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Allowing Spontaneous Breaking of Diffeomorphism Invariance

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G. Policastro[1] RCH[2]

[1]CNRS Paris

[2]LMU München

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Quantum Gravity

- Ultimate pinnacle of unification program
- Naively: General Relativity is not renormalisable
- Fundamentally: Space-time, the stage of quantum physics is itself dynamical and might dissolve at small length scales
- Two major approaches: String Theory and Canonical/Loop Quantum Gravity

Strings

The big player:

- Origin in particle physics
- Scattering amplitudes, S-matrix
- Space-time is a secondary concept

Not just quantum gravity:

- All interactions including gauge fields
- Consistent only in higher dimensions
- More objects: Branes
- Intricate web of mathematical relations
- Usually formulated perturbatively around simple backgrounds

Loop Quantum Gravity

The Gallic village

- Origin in General Relativity
- Focus on geometric description: (Generalized) Wilson Loops as gauge invariant variables.
- Canonical treatment of *new variables* for pure 4d gravity possibly including matter
- No background assumed, operator algebraic methods.

But:

- No semi-classical expansion
- Flat space physics cannot be recovered (yet?)

The bosonic string as a testing case

As the starting points and mathematical descriptions are very different, there is only little mutual understanding.
Thomas Thiemann proposed to study the world-sheet theory of the bosonic string (the $U(1)^d$ current algebra) using methods from both camps.

- Simple, 1+1 dimensional field theory
- Quadratic, thus solvable
- Classically diffeomorphism invariant

Thiemann's results

Using an approach modelled on Loop Quantum Gravity, Thiemann presented a quantisation with unexpected features including

- No conformal anomaly ("central charge")
- No critical dimension
- No tachyon even in the bosonic theory

The bosonic string

The string can be described as a map $X\colon \mathbb{R}\times S^1 \to \mathbb{R}^d$ with action

$$\mathcal{S} = \int d\tau \int_0^{2\pi} dx\ (-\partial_\tau X \partial_\tau X + \partial_x X \partial_x X)$$

Poisson brackets: $\{X(x,0), \partial_\tau X(x',0)\} = \delta(x-x')$
Define currents $j^\pm(x) = \partial_\tau X(x,0) \pm \partial_x X(x,0)$ to obtain decoupled systems $\{j^+, j^-\} = 0$ and

$$\{j^\pm(x), j^\pm(x')\} = 2\partial_x \delta(x-x')$$

Smear with real functions $f\colon S^1 \to \mathbb{R}$ to obtain regular

$$j[f] = \int_0^{2\pi} dx f(x) j(x).$$

String theory conventionally use a Fourier basis $f(x) = \sum_n f_n e^{inx}$, for Thiemann characteristic functions $f(x) = \chi_{[a,b]}$ model LQG holonomies.

The symplectic structure

The Poisson brackets yield

$$\sigma(f,g) = \{j[f], j[g]\} = \int_0^{2\pi} dx\, f(x)\partial_x g(x) = \int f dg$$

This parametrized string is invariant under reparametrizations of the circle:
Diffeomorphisms $S : S^1 \to S^1$ act as “gauge transformations”
$S : f(x) \mapsto (Sf)(x) = f(S(x))$ and leave the symplectic structure invariant:

$$\sigma(Sf, Sg) = \sigma(f,g)$$

Interlude: Quantization

It is useful to divide the quantization procedure into two parts:

- Promotion of the classical, commuting observables to a quantum (C*) algebra $(\mathcal{A}, \|\cdot\|, *)$ with commutation relations.
- Finding representations as operators on Hilbert spaces.

Stone von Neumann Theorem: In quantum mechanics there is only one representation (up to unitary equivalence, i.e. changes of basis) of the canonical commutation relations. In quantum field theory, typically, there are several inequivalent representations ($\rightarrow$ super-selection sectors).

The GNS construction

Given a C*-algebra and an expectation value functional (“state”) $\omega\colon \mathcal{A} \to \mathbb{C}$ with $\omega(\mathbf{1}) = \mathbf{1}$ and $\omega(A^*A) \geq 0$ one can construct a Hilbert space on which the observables act as operators:

- Define the ideal $\mathcal{J} = \{A \in \mathcal{A} | \omega(A^*A) = 0\}$
- As a vector space $\mathcal{H} = \mathcal{A}/\mathcal{J}$
- There is a natural action $\rho(A)|B\rangle = |AB\rangle$.
- The scalar product is given in terms of $\langle A|B\rangle = \omega(A^*B)$

Quantum mechanics: Harmonic Oscillator

From $[x, p] = i\hbar$ it follows that not both x and p can be bounded operators. Thus it is more convenient to deal with Weyl operators

$$W(z) = W(u + iv) = e^{i(ux + vp)}$$

Analogous: Consider symplectomorphisms instead of their generators. x and p can be recovered as derivatives of $W(z)$ at $z = 0$.

CCR now read $W(z_1)W(z_2) = e^{\frac{i}{2}\mathfrak{Im}(z_1\bar{z_2})}W(z_1 + z_2)$. Ground state of harmonic oscillator is the Fock state $\omega_F(W(z)) = e^{-\frac{1}{4}|z|^2}$

Weyl operators for the string

We can copy this procedure for the string and define operators $W(f)$ that obey

$$W(f)W(g) = e^{\frac{i}{2}\sigma(f,g)}W(f+g).$$

As the symplectic structure σ is invariant under diffeomorphisms, these can be promoted to automorphisms of the quantum algebra:

$$\alpha_S(W(f)) = W(Sf)$$

These obey $\alpha_{S_1} \circ \alpha_{S_2} = \alpha_{S_1 \circ S_2}$.

Implementing the symmetries

As physics is invariant under reparametrizations S of the circle, these have to implemented as unitary operators $U(S)$ on the Hilbert space:

$$U(S)^{-1}\rho(W(f))U(S) = \rho(W(S^{-1}f))$$

This would lead to the physical Hilbert space as the invariant subspace. If the GNS state ω is invariant $\omega \circ \alpha_S = \omega$ the existence of the unitary operators is automatic: $U(S)|a\rangle = |\alpha_S(a)\rangle$
If the state is not invariant, these operators fail to be unitary.
There might be other unitary implementers, but in general they do not obey the group property $U(S_1)U(S_2) = U(S_1 \circ S_2)$.

Gupta Bleuler and the Fock state for the string

To write down the ground state for the string, we have to introduce extra structure that encodes the difference between positive and negative energies (particles and anti-particles).
This is done with a complex structure $J : f \mapsto Jf$ that obeys $J^2 = -\mathbf{1}$, $\sigma(Jf, g) = -\sigma(f, Jg)$ and $\sigma(f, Jf) \geq 0$.
In our case we take $(Jf)(x) = \frac{1}{2\pi} \int_0^{2\pi} dy f(y) \cot \frac{1}{2}(y - x)$.
After a Fourier transform one sees that this multiplies positive frequency modes by i and negative frequency modes by $(-i)$.
With J we can define the Fock state

$$\omega_F(W(f)) = e^{-\frac{1}{4}\sigma(f, Jf)}$$

This state is only invariant under diffeomorphisms S that commute with J.

Properties of ω_F

- This state is differentiable with respect to f. Thus the field and creation/annihilation operators can exist in the Hilbert space.
- For $S = e^A$, the problematic part of A is $A_2 = \frac{1}{2}(A + JAJ)$.
- $U(e^{A_2})$ can still be defined but does not obey the group property: There is a phase that infinitesimally reads $[dU(A), dU(B)] = dU([A, B]) + \mathrm{Tr}([A_2, B_2])\mathbf{1}$
- This is the manifestation of the conformal anomaly (central charge). The invariant Hilbert space is only $\{0\}$.
- This can be repaired by tensoring 26 of these X-theories with a bc-Fadeev-Popov ghost system which has central charge -26. This is the technical reason that the bosonic string lives in 1+25 dimensions.

The Polymer State

Thiemann, motivated by the usual choice in loop quantum gravity (the "Ashtekar-Lewandowsky-measure"), however uses the state

$$\omega_P(W(f)) = \begin{cases} 1 & \text{if } f = 0 \\ 0 & \text{else} \end{cases}$$

- This state is trivially invariant under all reparametrizations of the circle, there is no anomaly and all dimensions are critical.
- However, the Hilbert space representation is not continuous, thus the field operators cannot be defined as derivatives.
- Furthermore the vectors $|W(f)\rangle$ become orthogonal for different f's, there is no overlap.

The polymer state of the harmonic oscillator

To understand the physical consequences, let us treat the harmonic oscillator analogously $\omega_P(W(z)) = \begin{cases} 1 & \text{if } z = 0 \\ 0 & \text{else} \end{cases}$

- The Hilbert space is the non-separable Bohr compactification of $\mathbb{C}$, wave functions vanish nearly everywhere.
- The time evolution $U(t)|W(z)\rangle = |W(e^{it}z)\rangle$ has only the ground state as eigenstate.
- The Hamiltonian is not defined, but we can "approximate" it by the hermitian
$$H_\epsilon = \frac{U(\epsilon) - U(-\epsilon)}{2i\epsilon}$$
- For generic $\epsilon > 0$ and wave functions $|\phi\rangle$ this has expectation values $\langle\phi|H_\epsilon|\phi\rangle = 0$ and $\langle\phi|H_\epsilon^2|\phi\rangle \sim \frac{1}{\epsilon^2}$.
- Thus the spectrum of H_ϵ is unbounded from below and all states contain contributions of arbitrarily large energies.

Conclusions

- We treated the usual Fock and the Loop inspired quantization of the string in a common framework.
- The difference lays in the choice of state.
- Diffeomorphisms can be implemented in the Fock space but lead to a central charge that is only cancelled in 1+25 dimensions.
- The polymer state is always invariant under diffeomorphisms but it is non continuous.
- Thus the field and creation/annihilation operators do not exist in the too singular Hilbert space.
- The case of the harmonic oscillator shows that this state is so singular that is has unphysical properties.