Non-abelian vs. non-commutative

hep-th/0503104

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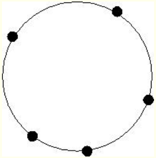
Outline

NC Geometry from open strings

With constant $B_{\mu\nu}$:



$$\sim \int d(\text{moduli}) \langle e^{ik_1 X(x_1)} \cdots e^{ik_n X(x_n)} \rangle \times$$

$$\times\, e^{ik_1 \theta k_2} e^{ik_2 \theta k_3} \cdots e^{ik_n \theta k_1}$$

Replace all products by $*$-products in the effective action

$$(f * g)(x) = e^{\pi \theta^{mn} \partial_{x^m} \partial_{y^n}} f(x) g(y) \Big|_{y \to x}$$

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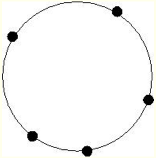

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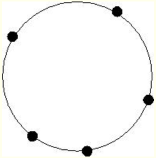

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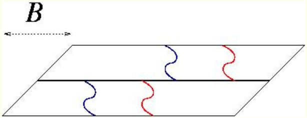
- Under T-duality, momentum becomes winding. The tilted torus illustrates the shift proportional to the winding:



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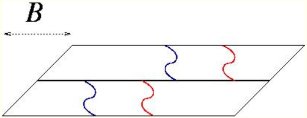
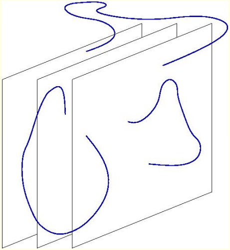


D-brane actions



- For a stack of N D-branes, open strings have labels S_{ab} indicating the branes the two ends are attached to
- Low energy action: $U(N)$ Yang-Mills with adjoined scalars X^i_{ab}
- Diagonal entries X^i_{aa} denote the position of brane a in the transverse direction
- Thus $\frac{1}{N}\mathrm{tr}(X^i)$ is the center of mass position
- The centre of mass kinematics decouples. CM-frame: $\tilde{X}^i = X^i - \frac{1}{N}\mathrm{tr}(X^i)\mathbb{1}$

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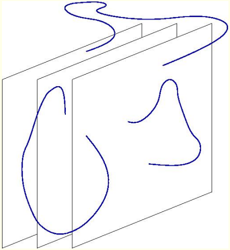

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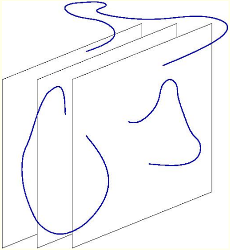

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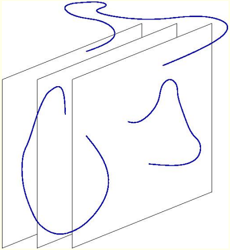

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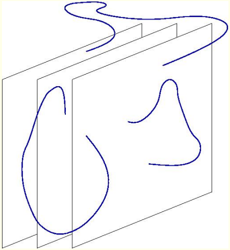

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Non-commutative YM

- Field strength

$$F_{mn} = \partial_m A_m - \partial_n A_m + A_m * A_n - A_n * A_m = \partial_m A_m - \partial_n A_m + [A_m, A_n]_*$$

- Even the $U(1)$ theory is now interacting.
- In terms of Lie algebra generators T^a:

$$[A_m, A_n]_* = \frac{1}{2}\{A_m^a, A_n^b\}_* [T^a, T^b] + \frac{1}{2}[A_m^a, A_n^b]_* \{T^a, T^b\}$$

- $\{T^a, T^b\}$ is not defined in the abstract Lie algebra.
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Closure of the algebra

- For representation matrices, in general the algebra is not closed:

$$\{\rho(T^a), \rho(T^b)\} = \rho(???)$$

- It only closes for structure groups $\otimes_\alpha U(N_\alpha)$ as their adjoint representation consist of *all* hermitian (block-)matrices.
- What about other groups: $SU(N)$ or $SO(N)$ or $Sp(N)$ or exceptional groups?
- What about the center of mass system $SU(N)$? (tangential $B_{\mu\nu}$ preserves translation invariance!)

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Not the center of mass

In the NC theory, we cannot just take the matrix trace:

- As $\{\mathbf{1}, \mathbf{1}\}_* \neq 0$, the matrix trace is an interacting degree of freedom.
- As $\{1, \tilde{X}^i\}_* \neq 0$ and $\mathrm{tr}(\{\tilde{X}^i, \tilde{X}^j\}_*) \neq 0$, the matrix trace does not decouple from the internal brane dynamics.

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Non-commutative torus

- Compactify non-commutative coordinates.
- Simplest case: Non-commutative torus. x^1 and x^2 periodic with 2π.
- Fourier-decompose all fields

$$f(x^1, x^2) = \sum_{mn} f_{mn} e^{inx^1} e^{imx^2} = \sum_{mn} f_{mn} U^m V^n$$

- Non-commutative algebra is expanded in terms of $U = e^{ix^1}$ and $V = e^{ix^2}$:

$$U * V = V * U e^{-2\pi i \theta}$$

- θ is now dimensionless. We are interested in the case of rational $\theta = p/q$: Translations commensurate with periodicity of torus.

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The central $U(1)$

- From $UV = VUe^{-2\pi ip/q}$ we make the crucial observation that U^q and V^q are central, they $*$-commute with all functions.
- Therefore we can define the proper $*$-analogue of the matrix trace:
$$\mathrm{tr}_*(X^i) = \sum_{mn} \mathrm{tr}(X^i_{qm,qn})\, U^{qm} V^{qn}$$
- This commutes with all other fields both in the Lie algebra and in the $*$-product sense: $[\mathrm{tr}_*(X^i)\mathbb{1}, X^j]_* = 0$.
- Similarly, one checks $\mathrm{tr}_*([X^i, X^j]_*) = 0$.
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The non-commutative $SU(N)$

- The remainder $\tilde{X}^i = X^i - \frac{1}{N}\mathrm{tr}_*(X^i)\mathbb{1}$ is the NC analogue of the $SU(N)$ center of mass theory.
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- $\tilde{X}^i$ are *not* $SU(N)$ valued functions but the restriction from $U(N)$ is non-local.
- This is a restriction of the gauge group rather than the structure group.
- This resonates well with the NC shift of focus from points to functions on a space.
- It is tempting to define the NC version of $SU(N)$ as the image of $[\cdot, \cdot]_*$ in more general situations.

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The (other) T-dual picture

The D1 for $\theta = 1/3$

- The D2-brane with $\theta = p/q$ is mapped to a D1 brane that wraps p and q times the cycles of a dual torus.
- The D1 "comes around q times".
- Only every qth mode describes the collective motion of the total D1.

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The (other) T-dual picture

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- The D2-brane with $\theta = p/q$ is mapped to a D1 brane that wraps p and q times the cycles of a dual torus.
- The D1 “comes around q times”.
- Only every qth mode describes the collective motion of the total D1.

The Morita-equivalent picture

- The operators U and V with $UV = VUe^{-2\pi ip/q}$ have a commutative, non-abelian representation as well:

$$U = \begin{pmatrix} 1 & & & \\ & e^{2\pi i \frac{p}{q}} & & \\ & & \ddots & \\ & & & e^{2\pi i (q-1)\frac{p}{q}} \end{pmatrix} e^{iy^1/q}, \quad V = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 1 & & & \end{pmatrix} e^{iy^2/q}$$

  with commuting coordinates y^1 and y^2.
- These generate all hermitian $q \times q$ matrices on a larger torus.
- Together with the original $U(N)$ gauge group, the theory becomes a commutative $U(N) \otimes U(q) = U(qN)$ theory.
- Our center of mass $U(1)$ is the center of this gauge group and the internal dynamics is in $S(U(N) \otimes U(q))$ and tr_* is the trace in $U(qN)$.

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A related problem: $SU(N)$ for the NC deformed plane

- In general, (varying) θ can be treated as a formal parameter in formal power series.
- Idea: The sets of commutative and non-commutative gauge orbits are equivalent.
- There is a connection dependant map from a commutative theory to a theory where gauge transformations $\Lambda_\alpha(a_\mu)$ act via $*$-products.
- SW-condition: $A(a) + \partial\Lambda_\alpha(a) + [A(a), \Lambda_\alpha(a)]_* = A(a + \partial\alpha + [a, \alpha])$.
- We can get a grip on this relation by using the Ansatz.

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\begin{aligned}
A(a) &= a + O(\theta) \\
\Lambda_\alpha(a) &= \alpha + O(\theta)
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The non-commutative gauge group

- The fact that the map depends on the commutative connection implies for the group law in the gauge group:

$$G_{g_1}(ag_2) * G_{g_2}(a) = G_{g_1 g_2}(a)$$

- Infinitessimally

$$[\Lambda_\alpha(a), \Lambda_\beta(a)]_{NC} = [\Lambda_\alpha(a), \Lambda_\beta(a)]_* + \delta_\alpha \Lambda_\beta(a) - \delta_\beta \Lambda_\alpha(a) = \Lambda_{[\alpha,\beta]}(a)$$

- Again, this is a non-local modification of the gauge group rather than the structure group.
- One can restrict the map $A(a)$ to $a \in SU(N)$
- Here as well, the NC $SU(N)$ is defined in terms of a new "commutator".
- This so far only generalises the tracelessness condition. Possibly the generalization of the trace itself exists.

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$SU(N)$ in two NC theories

- Both definitions define the restriction in terms of the gauge group.
- Both use an expansion (grading) for their definition: U^n and V^m *viz.* θ (or $\hbar$).
- There is a difference in the number of local degrees of freedom: $N^2 - \frac{1}{q}$ vs. $N^2 - 1$.
- The definition in terms of tr_* works directly only for $T_{\frac{p}{q}}$ but it sees "global effects" like $[U^q, V]_* = [U, V^q]_* = 0$.

Prospect: Apply the non-commutative vs. non-abelian strategy to more problems. E.g. non-abelian DBI actions, NC string compactification, non-abelian gerbes and membranes, etc.

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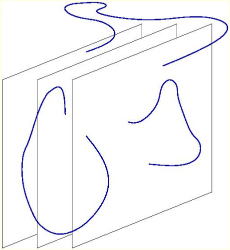
Yang-Mills and Dirac-Born-Infeld



The description in terms of Yang-Mills-Theory is good for $\alpha' \to 0$. More precisely

- $\alpha' F \ll 1$
- $\alpha' p^2 \ll 1$. This means, it is exact only for $\partial F = 0$.

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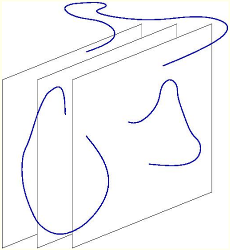

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Non-abelian DBI

In the non-abelian version F is not gauge invariant. The condition $\partial F = 0$ has to be replaced by $DF = 0$.

$$\Longrightarrow DDF = 0 \Longrightarrow [D, D]F = 0 \Longrightarrow [F, F] = 0$$

We are effectively back in the abelian case. This is the symmetrized trace prescription.
Of course, one can in principle compute corrections in $\alpha' p^2$.

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