The hierarchy of Minkowski valuations and the morphometry of cosmic structure

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Abstract. Minkowski valuations provide a unique morphometric description of cosmic structure. We introduce the hierarchy of Minkowski valuations and exemplify their discriminative power using the PSCz galaxy catalogue and clusters of galaxies from simulations.

1 Introduction: The hierarchy of Minkowski valuations

For mining the sky one needs intelligent tools that separate the gold and the smut found within the data. In this spirit, the Minkowski valuations supply a set of robust morphological descriptors capable of uncovering the geometry and connectivity both in point sets and pixelised maps. Mathematically, the Minkowski valuations are extensions of the scalar Minkowski functionals introduced into cosmology by [1] and already applied to a number of cosmological data sets, among them the Abell/ACO cluster catalogue [2], the IRAS 1.2Jy galaxy catalogue [3] and the COBE DMR maps of the cosmic microwave background [4]. From a mathematical point of view, the Minkowski

d	2	3
M_0	area	volume
M_1	circumference	surface
M_2	Euler characteristic	mean curvature
M_3	–	Euler characteristic

Table 1. The geometrical interpretation of the scalar Minkowski functionals M_μ, $\mu = 0, \dots, d$ in two and three dimensions.

valuations map patterns in Euclidean space onto scalars, vectors and tensors. They can be described axiomatically by a number of requirements that set reasonable standards for a morphological description. Roughly speaking, these requirements generalise the notion of the volume as the very basic geometric descriptor. In particular, the Minkowski valuations transform in a

well-defined and simple way, if the pattern under consideration is moved in space. Characterisation theorems [5–7] make sure that only a finite number of independent Minkowski valuations exist for each tensor rank.
The geometrical meaning of the Minkowski valuations is also intuitive, at least if the number of dimensions and the tensor rank are not too large. The scalar Minkowski functionals in two and three dimensions correspond to well-known geometric quantities summarised in Table 1. All of them can be expressed either as volume or surface integrals over the pattern; in the latter case, the surface element is weighted with the local curvatures. – The higher-order Minkowski valuations [8] are spatial moments of the scalar functionals, and generalise these integrals by weighting the volume and surface elements with appropriate powers of the position vector and the normal vector. Thus, the Minkowski valuations of first rank (vectors) are points in space which generalise the concept of a centre of mass, and the second-rank tensors extend the notion of an inertia tensor.

The descriptive power of the Minkowski functionals can be illustrated us-

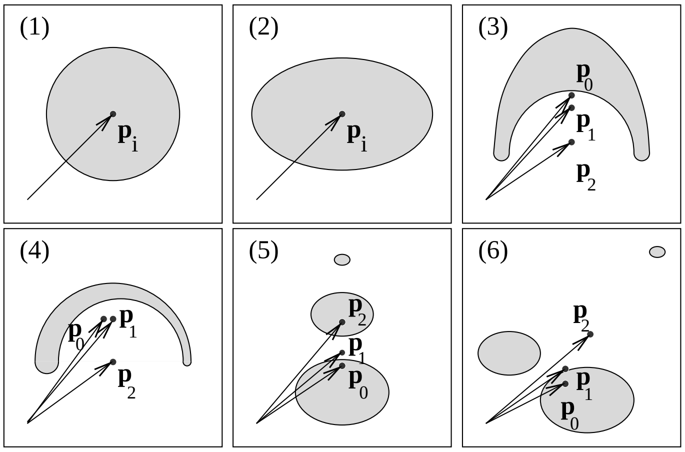

Fig. 1. Examples of the three curvature centroids $\mathbf{p}_0$, $\mathbf{p}_1$, and $\mathbf{p}_2$ in two dimensions for a number of patterns.

ing the simple two-dimensional patterns displayed in Figure 1. For a disk (panel 1), all vectors (more precisely the normalised curvature centroids) coincide with the symmetry centre, and all tensors are isotropic. For an ellipse (panel 2), the tensors become anisotropic indicating the ellipticity and the orientation of the ellipse, but because of the point symmetry, the curvature centroids still coincide. For an axisymmetric pattern (panel 3) they begin to fan out along the symmetry axis. In the generic case (panel 4), the curvature centroids span a triangle; its size can serve as a measure of the anisotropy present within the data. Panels 5 and 6 of Figure 1 illustrate that

the Minkowski valuations are capable of unfolding the geometry of patterns with several components. Here, every clump is weighted according to its morphological properties. Note, that the scalar Minkowski functionals simply add up for the various individual subclumps and fail to capture their relation.
In summary, we can say that the Minkowski valuations are sensitive to the topology, geometry and symmetry of a pattern. This makes them suitable for a number of cosmological applications, two of which we shall study in the following.

2 Applications I: Fluctuations within the PSCz survey

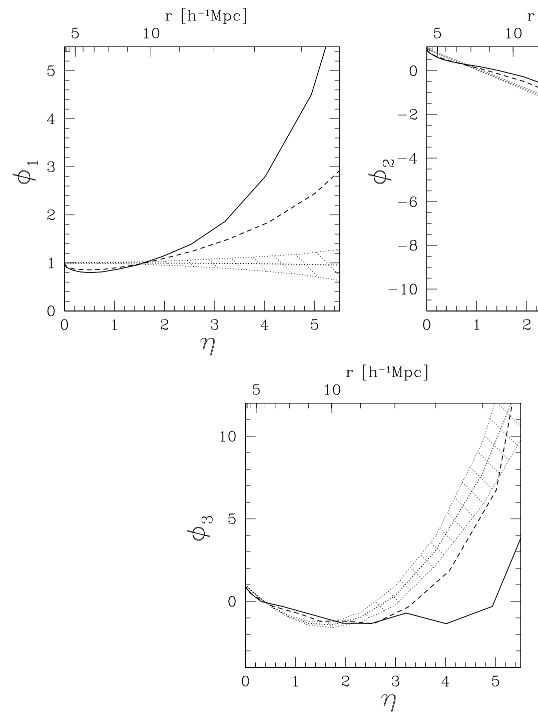

Fig. 2. The scalar Minkowski functionals of a volume-limited subsample of the PSCz survey with $100h^{-1}Mpc$ depth. We show three rescaled Minkowski functionals, namely the surface area ϕ_1, the mean integrated curvature ϕ_2, and the Euler characteristic ϕ_3, versus the radius r of the Boolean grains. The southern (solid line) and northern (dashed line) part of the catalogue are considered separately. In comparison to shaded areas indicating the average and standard deviation of a binomial process the differences between the two parts of the PSCz survey are a lot larger. Obviously, there are significant morphological fluctuations between the southern and the northern part.

Several large galaxy catalogues are expected to be released in the next few years. One of the main physical questions to be answered with the aid of such catalogues is the problem of cosmic variance: What depth must a survey reach to be a fair sample of the Universe?
Within the IRAS 1.2Jy catalogue large morphological fluctuations were found between the northern and the southern part [3]. Furthermore, it seemed as if the simulations of that time could not reproduce them. A follow-up study [10] uses a deeper IRAS catalogue, the PSCz survey [9]. Since geometric descriptors are always sensitive to the point density of a sample, we consider volume-limited subsamples of different depths. The galaxy positions in redshift space are interpreted as a realization of a three-dimensional point process and can be compared with simple models, e.g. the binomial process where no spatial clustering is present [11]. To make the data accessible to a meaningful analysis with the Minkowski functionals, we decorate the points with balls ("Boolean grains") of radius r and calculate the scalar Minkowski functionals of the union set of all these balls. Varying the size of the Boolean grains, we obtain the Minkowski functionals as functions of the radius r. We consider the northern and the southern part separately and use the difference between the two as a rough estimate of the morphological fluctuations on scales of the order of the survey depth. Results can be seen in Figure 2.
Obviously, large fluctuations between the two parts of the PSCz survey exist. In particular, the southern part is much more clustered than the northern part. So the results obtained from the IRAS 1.2Jy catalogue [3] are confirmed by the PSCz survey. Note that the differences between the two parts only show up at radii beyond $10h^{-1}Mpc$. The binomial process with the same number density is completely incompatible with both the north and the south. Further analysis [10] shows that these significant fluctuations of the Minkowski functionals are not parallelled by two-point measures such as σ^2, the fluctuations of galaxy number counts in spheres. The reason is that Minkowski functionals contain correlations of any order.
In [10] we discuss larger volumes, the question whether the fluctuations observed can be understand by means of present-day simulations, and analytical models.

3 Applications II: The morphological evolution of galaxy clusters

Another example of cosmological data sets are X-ray maps of galaxy clusters. Since the formation of galaxy clusters depends on the cosmological Friedmann-Lemaître background, their abundance and in particular their inner state can help us to constrain the values of the cosmological parameters [12]. Thus, it is important to characterize the substructure within galaxy clusters quantitatively.
Here, we use simulated galaxy clusters [13] to probe the average cluster mor-

phology in different background models. To apply Minkowski valuations, here the scalar Minkowski functionals and their vector-valued counterparts, we smooth the pixelised maps slightly and investigate the excursion sets, which comprise all pixels above a certain density threshold. To get a number of simple descriptors, we sum up the Minkowki valuations over several surface brightness thresholds. So we get robust structure functions, which also distinguish between different types of substructure. We consider the clumpiness C, the degree of asymmetry A_1 and the amount of morphological shift in space S_1.

In Figure 3, we show the morphological evolution of large simulated cluster samples traced by the structure functions. As background models, we consider two models with a high cosmological density parameter Ω_m (CHDM and CDM) and a low-Ω_m model (ΛCDM).

The results show that indeed the averaged morphology of galaxy clusters is

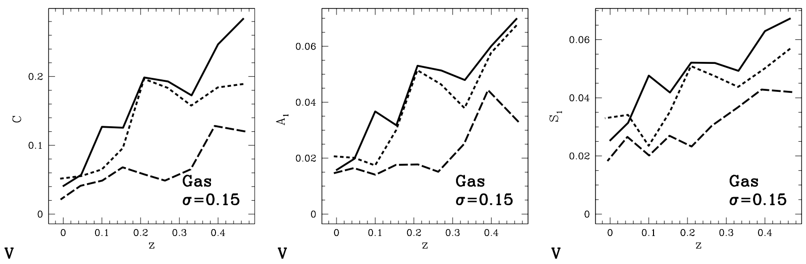

Fig. 3. The averaged morphological evolution of galaxy clusters for three cosmological models: CDM: solid line; LDM: long dashed line; CHDM: short dashed line. We show the three structure functions clumpiness C, asymmetry A_1 and shift of morphological properties S_1 as a function of redshift. Low values of the structure functions indicate symmetric and substructure-poor clusters. The smoothing length applied was $\sigma = 0.15h^{-1}Mpc$.

much more relaxed and symmetric in the low-Ω model. Note, that the morphological evolution seems to be even more interesting than the morphological state at redshift $z = 0$. In [14] we show, that the aspects of substructure featured by the Minkwoski valuations are correlated to the intrinsic dynamical state of a cluster.

4 Summary and Outlook

The hierarchy of Minkowski valuations supplies a number of sensitive tools to unfold the structure of spatial patterns. Their application to data can draw from several advantages such as the intuitive geometrical meanings of the Minkowski valuations, their uniqueness and completeness in terms of an axiomatical description, and their discriminative power. The examples shown

here illustrate that Minkowski valuations reveal morphological fluctuations within galaxy catalogues and diagnose the inner state of individual galaxy clusters through a quantitative analysis of the substructure.
Let us finally note that the Minkowski valuations can be evaluated with moderate computational requirements for the upcoming large-scale structure surveys and CMB maps. Hence, we recommend them as standard equipment for every sky miner.

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