{(F13)}$$

The last line follows because $\varphi_\eta(x) = \varphi_\eta(x+L)$ so that $O_2(x, x+\bar n L) = \frac{1}{L} e^{i\pi\bar n\delta_b}$. Thus, we have reproduced the anti-commutation relations Eq. (8).

G Point-splitting

We discuss the regularization technique of "point-splitting". We introduce the general concept of operator product expansions to explain why point-splitting an operator product regularizes it, then explain why this is usually equivalent to normal-ordering, and finally illustrate the care needed when using bosonization to evaluate point-split products of fermion fields.

G.1 Operator product expansions

Consider the product $\psi^\dagger_\eta(z+a)\,\psi_\eta(z)$ of two fermion fields, with $z = \tau + ix$ and $a > 0$ a real constant. When $a \to 0$, the result diverges, because the product is not normal-ordered. To calculate the divergence, one simply has to normal order it explicitly:

$$\psi^\dagger_\eta(z+a)\psi_\eta(z) = \sum_{k\neq 0} e^{-k(z+a)} \tfrac{2\pi}{L} \sum_{k'} e^{k'(z+a)} c^\dagger_{k'-k\eta} e^{-k'z} c_{k'\eta} + \tfrac{2\pi}{L} \sum_{k'} e^{k'a} c^\dagger_{k'\eta} c_{k'\eta} \quad \text{(G1)}$$

$$\stackrel{a\to 0}{\longrightarrow} \quad \sum_{q>0} \left(\tfrac{2\pi q}{L}\right)^{1/2} \left(e^{-qz} i b_{q\eta} - e^{qz} i b^\dagger_{q\eta}\right) + \tfrac{2\pi}{L}\widehat{N}_\eta + \tfrac{2\pi}{L} \sum_{k'\leq 0} e^{k'a} \quad \text{(G2)}$$

$$= \quad i\partial_z \phi_\eta(z) + \tfrac{2\pi}{L}\widehat{N}_\eta + \left[\tfrac{1}{a} + \tfrac{\pi}{L}(1-\delta_b) + \text{Order}(\tfrac{a}{L^2})\right] \quad \text{(G3)}$$

Since the first ($k \neq 0$) term in Eq. (G1) is normal-ordered, it is possible to set $a = 0$ there. However, in the second ($k = 0$) term we first have to normal order, producing the $\sum_{k'}$ in Eq. (G2), which diverges for $a \to 0$. The bracketed terms in Eq. (G3) are its order a^{-1} and a^0 contributions. Eq. (G3) agrees with Eq. (92), but, since we included terms of order $1/L$, includes finite-size corrections that were neglected in the latter.

Eq. (G3) is an example of a so-called *operator product expansion* (OPE). In general, when the product $O_i(z)\,O_j(z')$ of two quantum fields is evaluated at points z and z' that are very close to each other, the result diverges if the product is not normal-ordered, typically as some power $(z - z')^{-\nu}$. By bringing the product $O_i(z)O_j(z')$ into normal-ordered form, one can generally express it as a linear combination of other (normal-ordered) fields in the theory, the coefficients being functions of $z - z'$. In the limit $z \to z'$, a Laurent-expansion of these functions in powers of $(z-z')$ yields an OPE of the general form

$$O_i(z)O_j(z') \stackrel{z\to z'}{\longrightarrow} \sum_k \frac{C_{ijk}\, O_k(z')}{(z-z')^{\Delta_i+\Delta_j-\Delta_k}} \,. \tag{G4}$$

The exponents Δ_j are known as the *scaling dimensions* of the fields $O_j(z)$, and the C_{ijk} are c-number coefficients. An OPE succinctly summarizes the short-distance behavior of a theory. For example, the leading ultraviolet behavior of correlation functions can directly be read off from Eq. (G4): $\langle O_i(z)O_j(z')\rangle \to C_{ij1}(z-z')^{-(\Delta_i+\Delta_j)}$, since $\langle O_k(z')\rangle = \delta_{k1}$ is only non-zero if $O_k = \mathbf{1}$, the unit operator, for which $\Delta_1 = 0$. It follows that the fermion fields ψ and $\psi^\dagger$, for which $\langle \psi(z)\psi^\dagger(0)\rangle = z^{-1}$, have scaling dimension $\Delta_\psi = \Delta_{\psi^\dagger} = 1/2$.

G.2 Point splitting versus normal ordering

In field theory, it is popular to regularize divergent products of two fields at the same point by adopting the so-called *point-splitting* prescription, denoted by : :, which evaluates the product at points a short distance apart, and then subtracts the divergence:

$$:O_i(z)O_j(z): \equiv\; O_i(z+a)O_j(z) - {}_0\langle\vec{0}|O_i(z+a)O_j(z)|\vec{0}\rangle_0 \,. \tag{G5}$$

Note that we chose to use here the same regularization parameter a as the one introduced for boson fields in Eq. (33); this is necessary if one wants to reproduce point-split products of fermion fields by using the bosonization formula, as shown in Section G.3.

This is incorrect: see footnote [69] of Zarand & von Delft.
Instead, firstset a=0, then take x0 -> 0.

From Eqs. (G3) and (37), we find

$$:\psi^\dagger_\eta(z)\psi_\eta(z): = \; i\partial_z\phi_\eta(z) + \tfrac{2\pi}{L}\widehat{N}_\eta \;=\; {}^*_*\psi^\dagger_\eta(z)\psi_\eta(z)^*_* \;=\; \rho_\eta(z)\,, \tag{G6}$$

showing that the point-split and normal-ordered versions of the electron density agree. This illustrates the fact that point-splitting simply subtracts a constant that would not have arisen at all had we started with a normal-ordered expression. Therefore, in most cases point-splitting and normal ordering are equivalent ways of regularizing. There are, however, exceptions: if the term that diverges as $a \to 0$ is not a c-number but an operator, such as $O_k(z')$ in the general OPE (G4), and if the expectation value of this operator is zero, then the point-splitting prescription does not succeed in subtracting this divergence.

Point-splitting is a popular regularization scheme in field-theoretical texts, because field theorists typically "know" from experience the various standard OPEs of a given theory, so that it is a simple matter to subtract out the appropriate divergences. However, if one is less familiar with standard OPEs than a an experienced field-theorist, one would have to derive them first by normal-ordering all products of fields. But then one might as well simply adopt the prescription that from the beginning only normal-ordered products of fields at the same point are to be used, thereby eliminating the need to point-split. For the purposes of this tutorial, written for "non-field-theorists", we always adopt the latter procedure.

G.3 Evaluating point-split products of Fermion Fields using Bosonization

As a consistency check on the bosonization rules, we now check that point-split products of fermion fields can also be calculated using their bosonized versions (62) or (91).

Density:— Eq. (G3) can be rederived as follows [we abbreviate $\tilde{\mathcal{N}}_\eta \equiv \frac{2\pi}{L}(\widehat{N}_\eta - \frac{1}{2}\delta_b)$]:

$$\begin{aligned}
\psi^\dagger_\eta(z+a)\psi_\eta(z) &= \tfrac{2\pi}{L} e^{\tilde{\mathcal{N}}_\eta(z+a)} e^{i\varphi^\dagger_\eta(z+a)} e^{i\varphi_\eta(z+a)} e^{-\tilde{\mathcal{N}}_\eta z} e^{-i\varphi^\dagger_\eta(z)} e^{-i\varphi_\eta(z)} && \text{(G7)}\\
&= \tfrac{2\pi}{L} e^{\tilde{\mathcal{N}}_\eta a} e^{i(\varphi^\dagger_\eta(z+a)-\varphi^\dagger_\eta(z))} e^{i(\varphi_\eta(z+a)-\varphi_\eta(z))} e^{[\varphi_\eta(z+a),\varphi^\dagger_\eta(z)]} && \text{(G8)}\\
&= \tfrac{2\pi}{L}\Big[1 + ia\partial_z(\varphi^\dagger_\eta(z)+\varphi_\eta(z)) + a\tilde{\mathcal{N}}_\eta\Big]\left(\tfrac{L}{2\pi a}+\tfrac{1}{2}\right) && \text{(G9)}\\
&= i\partial_z\phi_\eta(z) + \tfrac{2\pi}{L}(\widehat{N}_\eta - \tfrac{1}{2}\delta_b) + \tfrac{1}{a} + \tfrac{\pi}{L} + \text{Order}(\tfrac{a}{L^2})\ . && \text{(G10)}
\end{aligned}$$

To get Eq. (G8), we normal-ordered Eq. (G7), using Eq. (C6). For Eq. (G9), we Taylor-expanded the normal-ordered expressions in a, and evaluated the boson commutator using Eq. (40). Note that the latter had to be expanded to next-to-leading order in a, giving $\left(\frac{L}{2\pi a}+\frac{1}{2}\right)$, in order to correctly reproduce the subleading (non-diverging) terms of Eq. (G3). Note also that in (G9) the cancellation of the $1/a$ arising from the boson commutator and the linear a factors arising from expanding functions of $z+a$ *only* occurs if we use the same short-distance regularization parameter a when point-splitting as the one used for the boson fields.

Hamiltonian:— Next we consider the Hamiltonian $H_{0\eta}$. In analogy to Eq. (G1), we have

$$-\psi^\dagger_\eta(z+a)\partial_z\psi_\eta(z) = \sum_{k\neq 0} e^{-k(z+a)}\tfrac{2\pi}{L}\sum_{k'} k' e^{k'(z+a)} c^\dagger_{k'-k\eta} e^{-k'z} c_{k'\eta} + \tfrac{2\pi}{L}\sum_{k'} e^{k'a} k' c^\dagger_{k'\eta} c_{k'\eta}\ . \tag{G11}$$

It follows immediately that

$$-\int_{-L/2}^{L/2} \tfrac{dx}{2\pi} :\psi^\dagger_\eta(z)\partial_z\psi_\eta(z): = \sum_k k \,{}^*_* c^\dagger_{k\eta} c_{k\eta}\,{}^*_* \ = \ H_{0\eta}\ , \tag{G12}$$

since $\int_{-L/2}^{L/2}\frac{dx}{2\pi}e^{-ikx} = 0$ for $k \neq 0$, and the point-splitting subtraction eliminates the non-normal-ordered contributions in the second term. Comparison with Eq. (66) shows that the normal-ordered and point-split fermionic versions of $H_{0\eta}$ are equal.

Next we show that the point-split bosonic version (70) of $H_{0\eta}$ can be obtained from Eq. (G12) by diligent use of the bosonization formula. Proceeding as in Eq. (G7), we find

$$\begin{aligned}
&-\psi^\dagger_\eta(z+a)\partial_z\psi_\eta(z)\\
&= \tfrac{2\pi}{L} e^{\tilde{\mathcal{N}}_\eta(z+a)} e^{i\varphi^\dagger_\eta(z+a)} e^{i\varphi_\eta(z+a)} e^{-\tilde{\mathcal{N}}_\eta z} e^{-i\varphi^\dagger_\eta(z)} e^{-i\varphi_\eta(z)} \Big\{\tilde{\mathcal{N}}_\eta + i\partial_z\phi_\eta(z) + [\partial_z\varphi^\dagger_\eta(z),\varphi_\eta(z)]\Big\} && \text{(G13)}\\
&= \tfrac{2\pi}{L} e^{\tilde{\mathcal{N}}_\eta a} e^{i(\varphi^\dagger_\eta(z+a)-\varphi^\dagger_\eta(z))} e^{i(\varphi_\eta(z+a)-\varphi_\eta(z))}\left(\tfrac{L}{2\pi a}+\tfrac{1}{2}\right)\Big\{\tilde{\mathcal{N}}_\eta + i\partial_z\phi_\eta(z) - \left(\tfrac{1}{a}-\tfrac{\pi}{L}\right)\Big\} && \text{(G14)}\\
&= \Big[1 + a\Big(\tilde{\mathcal{N}}_\eta + i\partial_z\phi_\eta(z)\Big) + \tfrac{1}{2}a^2\Big(\tilde{\mathcal{N}}^2_\eta + 2\tilde{\mathcal{N}}_\eta\, i\partial_z\phi_\eta(z) + {}^*_*(i\partial_z\phi_\eta(z))^2{}^*_* + i\partial^2_z\phi_\eta(z)\Big)\Big]\\
&\qquad \times \Big\{(\tilde{\mathcal{N}}_\eta + i\partial_z\phi_\eta(z))\left(\tfrac{1}{a}+\tfrac{\pi}{L}\right) - \Big(\tfrac{1}{a^2}+\tfrac{\pi^2}{L^2}\Big)\Big\}\ . && \text{(G15)}
\end{aligned}$$

The commutator in Eq. (G13) arises from commuting the $i\partial_z\varphi^\dagger_\eta(z)$, produced by differentiating an exponent, to the left past $e^{-i\varphi_\eta(z)}$ using Eq. (C3). Using Eq. (40) it is evaluated to subleading order in $1/a$, giving $-\left(\frac{1}{a}-\frac{\pi}{L}\right)$. The factor $\left(\frac{L}{2\pi a}+\frac{1}{2}\right)$ in Eq. (G14) comes from normal-ordering the exponentials of Eq. (G13), as in Eq. (G9). Since the leading divergence is $1/a^2$, we had to Taylor-expand all exponentials to order a^2. It is now merely a matter of algebra to verify that

$$-\int_{-L/2}^{L/2}\tfrac{dx}{2\pi} :\psi^\dagger_\eta(z)\partial_z\psi_\eta(z): = \int_{-L/2}^{L/2}\tfrac{dx}{2\pi}\tfrac{1}{2} :(i\partial_z\phi_\eta(z))^2: + \tfrac{2\pi}{L}\tfrac{1}{2}\widehat{N}_\eta(\widehat{N}_\eta + 1 - \delta_b) \ = \ H_{0\eta}, \tag{G16}$$

which is the point-split version of the boson Hamiltonian (70). To this end, note that the $\int dx$ integral gives zero for all expressions linear in $\partial_z\phi$ and $\partial_z^2\phi$, because ϕ is periodic in x. Furthermore, we used the fact, easily verified, that $:(i\partial_z\phi_\eta(z))^2: = {}^*_*(i\partial_z\phi_\eta(z))^2{}^*_*$ (which sometimes is also written as $\frac{1}{2}:\Pi^2(z) + (i\partial_z\phi(z))^2:$, compare footnote 6).

The above way of deriving the bosonic form (G16) of $H_{0\eta}$ is often used in the field-theoretical approach to bosonization. Note that it is considerably more arduous than the derivation given in Section 7. We included it here for two reasons: Firstly, to illustrate how careful one needs to be if one wants to correctly produce $1/L$ terms using point splitting methods; and secondly, because when performed in reverse order it forms the rigorous basis for refermionizing a bosonic Hamiltonian, as in Eqs. (138-139).

H Free Green's functions

We calculate the free fermion and boson Green's functions $\langle \mathcal{T}\psi_\eta\psi_\eta^\dagger\rangle$ and $\langle \mathcal{T}\phi_\eta\phi_\eta\rangle$, for $L \neq \infty$ at $T = 0$, and also for $L \to \infty$ at $T \neq 0$.

We shall calculate the desired time-ordered Green's functions in the imaginary-time Heisenberg picture, in which they depend on the imaginary-time variable $\tau \in (-\beta, \beta]$. Real time-ordered Green's functions can obtained from these by simply analytically continuing $\tau \to it$ (or, for $v_F\hbar \neq 1$, $\tau \to iv_F\hbar t$ and $\beta \to v_F\hbar\beta$).

The linear dispersion of the Hamiltonian of Eqs.(65) and (69) implies the following thermal expectation values and imaginary-time development of the fermion and boson operators:

$$\langle c^\dagger_{k\eta} c_{k'\eta'}\rangle = \frac{\delta_{\eta\eta'}\delta_{kk'}}{e^{\beta k}+1}, \qquad c_{k\eta}(\tau) = e^{-k\tau}c_{k\eta}, \qquad c^\dagger_{k\eta}(\tau) = e^{k\tau}c^\dagger_{k\eta}, \tag{H1}$$

$$\langle b^\dagger_{q\eta} b_{q'\eta'}\rangle = \frac{\delta_{\eta\eta'}\delta_{qq'}}{e^{\beta q}-1}, \qquad b_{q\eta}(\tau) = e^{-q\tau}b_{q\eta}, \qquad b^\dagger_{q\eta}(\tau) = e^{q\tau}b^\dagger_{q\eta}. \tag{H2}$$

H.1 The limit $T = 0$ for $L \neq \infty$

For $T = 0$, the factors in (H1-H2) reduce to $(e^{\beta k}+1)^{-1} = \theta(-k)$ and $(e^{\beta q}-1)^{-1} = -\theta(-q)$, hence correlation functions are easy to evaluate.

H.1.a Fermion correlation function

For fermions fields, $-G^>_{\eta\eta'}(\tau, x) \equiv \langle\psi_\eta(\tau,x)\psi^\dagger_{\eta'}(0,0)\rangle$ and $G^<_{\eta\eta'}(\tau,x) \equiv \langle\psi^\dagger_{\eta'}(0,0)\psi_\eta(\tau,x)\rangle$ (defined only for $\tau \gtrless 0$, respectively) are two distinct functions, that must be calculated independently. The time-ordered Green's function $G_{\eta\eta'}(\tau,x)$ is a convenient combination of both, which can be evaluated as follows (with $\sigma \equiv \mathrm{sgn}(\tau)$ and $y \equiv e^{-\frac{2\pi}{L}(\sigma\tau+\sigma i x+a)}$):

$$\begin{aligned}
-G_{\eta\eta'}(\tau,x) &\equiv \theta(\tau)G^>_{\eta\eta'}(\tau,x) + \theta(-\tau)G^<_{\eta\eta'}(\tau,x) && \text{(H3)}\\
&= \delta_{\eta\eta'}\Big[\theta(\tau)\tfrac{2\pi}{L}\sum_{k>0} e^{-k(\tau+ix+\sigma a)} - \theta(-\tau)\tfrac{2\pi}{L}\sum_{k<0} e^{-k(\tau+ix+\sigma a)}\Big] && \text{(H4)}\\
&= \delta_{\eta\eta'}\tfrac{2\pi}{L}\sigma y^{-\sigma\delta_b/2}\sum_{n=1}^{\infty} y^n = \delta_{\eta\eta'}\tfrac{2\pi}{L}\sigma\frac{y^{-(\sigma\delta_b+1)/2}}{y^{-1/2}-y^{1/2}} && \text{(H5)}\\
&= \frac{\delta_{\eta\eta'}e^{\frac{\pi}{L}(\delta_b+\sigma)(\tau+ix)}}{\frac{L}{\pi}\sinh[\frac{\pi}{L}(\tau+ix+\sigma a)]} \stackrel{L\to\infty}{\longrightarrow} \frac{\delta_{\eta\eta'}}{\tau+ix+\sigma a}. && \text{(H6)}
\end{aligned}$$

For (H4) we inserted the definition (3) of ψ_η, simplified using (H1) and inserted a factor $e^{-k\sigma a}$ to ensure convergence[19] when $\tau \to 0$. In the k sums, we took $k = \frac{2\pi}{L}(n_k - \delta_b/2)$, see (2).

H.1.b Boson correlation function

For boson fields, $-\mathcal{G}^{>}_{\eta\eta'}(\tau,x) \equiv \langle \phi_\eta(\tau,x)\phi_{\eta'}(0,0)\rangle$ and $-\mathcal{G}^{<}_{\eta\eta'}(\tau,x) \equiv \langle \phi_{\eta'}(0,0)\phi_\eta(\tau,x)\rangle$ (defined only for $\tau \gtrless 0$, respectively) are in fact *not* independent, since $\mathcal{G}^{<}_{\eta\eta'}(\tau,x) = \mathcal{G}^{>}_{\eta'\eta}(-\tau,-x)$. Hence, the time-ordered Green's function can be evaluated as follows:

$$-\mathcal{G}_{\eta\eta'}(\tau,x) \equiv \theta(\tau)\mathcal{G}^{>}_{\eta\eta'}(\tau,x) \;+\; \theta(-\tau)\mathcal{G}^{<}_{\eta\eta'}(\tau,x) \;=\; \mathcal{G}^{>}_{\eta\eta'}(\sigma\tau,\sigma x) \tag{H7}$$

$$= \delta_{\eta\eta'} \sum_{q>0}^{\infty} \frac{1}{n_q} e^{-q(\sigma\tau+\sigma i x + a)} = \delta_{\eta\eta'} \sum_{n_q=1}^{\infty} \frac{1}{n_q} y^{n_q} = -\delta_{\eta\eta'} \ln(1-y) \tag{H8}$$

$$= -\delta_{\eta\eta'} \ln\left(1 - e^{-\frac{2\pi}{L}(\sigma\tau+\sigma i x+a)}\right) \stackrel{L\to\infty}{\longrightarrow} -\delta_{\eta\eta'} \ln\left[\tfrac{2\pi}{L}(\sigma\tau+\sigma i x + a)\right]. \tag{H9}$$

For (H8) we inserted the definition (34) of ϕ_η and simplified using (H2).

H.2 The limit $L \to \infty$ for $T \neq 0$

We next consider the continuum limit $L \to \infty$ for $T \neq 0$, in which the $1/L$ terms in Eq. (69) can be neglected, and discrete sums can be treated as integrals, $\frac{2\pi}{L}\sum_{n_k} \to \int dk$. (In particular, if also $T \to 0$, the order of limits considered here is $L/\beta \to \infty$.) Calculating the free fermion and boson Green's functions is again straightforward, though for $T \neq 0$ the integrals we shall encounter are non-trivial (readers that do not enjoy doing contour integrals can look them up in [42]).

H.2.a Fermion correlation function

Starting from (H3), simplifying using (H1) and again inserting the convergence[19] factor $e^{-k\sigma a}$, we proceed as follows (with $\sigma \equiv \mathrm{sgn}(\tau)$ and $\bar y \equiv e^{-2\pi i(\sigma\tau+\sigma i x+a)/\beta}$):

$$-G_{\eta\eta'}(\tau,x) = \delta_{\eta\eta'} \int_{-\infty}^{\infty} dk\, \frac{e^{-k(\tau+ix+\sigma a)}}{\sigma(1+e^{-\sigma\beta k})} \tag{H10}$$

$$= \delta_{\eta\eta'}\,(2\pi i/\beta) \left[-\theta(x) \sum_{\bar n=-\infty}^{0} \bar y^{(\bar n - 1/2)\sigma} \;+\; \theta(-x) \sum_{\bar n=0}^{\infty} \bar y^{(\bar n+1/2)\sigma}\right] \tag{H11}$$

$$= \frac{\delta_{\eta\eta'}}{\frac{\beta}{\pi}\sin[\frac{\pi}{\beta}(\tau+ix+\sigma a)]} \quad \stackrel{T\to 0}{\longrightarrow} \quad \frac{\delta_{\eta\eta'}}{\tau+ix+\sigma a} \tag{H12}$$

The integral was done using contour methods, by closing it along a semi-circle in the lower (upper) half of the complex k plane for $x > 0$ ($x < 0$). The poles at $k = 2\pi i(\bar n + 1/2)/\beta$ have residues $\bar y^{(\bar n+1/2)\sigma}/\beta$; summing their contributions readily yields the $1/\sin$ behavior.

[19] It is not *a priori* clear that the regularization parameter a to be used in the cut-off factor $e^{-k\sigma a}$ in Eqs. (H4) or (H10) for the fermion correlator must be precisely the same a as the one occuring in the cut-off factor $e^{-qa/2}$ introduced for the boson fields in Eq. (33). In Section 8 we checked that it must indeed be the same, else the results (H6) or (H12) for the fermion correlator would not be consistent with those obtained by evaluating these correlators via bosonization [see (81) and (80) or (78) and (74)].

H.2.b Boson correlation function

Starting from (H7) and simplifying using (H2), we obtain

$$-\mathcal{G}_{\eta\eta'}(\tau,x) \quad = \quad \delta_{\eta\eta'} \int_{\frac{2\pi}{L}}^{\infty} dq \, \frac{e^{-qa}}{q} \left[\frac{e^{-q(\sigma\tau+\sigma ix)}}{(1-e^{-\beta q})} + \frac{e^{q(\sigma\tau+\sigma ix)}}{(e^{\beta q}-1)} \right] . \tag{H13}$$

Since the original discrete sum $\sum_{q>0}$ does not include $q = 0$, we took the lower integration limit in Eq. (H13) at $q = 2\pi/L$ to regularize the infrared-divergence at $q = 0$. When $L \to \infty$, the integral diverges as $\ln(L/a)$ (as is particularly obvious for $\tau = x = 0$ and $T = 0$). To be able to perform the integral by contour methods and nevertheless correctly keep track of this divergent constant, we shall proceed as follows: We take the lower integration limit in Eq. (H13) at $q = 0$ and regularize the divergence using the principle-value prescription, which gives a finite (L-independent) function of $(\tau + ix)/\beta$. To this we add a (diverging) L-dependent constant C, whose value we shall find at the end by requiring that the final result agree with $-\mathcal{G}^{(T=0,L\neq\infty)}_{\eta\eta'}(0,0)$. The integral (H13) can then be evaluated as follows:

$$-\mathcal{G}_{\eta\eta'}(\tau,x) = \delta_{\eta\eta'} \Big[\mathcal{P}\!\!\int_{-\infty}^{\infty} dq \, \frac{e^{-q(\sigma\tau+i\sigma x+a)}}{q(1-e^{-\beta q})} \; + \; C \Big] \tag{H14}$$

$$= \quad \delta_{\eta\eta'} \Big[-\theta(\sigma x) \Big(\tfrac{1}{2} \ln \bar y + \sum_{\bar n=-\infty}^{-1} \frac{\bar y^{\bar n}}{\bar n} \Big) + \theta(-\sigma x) \Big(\tfrac{1}{2} \ln \bar y + \sum_{\bar n=1}^{\infty} \frac{\bar y^{\bar n}}{\bar n} \Big) + C \Big] \tag{H15}$$

$$= \quad \delta_{\eta\eta'} \Big[-\theta(\sigma x) \Big(\ln \bar y^{1/2} + \ln(1-\bar y^{-1}) \Big) - \theta(-\sigma x) \Big(\ln \bar y^{-1/2} + \ln(1-\bar y) \Big) + C \Big] \tag{H16}$$

$$= \quad \delta_{\eta\eta'} \Big[-\ln \Big[\mathrm{sgn}(\sigma x) \left(\bar y^{1/2} - \bar y^{-1/2} \right) \Big] + C \Big] \tag{H17}$$

$$= \quad -\,\delta_{\eta\eta'} \, \ln \left(\tfrac{2\beta}{L} \sin[\tfrac{\pi}{\beta}(\sigma\tau + \sigma ix + a)] \right) . \tag{H18}$$

Since in (H13) the convergence factor e^{-qa} is needed only in the first term (and there only when $\tau = 0$), we replaced it in the second term by e^{qa} (which causes errors of at most $a/\beta \simeq 0$), since both terms can then be combined into the single $\int_{-\infty}^{\infty} dq$ integral of (H14). The $\int dq$ integral of (H14) can be done by contour methods, closing the contour along a semi-circle in the lower (upper) half of the complex q plane for $\sigma x > 0$ ($\sigma x < 0$). The poles of order one at $q = 2\pi i\bar n/\beta$ (with $\bar n \neq 0$) have residues $\bar y^{\bar n}/(2\pi i\bar n)$, where again $\bar y = e^{-i2\pi(\sigma\tau+\sigma ix+a)/\beta}$. The pole of order two at $q = 0$ has residue $-(\sigma\tau + i\sigma x + a)/\beta = (\ln \bar y)/(2\pi i)$, which is multiplied by $\frac{1}{2}$ in (H15) due to the principle-value prescription. The sums $\sum_{\bar n}$ directly yield the $\ln(1-\bar y^{\mp 1})$ terms of (H16). We find the constant C by requiring that (H17) be compatable with (H9) for $\tau = x = 0$, namely $-\mathcal{G}^{(T=0,L\neq\infty)}_{\eta\eta'}(0,0) = \delta_{\eta\eta'} \, \ln(L/2\pi a)$; this readily yields $C = \ln[-iL\,\mathrm{sgn}(\sigma x)/\beta]$, and hence also (H18).

That $\mathcal{G}_{\eta\eta'}(\tau,x)$ diverges as $L \to \infty$ is in itself no cause for concern, since it turns out that only those combinations of Green's functions are needed in which the divergence is subtracted out. For the combination most often encountered, namely

$$-\left[\mathcal{G}_{\eta\eta'}(\tau,x) - \mathcal{G}_{\eta\eta'}(0,0)\right] \quad = \quad \delta_{\eta\eta'} \, \ln \left(\frac{\pi a/\beta}{\sin[\frac{\pi}{\beta}(\sigma\tau + \sigma ix + a)]} \right) \tag{H19}$$

$$\stackrel{T\to 0}{\longrightarrow} \quad \delta_{\eta\eta'} \, \ln \left(\frac{a}{\sigma\tau + \sigma ix + a} \right) , \tag{H20}$$

it would not have been necessary to worry about the divergent term $-\ln(2\beta/L)$ term in (H18) at all. However, there are cases for which it *is* needed explicitly, e.g. when obtaining Eq. (95) from Eq. (94), where it yields the important L-dependent prefactor of Eq. (96).

As a final consistency check, note that using Eq. (H20), we can readily reproduce Eq. (45):

$$\partial_{x'} \langle \phi_\eta(0^+, x)\, \phi_\eta(0, x') \rangle \;=\; \frac{2ai}{a^2 + (x - x')^2} \;\stackrel{a\to 0}{\longrightarrow}\; 2\pi i\, \delta(x - x') \; . \qquad \text{(H21)}$$

I Finite-size diagonalization of backscattering Hamiltonian H'_+

We diagonalize the refermionized impurity backscattering term in a $g = \frac{1}{2}$ Tomonaga-Luttinger liquid of finite size L, exploiting its similarity to the 2-channel Kondo problem.

The refermionized H'_+ of Eq. (144), Section 10.C.2, which describes backscattering off an impurity in a Tomonaga-Luttinger liquid with coupling constant $g = \frac{1}{2}$, has the form

$$H'_+ \equiv U_+ H_+ U_+^{-1} = \Delta_L \tfrac{P}{8} + \sum_{\bar k} \left[\varepsilon_{\bar k *}{}^{*} c^\dagger_{\bar k} c_{\bar k *}{}^{*} + \sqrt{\Delta_L \Gamma} \left(c^\dagger_{\bar k} + c_{\bar k} \right) \left(i\sqrt{2}\, \alpha_d \right) \right] \; , \qquad \text{(I1)}$$

with $\{\alpha_d, \alpha_d\} = 1$, $\alpha_d^\dagger = \alpha_d$ and $\{c_{\bar k}, c^\dagger_{\bar k'}\} = \delta_{\bar k \bar k'}$. As pointed out by Furusaki [22], its form is related to that arising after bosonizing and refermionizing the 2-channel Kondo model, whose solution in Ref. [18] inspired that presented below.

Perhaps the cleanest and most instructive way to diagonalize H'_+ is to do so for finite L with[20] $\bar k = \frac{2\pi}{L}(n_{\bar k} - \frac{1}{2})$ (i.e. in the $P = 0$ subspace of Section 10.C.2) and take the continuum limit $L \to \infty$ at the end. Determining the unitary transformation from the $c_{\bar k}$'s and α_d to the "eigenoperators" $\tilde\alpha_\varepsilon$ that diagonalize H'_+ (and inverting this transformation) is easier for finite L than in the continuum limit, since the discrete state α_d can easier be kept track of if all states are discrete than if the $\bar k$'s form a continuum. Moreover, one sees explicitly how each exact eigenvalue ε develops from its unperturbed value $\varepsilon_{\bar k}$ as the scattering interaction is turned on (the energy shift being of order Δ_L). This is useful and instructive, but not possible if $L \to \infty$ is taken from the outset, since then the spectrum is dense and shifts of order Δ_L are negligible.

We begin by making a further transformation to a new set of fermions $\alpha_{\bar k}$ and $\beta_{\bar k}$,

$$\begin{pmatrix} \alpha_{\bar k} \\ \beta_{\bar k} \end{pmatrix} = \frac{1}{\sqrt 2} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} c_{\bar k} \\ c^\dagger_{-\bar k} \end{pmatrix}, \qquad \begin{pmatrix} c_{\bar k} \\ c^\dagger_{-\bar k} \end{pmatrix} \;\equiv\; \frac{1}{\sqrt 2} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} \alpha_{\bar k} \\ \beta_{\bar k} \end{pmatrix}, \qquad \text{(I2)}$$

which have the following properties (the index n takes the values $\bar k$ and d, with $-d \equiv d$):

$$\alpha^\dagger_n = \alpha_{-n}, \quad \beta^\dagger_{\bar k} = \beta_{-\bar k}, \quad \{\alpha_n, \alpha_{-n'}\} = \delta_{nn'}, \quad \{\beta_{\bar k}, \beta_{-\bar k'}\} = \delta_{\bar k \bar k'}, \quad \{\alpha_n, \beta_{\bar k'}\} = 0 \, . \qquad \text{(I3)}$$

These have the advantage that the $\beta_{\bar k}$ decouple completely, since Eq. (I1) becomes:

$$H'_+ = \Delta_L \tfrac{P}{8} + \sum_{\bar k > 0} \varepsilon_{\bar k} \left(\alpha^\dagger_{\bar k} \alpha_{\bar k} + \beta^\dagger_{\bar k} \beta_{\bar k} \right) + i\, 2\sqrt{\Delta_L \Gamma} \sum_{\bar k > 0} \left(\alpha^\dagger_{\bar k} + \alpha_{\bar k} \right) \alpha_d ; . \qquad \text{(I4)}$$

We seek a set of orthonormal fermions, $\{\tilde\alpha_\varepsilon, \tilde\alpha^\dagger_{\varepsilon'}\} = \delta_{\varepsilon\varepsilon'}$, that diagonalize H'_+, i.e. for which

$$H'_+ \equiv \sum_{\varepsilon > 0} \varepsilon\, \tilde\alpha^\dagger_\varepsilon \tilde\alpha_\varepsilon \;+\; \sum_{\bar k > 0} \varepsilon_{\bar k}\, \beta^\dagger_{\bar k} \beta_{\bar k} \;+\; E'_G, \qquad \text{(I5)}$$

[20] The case $\bar k = \frac{2\pi}{L} n_{\bar k}$ (i.e. the $P = 1$ subspace) can be treated analogously, but requires a bit more care due to the presence of a $\bar k = 0$ state that does not arise for $P = 0$, see [17, 18].

where the constant E'_G represents a shift in ground state energy due to the interaction. Now, when diagonalizing, it is convenient to use as independent operators not the set[21] $\{\alpha_d;\, \alpha_{\bar k}, \alpha^\dagger_{\bar k}, \forall \bar k > 0\}$ but the set $\{\alpha_d;\, \alpha_{\bar k}, \forall \bar k \gtrless 0\} \equiv \{\alpha_n\}$ (which by the first of Eqs. (I3) is equivalent to the former), since then the scattering term in Eq. (I6) looks simpler:

$$H'_+ = \Delta_L \tfrac{P}{8} + \sum_{\bar k} \Big[\tfrac{1}{2} \varepsilon_{\bar k} \big(\alpha_{-\bar k} \alpha_{\bar k} - \theta(-\bar k) \big) + i\, 2 s_B \sqrt{\Delta_L \Gamma} \alpha_{\bar k}\, \alpha_d \Big] + \sum_{\bar k > 0} \varepsilon_{\bar k}\, \beta^\dagger_{\bar k} \beta_{\bar k} \; . \tag{I6}$$

Analogously, we define

$$\tilde\alpha_{-\varepsilon} \equiv \tilde\alpha^\dagger_\varepsilon \; , \qquad \text{with} \qquad \{\tilde\alpha_\varepsilon, \tilde\alpha_{-\varepsilon'}\} \equiv \delta_{\varepsilon\varepsilon'} \tag{I7}$$

($\tilde\alpha_0$ will turn out to be the Majorana fermion that α_d develops into when the interaction is turned on), and use not the set $\{\tilde\alpha_0;\, \tilde\alpha_\varepsilon, \tilde\alpha^\dagger_\varepsilon, \forall \varepsilon > 0\}$, but instead the set $\{\tilde\alpha_0;\, \tilde\alpha_\varepsilon, \forall \varepsilon \gtrless 0\} \equiv \{\tilde\alpha_\varepsilon\}$ (and below $\sum_\varepsilon$ sums over *all* these ε). Then the desired diagonal form of H'_+ is:

$$H'_+ \equiv \sum_\varepsilon \tfrac{1}{2}\, \varepsilon \left(\tilde\alpha_{-\varepsilon} \tilde\alpha_\varepsilon - \theta(-\varepsilon) \right) \; + \; E'_G \; + \; \sum_{\bar k > 0} \varepsilon_{\bar k}\, \beta^\dagger_{\bar k} \beta_{\bar k} \; . \tag{I8}$$

Since H'_+ is quadratic, the α's and $\tilde\alpha$'s are linearly related, hence we make the Ansatz (A is a matrix, with $A^\dagger_{\varepsilon n} \equiv A^*_{n\varepsilon}$)

$$\tilde\alpha_\varepsilon = \sum_{n = d, \bar k} A^\dagger_{\varepsilon n}\, \alpha_n \; , \qquad \text{with} \quad (A^\dagger_{\varepsilon n})^* = A^\dagger_{-\varepsilon -n} = A^*_{-n-\varepsilon}, \quad \text{to ensure} \quad \tilde\alpha^\dagger_\varepsilon \equiv \tilde\alpha_{-\varepsilon} \; . \tag{I9}$$

Inserting the first of Eq. (I9) into Eq. (I7) shows that

$$\sum_n A^\dagger_{\varepsilon n} A_{n\varepsilon'} = \delta_{\varepsilon\varepsilon'} \qquad \big(\text{i.e. } A^\dagger_{\varepsilon n} = (A^{-1})_{\varepsilon n}\big) \; , \qquad \text{thus} \qquad \alpha_n = \sum_\varepsilon A_{n\varepsilon}\, \tilde\alpha_\varepsilon \tag{I10}$$

is the inverse transformation of Eq. (I9). To determine the coefficients $A^\dagger_{\varepsilon n}$, insert Ansatz (I9) into the Heisenberg equation $\varepsilon \tilde\alpha_\varepsilon = [\tilde\alpha_\varepsilon, H'_+]$ [which follows from (I8)]. This yields

$$\varepsilon A^\dagger_{\varepsilon d} \;=\; i\, 2 \sqrt{\Delta_L \Gamma} \sum_{\bar k} A^\dagger_{\varepsilon \bar k} \; , \qquad A^\dagger_{\varepsilon \bar k} = - \frac{i 2 \sqrt{\Delta_L \Gamma} A^\dagger_{\varepsilon d}}{\varepsilon - \varepsilon_{\bar k}} \; , \qquad \text{implying} \tag{I11}$$

$$\frac{\varepsilon}{4\Gamma} \;=\; S_1(\varepsilon) \; , \qquad \text{where} \qquad S_1(\varepsilon) \equiv \Delta_L \sum_{\bar k = -\infty}^{\infty} \frac{1}{\varepsilon - \varepsilon_{\bar k}} = -\pi \tan(\pi \varepsilon / \Delta_L) \tag{I12}$$

(the latter equality is a standard identity for $\varepsilon_{\bar k} = \Delta_L (n_{\bar k} - \frac{1}{2})$, $n_{\bar k} \in \mathbb{Z}$). The first of Eq. (I12) is an eigenvalue equation determining the allowed ε's as functions of Γ. Analyzing it (e.g. graphically, cf. [[18]]) shows that (apart from one $\varepsilon = 0$ solution) each ε_k is shifted to a corresponding $\varepsilon(\bar k) \equiv \varepsilon_{\bar k} + \text{sgn}(\varepsilon) \delta_{\bar k} \Delta_L$, where the shift $\delta_{\bar k} \simeq \frac{1}{2}$ if $|\varepsilon_{\bar k}| \ll \Gamma$, and $\delta_{\bar k} \simeq 0$ if $|\varepsilon_{\bar k}| \gg \Gamma$. This identifies Γ as the cross-over scale below or above which the spectrum is strongly or weakly shifted, respectively.

$A^\dagger_{\varepsilon d}$ can be determined as follows from the first of Eq. (I10), with $\varepsilon = \varepsilon'$:

$$1 \;=\; \sum_n A^\dagger_{\varepsilon n} A_{n\varepsilon} = |A^\dagger_{\varepsilon d}|^2 \left[1 + 4\Gamma S_2(\varepsilon) \right] \; , \qquad \text{where} \tag{I13}$$

$$S_2(\varepsilon) \;\equiv\; \Delta_L \sum_{\bar k} \frac{1}{(\varepsilon - \varepsilon_{\bar k})^2} = - \frac{\partial S_1(\varepsilon)}{\partial \varepsilon} = \frac{\pi^2}{\Delta_L} \left[1 + \tan^2(\pi\varepsilon/\Delta_L) \right] = \frac{1}{\Delta_L} \left[\pi^2 + \frac{\varepsilon^2}{16\Gamma^2} \right]. \tag{I14}$$

[21] Working only with $\alpha_{\bar k}$'s having $\bar k > 0$ would have required instead of Eq. (I9) the more cumbersome Bogoljubov-like Ansatz $\tilde\alpha_\varepsilon \equiv A^\dagger_{\varepsilon d} \alpha_d + \sum_{\bar k > 0} (A^\dagger_{\varepsilon \bar k} \alpha_{\bar k} + \bar A^\dagger_{\varepsilon \bar k} \alpha^\dagger_{\bar k})$.

The second and third equalities follow from the first and second for $S_1(\varepsilon)$ in Eq. (I12), the fourth from the first of Eq. (I12).[22] Combining Eqs. (I14), (I13) and the second of (I11), gives

$$A_{d\varepsilon} = (A^\dagger_{\varepsilon d})^* = -i\,\mathrm{sgn}(\varepsilon)\left[\frac{4\Delta_L\Gamma}{4\Delta_L\Gamma+\varepsilon^2+(4\pi\Gamma)^2}\right]^{1/2}, \quad (\text{with } \mathrm{sgn}(\varepsilon=0)\equiv i)\,, \tag{I15}$$

$$A_{\bar k\varepsilon} = (A^\dagger_{\varepsilon\bar k})^* = \frac{i2\sqrt{\Delta_L\Gamma}A_{d\varepsilon}}{\varepsilon-\varepsilon_{\bar k}}. \tag{I16}$$

The phases in Eqs. (I15) and (I16) where chosen such that $A^*_{n\varepsilon} = A_{-n-\varepsilon}$ [as required by Eq. (I9)] and that $A_{\bar k\varepsilon}$ is real [the latter somewhat arbitrary choice ensures consistency with Ref. [18], with $(\alpha_{\bar k})_{here} = \frac{1}{\sqrt2}(\gamma_{\bar k+}+i\gamma_{\bar k-})_{there}$ and $(\alpha_d)_{here} = (\gamma_{d-})_{there}$].

With Eq. (I15), the desired unitary transformation that maps the refermionized H'_+ of Eq. (I1) into the diagonal form (I5) is complete.[23] That it indeed diagonalizes H'_+ can be checked explicitly by inserting the last of Eqs. (I10) for α_n into Eq. (I6). After some rearrangement and use of Eqs. (I12) and (I14) to do the $\bar k$ sums, one readily recovers Eq. (I8), and in the process finds that the ground state energy shift is $E'_G = \sum_{\bar k>0}\varepsilon_{\bar k} - \sum_{\varepsilon>0}\varepsilon$ (for details, see Ref. [18]).

To calculate ($T=0$) expectation values with respect to the ground state $|G'_B\rangle$ of H'_+, denoted by $\langle\,\rangle'$, of expressions involving the original operators $c_{\bar k}$ and α_d, one uses the inverse transformations [obtained from the last of Eq. (I10) and the second of Eq. (I2)]:

$$c_{\bar k} = \tfrac{1}{\sqrt2}(\alpha_{\bar k}+i\beta_{\bar k}) = \tfrac{1}{\sqrt2}\left(i\beta_{\bar k}+\sum_\varepsilon A_{\bar k\varepsilon}\tilde\alpha_\varepsilon\right), \qquad \alpha_d = \sum_\varepsilon A_{d\varepsilon}\tilde\alpha_\varepsilon\,, \tag{I17}$$

$$\langle\beta_{\bar k}\beta_{-\bar k'}\rangle' = \langle\beta_{\bar k}\beta^\dagger_{\bar k'}\rangle' = \delta_{\bar k\bar k'}\theta(\varepsilon_{\bar k'})\,, \quad \langle\tilde\alpha_\varepsilon\tilde\alpha_{-\varepsilon'}\rangle' = \langle\tilde\alpha_\varepsilon\tilde\alpha^\dagger_{\varepsilon'}\rangle' = \delta_{\varepsilon\varepsilon'}\theta(\varepsilon')\,, \quad [\theta(\varepsilon'=0)\equiv\tfrac12]. \tag{I18}$$

J Asymptotic analysis of various correlators

We evaluate explicitly the asymptotic $t\to\infty$ behavior of a number of correlators occuring in the refermionized theory of scattering off a Luttinger liquid of Sections 10.C and 10.D. Since they can all be expressed in terms of the fermionic operators β and $\tilde\alpha$, this is possible using Wick's theorem. The corresponding Feynman diagrams are shown in Fig. 4. Throughout this Appendix, we use the shorthand $c\equiv 4\pi\Gamma$.

J.1 The "total current" correlator $\langle\widehat N_+(t)\widehat N_+(0)\rangle'$

We describe the dispute between Fabrizio & Gogolin and Oreg & Finkelstein, mentioned in Section 10.D.3, regarding the calculation of the correlator $D_{\alpha_d}(t)$ in terms of the "total current" correlator $\langle\widehat N_+(t)\widehat N_+(0)\rangle'$. We believe that OF's critique of FG is unfounded; to illustrate our view, we confirm FG's calculation explicitly for $g=1/2$.

To evaluate $D_{\alpha_d}(t)$ for general g, FG [23] exploited the fact that $\alpha_d = e^{i\pi\widehat{\mathcal N}}/\sqrt2$ depends on the "total current" operator $2\widehat{\mathcal N} = (\widehat N_L - \widehat N_R)$, which they call J. They concluded that $D_{\alpha_d}(t)\sim t^{-1/2g}$ (as explained

[22] As consistency check, we verify that the first of Eq. (I10), divided by $A^\dagger_{\varepsilon d}A_{d\varepsilon'}$, also holds for $\varepsilon\neq\varepsilon'$ [the last equality follows from the first of Eq. (I12)]:

$$[A^\dagger_{\varepsilon d}A_{d\varepsilon'}]^{-1}\textstyle\sum_n A^\dagger_{\varepsilon n}A_{n\varepsilon'} = 1+4\Gamma\Delta_L\sum_{\bar k}\frac{1}{(\varepsilon-\varepsilon_{\bar k})(\varepsilon'-\varepsilon_{\bar k})} = 1+4\Gamma\Delta_L\sum_{\bar k}\frac{1}{\varepsilon'-\varepsilon}\left(\frac{1}{\varepsilon-\varepsilon_{\bar k}}-\frac{1}{\varepsilon'-\varepsilon_{\bar k}}\right) = 0\,.$$

[23] It is straightforward and instructive to verify directly that inserting the last of Eq. (I10) for $\tilde\alpha_n$ into the original form (I1) for H'_+ indeed yields the diagonal form (I8).

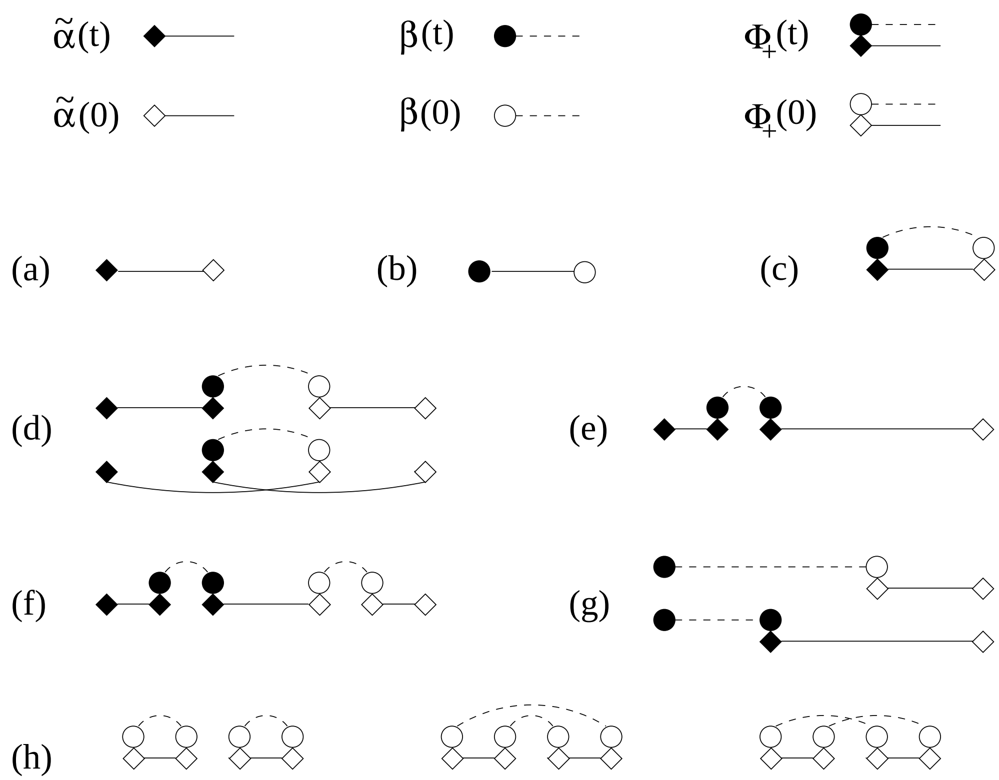

Figure 4: Feynman diagrams indicating the contractions that give the asymptotically leading contributions to the following correlators: (a) D_{α_d} of (156), the subleading term of D_Ψ of (J10), and also $D_{\Psi\alpha_d}$ of (J28); (b) D_β of (155) and the leading term of D_Ψ of (J10); (c) D_{Φ_+} of (164) and D_{N_+} of (J2); (d) D_{11} of (J16); (e) D_{20} of (J23); (f) D_{22} of (J26); (g) $D_{\Psi\Phi_+\alpha_d}$ of (J31); and (h) $\langle \hat{B}^4 \rangle$ of (J7).

below) by citing the result $D_{N_+}(t) \equiv \langle \widehat{N}_+(t)\widehat{N}_+(0)\rangle' \sim -(\ln t)/(2g\pi^2)$ from previous papers [35, 43]. In Ref. [35] FG had arrived at the latter result by assuming, following the RG results of Kane and Fisher [16], that *as far as current fluctuations are concerned* (which is of course all that matters for D_{N_+}), the effect of a backscattering impurity can be mimicked by using "open boundary conditions", $\Psi_{phys}(x=0) = \Psi_{phys}(x = L) = 0$, since both suppress current fluctuations. [Recall that current and density fluctuations are governed by Φ_+, $\widehat{N}_+$ and Φ_-, $\widehat{N}_-$, respectively, since $\tilde\rho_L(0) \mp \tilde\rho_R(0) = \sqrt{2}g^{\mp/2}\partial_x\Phi_\pm(x)|_{x=0} + 2\frac{2\pi}{L}\widehat{N}_\pm$, see (129)]. In their Reply [24], OF objected to this assumption of [35] (without commenting on [43]), pointing out that "cutting the wire" (i.e. open boundary conditions) is *not* fully equivalent to a backscattering impurity, since the latter affects only current but *not the density fluctuations* at the impurity site. As a general statement, this assertion is certainly correct: the density at $x = 0$ is clearly unaffected by backscattering, since $[H_B, \Phi_-] = 0$. [Free density fluctuations, in fact, are responsible for the decay of $D_F(t) \sim t^{-1/2g}$ of (180).] Nevertheless, in our opinion OF's critique is misguided, simply because D_{α_d} *depends solely on current fluctuations* (i.e. solely on Φ_+ and $\widehat{N}_+$ — though in the field-theoretical bosonziation formalism employed by FG and OF, this fact is perhaps not as obvious as here); but for the calculation of *current* fluctuations it is irrelevant whether *density* fluctuations are present or not, since the two types of fluctuations are completely decoupled ($[\Phi_+, \Phi_-] = 0$ and $[\widehat{N}_+, \Phi_-] = 0$). Therefore, FG's strategy in Ref. [35] for finding $\langle \widehat{N}_+(t)\widehat{N}_+(0)\rangle'$ is sound.

To illustrate our view and confirm FG's results, we now calculated D_{N_+} explicitly for $g = 1/2$. Using (145) and (147), the number operator $\widehat{N}_+$ of (136) (with $P = 0$) can be written as

$$\widehat{N}_+ = \sum_{\bar k>0} i(\alpha^\dagger_{\bar k}\beta_{\bar k} - \beta^\dagger_{\bar k}\alpha_{\bar k}) = \sum_{\bar k} i(\alpha_{-\bar k}\beta^\dagger_{\bar k}) = \sum_{\bar k\varepsilon} iA_{-\bar k,\varepsilon}\tilde\alpha_\varepsilon\beta_{\bar k}\ . \tag{J1}$$

Its correlator $D_{N_+}(t)$ [Fig. 4(c)] can thus be evaluated as follows:

$$D_{N_+}(t) \equiv \langle \widehat{N}_+(t)\widehat{N}_+(0)\rangle' = \sum_{\bar k\bar k'\varepsilon\varepsilon'} A_{-\bar k,\varepsilon}A^*_{-\bar k',\varepsilon'}\langle\tilde\alpha_\varepsilon(t)\beta_{\bar k}(t)\tilde\beta^\dagger_{\bar k'}(0)\alpha^\dagger_{\varepsilon'}(0)\rangle' \tag{J2}$$

$$= \sum_{\varepsilon,\bar k\geq 0}\theta(\varepsilon)|A_{-\bar k,\varepsilon}|^2 e^{-i(\varepsilon+\varepsilon_{\bar k})t} \overset{L\to\infty}{\longrightarrow} \frac{c^2}{\pi^2} P\!\!\!\int_{\Delta_L}^\infty d\varepsilon\, P\!\!\!\int_0^\infty d\varepsilon_{\bar k}\frac{e^{-i(\varepsilon+\varepsilon_{\bar k})t}}{(\varepsilon^2+c^2)(\varepsilon+\varepsilon_{\bar k})^2} \tag{J3}$$

$$= \begin{cases} -\frac{1}{\pi^2}\ln(\Delta_L/c) & (t=0); \\ -\frac{1}{\pi^2}\ln(r\Delta_L it) & (ct\gg 1, \Delta_L t \ll 1). \end{cases} \tag{J4}$$

When taking the limit $L \to \infty$ using (154), we cut off the low-energy divergence in the double integral of (J3) by $\Delta_L = v2\pi/L$, the level spacing for finite L. For $t = 0$, the logarithmic divergence of $\langle \widehat{N}_+^2\rangle'$ with system size when $L \to \infty$ reflects the fact that due to backscattering $\widehat{N}_+$ is *not conserved*. The divergence is sufficiently slow, however, that factors of order $\widehat{N}_+/L$ *can* be safely neglected when taking the continuum limit (as done in Section 10.D). The simplest way to find the asymptotic $ct \gg 1$ result is to first show that $\partial_{t^2}D_{N_+}(t) \sim (it)^{-2}$ using (158), then integrating twice w.r.t. t. This yields $-\ln(r\Delta_L it)$, where r is a constant of order unity, (and not, for example, $-\ln(rcit)$), since it is Δ_L (not c) which cuts off the low-energy divergence of $(\varepsilon+\varepsilon_{\bar k})^{-2}$ as long as $\Delta_L t \ll 1$, just as for the case $t = 0$. The numerical value of r depends on the precise way in which this infra-red cut-off is introduced.

These $g = 1/2$ results confirm those for general g of Gogolin and Prokof'ev [43], who found $\langle \widehat{N}_+^2\rangle' \sim \ln(Lc)$, and those of Fabrizio and Gogolin [35] for $D_{N_+}(t)$ mentioned above. FG used D_{N_+} in their Comment [23] on Oreg and Finkel'stein's work to calculate the correlator D_{α_d} for *general* g, by essentially proceeding as follows:

$$D_{\alpha_d}(t) \equiv \langle\alpha_d(t)\alpha_d(0)\rangle' = \tfrac{1}{2}\langle e^{i\pi\widehat{N}_+(t)}e^{-i\pi\widehat{N}_+(0)}\rangle' \text{ ``='' } \tfrac{1}{2}e^{\pi^2[D_{N_+}(t)-D_{N_+}(0)]} \sim \tfrac{1}{2}(rcit)^{-1/2g}\ . \tag{J5}$$

For $g = 1/2$ this yields $(ict)^{-1}$ behavior, in agreement with our result $D_{\alpha_d}(t) \sim (\pi cit)^{-1}$ of (157), but the numerical value of the prefactor (which FG did not specify) is $1/2r$ instead of $1/\pi$, i.e. it depends on the way the infrared cut-off in (J3) was performed. Actually, since the identity (76), which would make the "=" in (J5) a true equality, holds only for *free* bosonic operators (cf. the end of Appendix J.2), the prefactor in (J5) will be further renormalized by additional contributions, not contained in $e^{\pi^2[D_{N_+}(t)-D_{N_+}(0)]}$; these will depend on r but not on t and must evidently conspire to change the prefactor from $1/2r$ to $1/\pi$.

It is instructive to identify the nature of these additional contributions. For a free bosonic operator $\hat{B}$, the relation (75) can be proven as follows:

$$\langle e^{\hat{B}_0}\rangle = \sum_{n=0}^{\infty} \frac{1}{2n!}\langle \hat{B}_0^{2n}\rangle = \sum_{n=0}^{\infty} \frac{1}{2n!}\frac{(2n-1)!}{2^{n-1}(n-1)!}\langle \hat{B}_0^2\rangle^n = \sum_{n=0}^{\infty} \frac{1}{2^n n!}\langle \hat{B}_0^2\rangle^n = e^{\langle \hat{B}_0^2\rangle/2}. \tag{J6}$$

For the first equality we used $\langle \hat{B}_0^{2n+1}\rangle = 0$, and for the second evoked Wick's theorem to reduce $\langle \hat{B}_0^{2n}\rangle$ to a sum of $(2n-1)(2n-3)\dots$ identical terms, each equal to $\langle \hat{B}_0^2\rangle^n$. Relation (76) can be similarly proven, though the combinatorics is more involved.

Now, the bosonic operators $\hat{N}_+$ and Φ_+ are not free, but both of the general form $\hat{B} = \sum_{\varepsilon\bar{k}} B_{\varepsilon\bar{k}}\tilde{\alpha}_\varepsilon\beta_{\bar{k}}$, with non-trivial coefficients $B_{\varepsilon\bar{k}}$. When evaluating $\langle \hat{B}^{2n}\rangle$ using Wick's theorem for free fermions, one thus obtains two types of contributions: firstly, those containing only "pairwise" contractions, in which *both* operators from one pair $(\tilde{\alpha}\beta)$ are contracted with *both* operators from another pair; this yields $(2n-1)(2n-3)$ times $\langle \hat{B}^2\rangle^n = [\sum_{\varepsilon,\bar{k}\geq 0}|B_{\varepsilon\bar{k}}|^2]^n$, just as for free bosons. Secondly, there are "non-pairwise" contractions, in which $(\tilde{\alpha}\beta)$ is contracted with the $\tilde{\alpha}$ of one pair and the β of another, which has no analogue for free bosons. For example, in

$$\langle \hat{B}^4\rangle = 3\Bigg\{\Bigg[\sum_{\varepsilon,\bar{k}\geq 0}|B_{\varepsilon\bar{k}}|^2\Bigg]^2 + \sum_{\varepsilon,\bar{k},\varepsilon',\bar{k}'\geq 0}\Big[B_{\varepsilon\bar{k}}B_{-\varepsilon\bar{k}'}B^*_{-\varepsilon'\bar{k}'}B^*_{\varepsilon'\bar{k}} + B_{\varepsilon\bar{k}}B_{-\varepsilon\bar{k}'}B^*_{-\varepsilon'\bar{k}}B^*_{\varepsilon'\bar{k}'}\Big]\Bigg\} \tag{J7}$$

the first term arises from pairwise contractions, the second and third from non-pairwise contractions [see Fig. 4(h)]. It follows that in general

$$\langle e^{\hat{B}}\rangle = e^{\langle \hat{B}^2\rangle/2} + \dots\ , \qquad \langle e^{\hat{B}}e^{\hat{B}'}\rangle = e^{\langle \hat{B}\hat{B}' + (\hat{B}^2+\hat{B}'^2)/2\rangle} + \dots\ , \tag{J8}$$

where on the right-hand sides the exponentials and dots arise from pairwise and non-pairwise contractions, respectively. The latter will change the numerical value of the prefactor (J5), but not the leading asymptotic behavior t^{-1}. (It is straightforward but cumbersome to check this, by evaluating, for example, the non-pairwise contracted terms for $\langle \widehat{N}_+^2(t)\widehat{N}_+^2(0)\rangle$.)

J.2 Checking that $D_{\alpha_d V_{-1}}(t) \sim (it)^{-1}$

We check the result $D_{\alpha_d V_\lambda}(t) \sim (it)^{-1}$ for $\lambda = -1$, by relating $D_{\alpha_d V_{-1}}$ to $D_\Psi \equiv \langle \Psi_+(t)\Psi_+^\dagger(0)\rangle'$ and calculating the latter explicitly.

$$\begin{aligned} D_{\alpha_d V_{-1}}(t) &= \frac{1}{a}\langle e^{iH'_+t}\alpha_d e^{-i\Phi_+}e^{-iH'_+t}e^{i\Phi_+}\alpha_d\rangle' \\ &= \frac{1}{2a}\langle e^{iH'_+t}F_+e^{-i\Phi_+}e^{-iH'_+t}e^{i\Phi_+}F_+^\dagger\rangle' = \tfrac{1}{2}D_\Psi(t)\ . \end{aligned} \tag{J9}$$

We used (181) and (182) to trade $\alpha_d H'_+ \alpha_d$ for $\frac{1}{2} F_+ H'_+ F_+^\dagger$, then used (134) to identify $a^{-1/2} F_+ e^{-i\Phi_+}$ as Ψ_+. The correlator D_Ψ [Fig. 4(a,b)] can be evaluated as follows:

$$D_\Psi(t) \equiv \langle \Psi_+(t) \Psi_+^\dagger(0) \rangle' = \frac{2\pi}{L} \sum_{\bar{k}\bar{k}'} \frac{1}{2} \langle (\alpha_{\bar{k}}(t) + i\beta_{\bar{k}}(t)) \left(\alpha^\dagger_{\bar{k}'}(0) - i\beta^\dagger_{\bar{k}'}(0) \right) \rangle' \tag{J10}$$

$$= \frac{\Delta_L}{2v} \left[D_\beta(t) + \sum_{\varepsilon \geq 0} e^{-\varepsilon(it + a/v)} \theta(\varepsilon) \sum_{\bar{k}\bar{k}'} A_{\bar{k},\varepsilon} A^*_{\bar{k}',\varepsilon} \right] \tag{J11}$$

$$= \frac{1}{2v} \left[D_\beta(t) + \int_0^\infty d\varepsilon \, \frac{e^{-\varepsilon(it+a/v)} \varepsilon^2}{\varepsilon^2 + c^2} \right] = \begin{cases} \frac{1}{a}(1 - \pi a c/4) & (t = 0, \ ca \ll 1); \\ \frac{1}{2ivt} \left[1 + \mathcal{O}\left(\frac{1}{c^2 t^2} \right) \right] & (\Gamma t \gg 1). \end{cases} \tag{J12}$$

To obtain Eq. (J10), (J11) and (J12) we used, respectively, the first of Eqs. (134) and (145) for Ψ_+, (155) for D_β, (147) and (149) for $\alpha_{\bar{k}}(t)$ [with $c \equiv 4\pi\Gamma$], (151) to do the $\sum_{\bar{k}}$ sums in (J11), then (154) to take the continuum limit, and (158) for the asymptotic integral.

Eqs. (J9) and (J12) together evidently confirm that $D_{\alpha_d V_{-1}}(t) \sim (it)^{-1}$. Moreover, they also yield the leading prefactor of $D_{\alpha_d V_{-1}}$ in (190), namely $C_{-1} = 1/4$. This implies, via (189) and (157) that $a^{-1} \langle e^{i\lambda\Phi_+(t)} e^{-i\lambda\Phi_+(0)} \rangle'$ is proportional to Γ, which is consistent with what we would get from $a^{-1} e^{-\langle \Phi_+(0,0)^2 \rangle'} = (e^\gamma 4\pi\Gamma/v)$ [by (166)]. That the numerical prefactors obtained from $a^{-1} e^{-\langle \Phi_+(0,0)^2 \rangle'}$ and $a^{-1} \langle e^{i\lambda\Phi_+(t)} e^{-i\lambda\Phi_+(0)} \rangle'$ differ is of course no surprise, since according to the discussion preceding Eq. (J8) these two quantities are identically equal only for free boson fields.

Incidentally, it is straightforward to check that the result $D_{\alpha_d V_1} \sim (it)^{-1}$ can also be derived by writing $\alpha_d e^{-i\Phi_+} = e^{-i(\pi \widehat{N} + \Phi_+)}$ and using (J8), with $\hat{B} = \pi \widehat{N} + \Phi_+$.

J.3 Leading connected contributions to $\langle \alpha_d(t) \Phi_+^n(t) \Phi_+^{n'}(0) \alpha_d(0) \rangle'$

The result $D_{\alpha_d V_\lambda} \sim (it)^{-1}$ of (190) rests on the fact that each "connected" contribution to the correlator $D_{nn'}(t) \equiv \frac{i^{n-n'}}{n! n'!} \langle \alpha_d(t) \Phi_+^n(t) \Phi_+^{n'}(0) \alpha_d(0) \rangle'$ occuring in (188) asymptotically decays at least as fast as $1/t$ (most decay much faster), since it contains at least one contraction between two operators at times t and 0, which yields at least one factor of $1/t$. Here we illustrate this by considering the leading connected contributions to D_{11}, D_{20} and D_{22} explicitly.

Apart from (158), the following integrals will be found useful in the asymptotic evaluation of $D_{nn'}$:

$$I_1 = \mathcal{P}\!\!\int_0^\infty d\varepsilon \frac{\varepsilon}{(\varepsilon + \varepsilon')(\varepsilon^2 + c^2)} = \frac{c}{\varepsilon'^2 + c^2} \left[\frac{\pi}{2} + \frac{\varepsilon'}{c} \ln\left(\frac{|\varepsilon'|}{c} \right) \right], \tag{J13}$$

$$I_2 = \mathcal{P}\!\!\int_0^\infty d\varepsilon \frac{c}{(\varepsilon + \varepsilon')(\varepsilon^2 + c^2)} = \frac{c}{\varepsilon'^2 + c^2} \left[-\ln\left(\frac{|\varepsilon'|}{c} \right) + \frac{\pi\varepsilon'}{2c} \right], \tag{J14}$$

$$I_3 = \mathcal{P}\!\!\int_0^\infty d\varepsilon \frac{\varepsilon^n [\ln(\varepsilon/\bar{c})]^{\bar{n}} e^{-\varepsilon(it+a)}}{(\varepsilon^2 + c^2)^m} \sim \frac{n! [-\ln|\bar{c}t|]^{\bar{n}}}{c^{2m} (it)^{n+1}}, \tag{J15}$$

where I_3 assumes $n, \bar{n}, m \geq 0$ and integer, and $t/a, tc \gg 1$.

The leading connected contributions to D_{11} [Fig. 4(d)] are evaluated as follows:

$$D_{11}(t) \equiv \langle \alpha_d(t) \Phi_+(t) \Phi_+(0) \alpha_d(0) \rangle' \tag{J16}$$

$$\sim \sum_{\bar{k}\bar{k}'\varepsilon\varepsilon'\bar{\varepsilon}\bar{\varepsilon}'} A_{d\varepsilon} \Phi_{\bar{k},\bar{\varepsilon}} \Phi^*_{\bar{k}',\bar{\varepsilon}'} A^*_{d\varepsilon'} \langle \left(\tilde{\alpha}_\varepsilon \beta_{\bar{k}} \tilde{\alpha}_{\bar{\varepsilon}} \right)(t) \left(\tilde{\alpha}^\dagger_{\bar{\varepsilon}'} \beta^\dagger_{\bar{k}'} \tilde{\alpha}^\dagger_{\varepsilon'} \right)(0) \rangle \tag{J17}$$

$$= \sum_{\bar{k},\varepsilon,\varepsilon' > 0} e^{-i\bar{k}t} A_{d\varepsilon} \left[\Phi_{\bar{k},-\varepsilon} \Phi^*_{\bar{k},-\varepsilon'} - e^{-i(\varepsilon+\varepsilon')t} \Phi_{\bar{k},\varepsilon'} \Phi^*_{\bar{k},\varepsilon} \right] A^*_{d\varepsilon'}. \tag{J18}$$

Among the contractions that produced the first or second terms of (J18), there were one or three t-to-0 contractions (i.e. connecting operators at t and 0), respectively, yielding one or three oscillatory factors, respectively. Thus the first term is the one that decays slower when $t \to \infty$; to determine its asymptotics, we first do the sums on ε and ε' for a slightly more general expression (which occurs in the leading terms of all $D_{nn'}$):

$$F_{\bar{k},\bar{k}'} \equiv \sum_{\varepsilon,\varepsilon'>0} A_{d\varepsilon} \Phi_{\bar{k},-\varepsilon} \Phi^*_{\bar{k}',-\varepsilon'} A^*_{d\varepsilon'} \tag{J19}$$

$$= \not{\int}_0^\infty d\varepsilon\, d\varepsilon' \frac{\Delta_L\, c\, \varepsilon\, \varepsilon'}{\pi[\varepsilon^2+c^2][\varepsilon'^2+c^2]} \left[\frac{1}{\varepsilon_{\bar{k}}-\varepsilon} - \frac{\pi\varepsilon}{c}\delta(\varepsilon_{\bar{k}}-\varepsilon)\right] \left[\frac{1}{\varepsilon_{\bar{k}'}-\varepsilon'} - \frac{\pi\varepsilon'}{c}\delta(\varepsilon_{\bar{k}'}-\varepsilon')\right]$$

$$\sim \frac{\pi \Delta_L c^3}{4\,[\varepsilon^2_{\bar{k}}+c^2][\varepsilon^2_{\bar{k}'}+c^2]} \left[1 + \mathcal{O}\left(\frac{\varepsilon_{\bar{k}}}{c}\right) + \mathcal{O}\left(\frac{\varepsilon_{\bar{k}'}}{c}\right)\right] . \tag{J20}$$

After using I_1 of (J13) twice to perform the double integral, we retained only the term with the lowest powers of $\varepsilon_{\bar{k}}/c$ and $\varepsilon_{\bar{k}'}/c$, since [by (158)] it is the one giving the leading asymptotic behavior for D_{11}:

$$D_{11}(t) \sim \sum_{\bar{k}} F_{\bar{k},\bar{k}} e^{-i\varepsilon_{\bar{k}} t} \sim \int_0^\infty d\varepsilon_{\bar{k}}\, e^{-i\varepsilon_{\bar{k}} t} \frac{\pi c^3}{4\,[\varepsilon^2_{\bar{k}}+c^2]^2} \sim \frac{\pi}{4cit} . \tag{J21}$$

Thus leading term in $D_{11}(t)$ decays just as fast as the disconnected terms proportional to $D_{00} = D_{\alpha_d}(t) \sim 1/(cit)$, and hence its prefactor contributes to the prefactor C_λ in (190) for $D_{\alpha_d V_\lambda}$. For the second term of (J18), we do the $\varepsilon_{\bar{k}}$ before the $\varepsilon, \varepsilon'$ integrals (using (158) for all three integrals), obtaining

$$\not{\int}_0^\infty d\varepsilon\, d\varepsilon'\, d\varepsilon_{\bar{k}}\, \frac{c\, e^{-i(\varepsilon_{\bar{k}}+\varepsilon+\varepsilon')t}}{\pi\,[\varepsilon^2+c^2][\varepsilon'^2+c^2]} \left(\frac{\varepsilon'}{\varepsilon_{\bar{k}}+\varepsilon'}\right)\left(\frac{\varepsilon}{\varepsilon_{\bar{k}}+\varepsilon}\right) \sim \frac{1}{\pi(ict)^3} . \tag{J22}$$

This illustrates that the more contractions there are between times t and 0, the more powers of $1/(ct)$ are produced.

Next we consider D_{20} [Fig. 4(e)] , whose leading connected term differs from that of D_{11} only in the oscillatory factor (the 1/2! is cancelled by a combinatorical factor 2!):

$$D_{20}(t) \equiv \frac{1}{2!}\langle \alpha_d(t) \Phi^2_+(t) \alpha_d(0)\rangle' \sim \frac{2!}{2!} \sum_{\bar{k},\varepsilon,\varepsilon'>0} A_{d\varepsilon} \Phi_{\bar{k},-\varepsilon} \Phi^*_{\bar{k},-\varepsilon'} A^*_{d\varepsilon'}\, e^{-i\varepsilon' t} \tag{J23}$$

$$\sim \not{\int}_0^\infty d\varepsilon'\, d\varepsilon_{\bar{k}}\, d\varepsilon \frac{c\,\Delta_L\,\varepsilon\,\varepsilon'\, e^{-i\varepsilon' t}}{\pi\,[\varepsilon^2+c^2][\varepsilon'^2+c^2][\varepsilon_{\bar{k}}-\varepsilon][\varepsilon_{\bar{k}}-\varepsilon']} \tag{J24}$$

$$\sim \not{\int}_0^\infty d\varepsilon'\, \frac{c^2\,\varepsilon'\, \ln|\varepsilon'/c| e^{-i\varepsilon' t}}{2\,[\varepsilon'^2+c^2]^2} \sim -\frac{\ln|ct|}{2(cit)^2} . \tag{J25}$$

To obtain (J25) we did, in that order, the ε, $\varepsilon_{\bar{k}}$ and ε' integrals, using I_1, I_2 and I_3 of (J13), (J14) and (J15), respectively, keeping at each step only the asymptotically leading term. Evidently, D_{20} decays faster than $D_{00} = D_{\alpha_d}$ by a factor $\ln|ct|/(cit)$.

Finally, we consider the leading contribution to D_{22} [Fig. 4(f)], namely

$$D_{22}(t) \equiv \frac{1}{(2!)^2}\langle \alpha_d(t) \Phi^2_+(t) \Phi^2_+(0) \alpha_d(0)\rangle' \sim \frac{(2!)^2}{(2!)^2} \sum_{\bar{k},\bar{k}',\varepsilon>0} F_{\bar{k}\bar{k}'}\, \Phi_{-\bar{k},\varepsilon} \Phi^*_{-\bar{k}',\varepsilon}\, e^{-i\varepsilon t} \tag{J26}$$

$$\sim \not{\int}_0^\infty d\varepsilon\, \frac{\pi\, c^3\, \varepsilon^2\, (\ln|\varepsilon/c|)^2 e^{-i\varepsilon t}}{4\,[\varepsilon^2+c^2]^3} \sim \frac{\pi(\ln|ct|)^2}{4\,(cit)^3} . \tag{J27}$$

The $F_{\bar{k}\bar{k}'}$ in (J26) arises in the same way as in (J18); the $\varepsilon_{\bar{k}}$, $\varepsilon_{\bar{k}'}$ integrals and the ε integral can be done with I_2 and I_3 of (J14) and (J15), respectively.

These examples illustrate that the integrals that have to be done rapidly become very complicated when n, n' increase, so that it would be a daunting task to give a general formula for the leading asymptotic behavior of $D_{nn'}$. However, it also is evident that the leading term will always decay as least as fast as $\sim (it)^{-1}$, simply because it always contains at least one t-to-0 contraction.[24]

J.4 Leading contributions to $D_{LR}(t)$

We asymptotically evaluate the leading terms arising when $D_{LR}(t)$ of (192) is expanded in powers of Φ_+, namely $\langle \Psi_+(t)\alpha_d(0)\rangle' \sim (it)^{-2}$ and $\langle \Psi_+(t)[\Phi_+(0) - \Phi_+(t)]\alpha_d(0)\rangle' \sim (it)^{-1}$.

The calculation of the first of these [Fig. 4(a)] is analogous to that of $D_\Psi(t)$ of (J10):

$$D_{\Psi\alpha_d}(t) \equiv \sqrt{2/a}\,\langle \Psi_+(t) i\alpha_d(0)\rangle' = \sqrt{2\pi/aL}\sum_{\bar{k}} \langle [\alpha_{\bar{k}}(t) + i\beta_{\bar{k}}(t)]\, i\alpha_d(0)\rangle' \tag{J28}$$

$$= \quad i\sqrt{\Delta_L/va}\sum_{\varepsilon} e^{-\varepsilon(it+a/v)}\theta(\varepsilon)\sum_{\bar{k}} A_{\bar{k},\varepsilon}A^*_{d,\varepsilon}\langle \tilde{\alpha}_\varepsilon \tilde{\alpha}^\dagger_\varepsilon\rangle' \tag{J29}$$

$$= \quad -\sqrt{\frac{c}{\pi a v}}\int_0^\infty d\varepsilon\, \frac{e^{-\varepsilon(it+a/v)}\,\varepsilon}{\varepsilon^2+c^2} = \begin{cases} \sqrt{\frac{c}{\pi a v}}\ln\left(e^\gamma ca/v\right) & (t=0,\ ca\ll 1); \\ \sqrt{\frac{c}{\pi a v}}\frac{1}{(ct)^2}\left[1+\mathcal{O}\left(\frac{1}{ct}\right)\right] & (ct \gg 1). \end{cases} \tag{J30}$$

Here $\gamma = 0.577\ldots$ is Euler's constant, and the last line's asymptotic $ct \gg 1$ result follows from (158). Note that the non-zero result for $D_{\Psi\alpha}(t=0)$ implies by (144) that $\langle H'_B\rangle' \neq 0$, as expected.

Next we consider the correlator $D_{\Psi\Phi_+\alpha_d}(t)$, which is linear in Φ_+ [Fig. 4(g)]; it is non-zero, since the β in Φ_+ can be contracted with that in Ψ_+:

$$D_{\Psi\Phi_+\alpha_d}(t) \equiv \sqrt{2/a}\,\langle \Psi_+(t)\,[\Phi_+(0) - \Phi_+(t)]\,\alpha_d(0)\rangle' \tag{J31}$$

$$\sim \quad \sqrt{\Delta_L/va}\sum_{\bar{k}\bar{k}'\varepsilon\varepsilon'} i\,\Phi_{-\bar{k}',\varepsilon'}A^*_{d,\varepsilon}\langle \beta_{\bar{k}}(t)\left[(\beta^\dagger_{\bar{k}'}\tilde{\alpha}_{\varepsilon'})(0) - (\beta^\dagger_{\bar{k}'}\tilde{\alpha}_{\varepsilon'})(t)\right]\alpha^\dagger_\varepsilon(0)\rangle' \tag{J32}$$

$$\sim \quad -\sqrt{\frac{c}{\pi a v}}\, P\!\!\!\!\int_0^\infty d\varepsilon\, d\varepsilon_{\bar{k}}\, \left(e^{-i\varepsilon_{\bar{k}}t} - e^{-i\varepsilon t}\right)\,\frac{\varepsilon}{\varepsilon^2+c^2}\left[\frac{e^{-|\varepsilon-\varepsilon_{\bar{k}}|a/2v}}{\varepsilon-\varepsilon_{\bar{k}}} + \frac{\pi\varepsilon}{c}\delta(\varepsilon-\varepsilon_{\bar{k}})\right] \tag{J33}$$

$$\sim \quad -\sqrt{\frac{c}{\pi a v}}\left\{ P\!\!\!\!\int_0^\infty d\varepsilon_{\bar{k}}\,\frac{e^{-i\varepsilon_{\bar{k}}t}\,c}{\varepsilon^2_{\bar{k}}+c^2}\left[\frac{\pi}{2} - \frac{\varepsilon_{\bar{k}}}{c}\ln\left|\frac{\varepsilon_{\bar{k}}}{c}\right|\right] - P\!\!\!\!\int_0^\infty d\varepsilon\,\frac{e^{-i\varepsilon t}\,\varepsilon\,\ln|\varepsilon a/2v|}{\varepsilon^2+c^2}\right\} \tag{J34}$$

$$\sim \quad -\sqrt{\frac{\pi}{cav}}\,\frac{1}{(2it)}\left[1 + \frac{2\ln(2cvt^2/a)}{\pi cit}\right] \qquad (ct \gg 1). \tag{J35}$$

The first term of (J34) was obtained from the $e^{-i\varepsilon_k t}$ term of (J33) by doing the ε integral using I_1 of (J13), the second term of (J34) was obtained from the $e^{-i\varepsilon t}$ term of (J33) by doing the $\varepsilon_{\bar{k}}$ integral using $\int_0^\infty d\varepsilon e^{-a\varepsilon}/(\varepsilon - c) = -\ln|ac|$ for $ac \ll 1$. Eq. (J35) follows from (J34) by using (158) for the leading term and I_3 of (J15) for the logarithmic terms. – Remarkably, the $(it)^{-1}$ decay of $D_{\Psi\Phi_+\alpha_d}$ is slower than the $(it)^{-2}$ of $D_{\Psi\alpha_d}$. The reason is that the coefficients $C_{\bar{k}}$ in its $\sum_{\bar{k}} C_{\bar{k}}\langle \beta_{\bar{k}}(t)\beta^\dagger_{\bar{k}}(0)\rangle'$ contraction contain less powers of $\varepsilon_{\bar{k}}$ than the powers of ε contained in the coefficient C_ε arising in the contraction $\sum_\varepsilon C_\varepsilon\langle \tilde{\alpha}_\varepsilon(t)\tilde{\alpha}^\dagger_\varepsilon(0)\rangle'$ in (J29) for $D_{\Psi\alpha_d}$. [A similar observation applies for the β and $\tilde{\alpha}$ contributions to D_Ψ of (J10).]

[24] The only exception to this rule occurs for D_{N+} of (J2), for which two t-to-0 contractions yield $\ln t$; the reason why D_{N_+} is special is that it contains a factor $|A_{-\bar{k},\varepsilon}|^2 \sim (\varepsilon + \bar{k})^{-2}$, which produces an infrared divergence leading to $\ln t$. In contrast, however, the coefficients occuring when $\langle \alpha_d(t)\ldots\ldots\alpha_d(0)\rangle$ is involved are less infrared divergent (since $A_{d,\varepsilon} \to$ const for $\varepsilon \to 0$).

K Coulomb gas representation for D_B

We rederive Oreg and Finkel'stein's [21] exact mapping of the correlator D_B of (174) onto a difference of Coulomb gas partition functions, $D_B = Z_e - Z_o$, and confirm that their treatment of fermionic anti-commutation relations was correct, contrary to recent suggestions in the literature [23].

In this Appendix we write $F_\pm \equiv F_{L/R} \equiv F_\nu$, where $\nu = (L, R) = (+, -)$ (in contrast to our refermionization notation $F_+ \equiv F_R^\dagger F_L$ of Section 10.C.3). The backscattering term H_B of (128), with $\theta_B = 0$ for simplicity, and the $T = 0$, imaginary-time version of the correlator D_B of (174), with $\tau = it$ and $\tau \in [0, \infty]$, then read

$$H_B = \frac{v\lambda_B}{2\pi a} \sum_{\nu=\pm} F_\nu^\dagger F_{-\nu} e^{i\nu c\Phi_+} , \qquad \text{with} \quad c \equiv \sqrt{2g} , \tag{K1}$$

$$D_B(\tau) = \frac{1}{a} \sum_{\nu_0,\nu_\tau=\pm} \langle G_B | e^{(H_{0+}+H_B)\tau} F_{-\nu_\tau} e^{\frac{i}{2}\nu_\tau c\Phi_+} e^{-(H_{0+}+H_B)\tau} F_{\nu_0}^\dagger e^{\frac{i}{2}\nu_0 c\Phi_+} | G_B \rangle \tag{K2}$$

$$= \frac{\sum_{\nu_0,\nu_\tau=\pm} \langle 0_+ | \mathcal{T} \left\{ e^{-\int_0^\infty d\tau' \, H_B(\tau')} F_{-\nu_\tau}(\tau) e^{\frac{i}{2}\nu_\tau c\Phi_+(\tau)} F_{\nu_0}^\dagger(0) e^{\frac{i}{2}\nu_0 c\Phi_+}(0) \right\} | 0_+ \rangle}{a \, \langle 0_+ | \mathcal{T} e^{-\int_0^\infty d\tau' \, H_B(\tau')} | 0_+ \rangle} . \tag{K3}$$

In (K3) we wrote D_B in the $T = 0$, imaginary-time interaction representation [31], in which $\Phi_+(\tau) = e^{H_0\tau}\Phi_+ e^{-H_0\tau}$ and $F_\nu(\tau) = F_\nu$ (from (72), with $1/L$ terms neglected), and $|0_+\rangle$ is the ground state of H_{0+}. Expanding D_B in powers of H_B and keeping only connected terms (since disconnected ones are cancelled by the denominator), we readily obtain

$$D_B(\tau) = \frac{1}{a} \sum_{n=0}^\infty \left(\frac{v\lambda_B}{2\pi a} \right)^n \int_0^\infty d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{n-1}} d\tau_n \sum_{\nu_1\ldots\nu_n,\nu_\tau\nu_0=\pm} D^{\tau_1\ldots\tau_n,\tau}_{\nu_1\ldots\nu_n,\nu_\tau\nu_0} S^{\tau_1\ldots\tau_n,\tau}_{\nu_1\ldots\nu_n,\nu_\tau\nu_0} \tag{K4}$$

$$D^{\tau_1\ldots\tau_n,\tau}_{\nu_1\ldots\nu_n,\nu_\tau\nu_0} \equiv \langle 0_+ | \mathcal{T} \left\{ e^{i\nu_1 c\Phi_+(\tau_1)} \ldots e^{i\nu_n c\Phi_+(\tau_n)} \, e^{\frac{i}{2}\nu_\tau c\Phi_+(\tau)} e^{\frac{i}{2}\nu_0 c\Phi_+(0)} \right\} | 0_+ \rangle \tag{K5}$$

$$\begin{aligned} = \left(\frac{2\pi a}{L} \right)^{\frac{1}{2}Q^2} \exp c^2 \Big\{ & \tfrac{1}{4}\nu_\tau\nu_0 \ln\left(|\tau|/a + 1\right) + \sum_{i=1}^n \tfrac{1}{2}\nu_0\nu_i \ln\left(|\tau_j|/a + 1\right) \\ & + \sum_{i=1}^n \tfrac{1}{2}\nu_\tau\nu_i \ln\left(|\tau - \tau_j|/a + 1\right) + \sum_{i<j}^n \nu_i\nu_j \ln\left(|\tau_i - \tau_j|/a + 1\right) \Big\} \end{aligned} \tag{K6}$$

$$Q \equiv \sum_{j=1}^n \nu_j + \tfrac{1}{2}\nu_\tau + \tfrac{1}{2}\nu_0 \tag{K7}$$

$$S^{\tau_1\ldots\tau_n,\tau}_{\nu_1\ldots\nu_n,\nu_\tau\nu_0} \equiv (-1)^n \langle 0_+ | \mathcal{T} \left\{ \left(F_{\nu_1}^\dagger F_{-\nu_1} \right)(\tau_1) \ldots \left(F_{\nu_n}^\dagger F_{-\nu_n} \right)(\tau_n) \, F_{-\nu_\tau}(\tau) F_{\nu_0}^\dagger(0) \right\} | 0_+ \rangle \tag{K8}$$

$$= (-1)^{N_\tau} \delta_{Q,0} \tag{K9}$$

In (K4) we exploited the fact that all boson fields commute with all Klein factors to factorize each term into two factors, D and S, that depend only on Φ_+'s and F's, respectively. D of (K5) is a time-ordered expectation value of exponentials of free boson fields, which gives (K6) when evaluated using (94) [analogous to our derivation of (96)]. The argument of the exponential in (K6) can be interpreted as the potential energy of a "1-D Coulomb gas of charged particles", interacting with a logarithmic inter-particle potential $c^2\nu_1\nu_2 \ln(|\tau_1 - \tau_2|/a + 1)$, which has two charges $\frac{1}{2}\nu_0$ and $\frac{1}{2}\nu_\tau$ placed at positions 0 and τ, and n further charges $\nu_1, \ldots, \nu_n$ at positions $\tau_j \in (0, \infty)$, where all $\nu \in \pm$. The total charge of the configuration is Q of

(K7), and since the Q-dependent prefactor in (K9) is non-zero in the limit $a/L \to 0$ only if $Q = 0$, only "neutral" configurations of the Coulomb gas contribute.

This also follows from the Klein factor correlator S of (K8), which by inspection is non-zero only if it contains as many $F^\dagger_\pm$ as $F_\pm$ operators, which implies $Q = 0$ (illustrating our comments of Appendix D.1). For all neutral configurations, $S = 1$ or -1 if N_τ is even or odd, respectively, where N_τ is the number of charges ν_i occuring between 0 and τ, i.e. for which $\tau_i \in (0, \tau)$. To see this, one simply has to rearrange the F's under the time-ordering symbol in (K8) until they all "disappear" via $F^\dagger_\nu F_\nu = 1$, and count the number of minus signs produced by anticommuting an F_+ or $F^\dagger_+$ past an F_- or $F^\dagger_-$, as illustrated by the following simple examples:

$$
\begin{aligned}
N_\tau = 0: \quad & S^{\tau_1>\tau}_{+,--} &=& (-1)^1 \langle \Big[F^\dagger_+(\tau_1) F_-(\tau_1) \Big] F_+(\tau)\, F^\dagger_-(0) \rangle = 1 \\
N_\tau = 1: \quad & S^{\tau>\tau_1}_{+,--} &=& (-1)^1 \langle F_+(\tau) \Big[F^\dagger_+(\tau_1) F_-(\tau_1) \Big] F^\dagger_-(0) \rangle = -1 \\
N_\tau = 0: \quad & S^{\tau_1>\tau_2>\tau}_{+-,-+} &=& (-1)^2 \langle \Big[F^\dagger_+(\tau_1) F_-(\tau_1) \Big] \Big[F^\dagger_-(\tau_2) F_+(\tau_2) \Big] F_+(\tau)\, F^\dagger_+(0) \rangle = 1 \\
N_\tau = 1: \quad & S^{\tau_1>\tau>\tau_2}_{+-,-+} &=& (-1)^2 \langle \Big[F^\dagger_+(\tau_1) F_-(\tau_1) \Big] F_+(\tau) \Big[F^\dagger_-(\tau_2) F_+(\tau_2) \Big] F^\dagger_+(0) \rangle = -1
\end{aligned}
$$

It follows that $D_B(\tau) = Z_e(\tau) - Z_o(\tau)$, where Z_e and Z_o contain only configurations with $N_\tau = 1$ and -1, respectively. Thus Z_e and Z_o can be interpreted as the grand-canconical partition functions of a neutral Coulomb gas with fugacity $\frac{v\lambda_B}{2\pi a}$, two charges $\pm 1/2$ at positions 0 and τ, and either an even or an odd number of charges ± 1 between them, respectively (and arbitrarily many charges $\pm$ beyond τ).

This completes the exact mapping of D_B into a Coulomb gas representation derived by Oreg and Finkel'stein [21]. Evidently the minus sign in $Z_e - Z_o$ arose from the anti-commutativity of fermion operators, as emphasized by Oreg and Finkel'stein. Note that their derivation of it using field-theoretic bosonization [21, 25] is rather more involved than ours using constructive bosonization, which illustrates the benefits of using Klein factors. Their conclusions [21] about the asymptotic behavior of $Z_e - Z_o$ for $\tau \to \infty$, and our criticism thereof, are discussed in Section 10.D.3.

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