Self-adjoint extentions of momentum on the interval

Notes for Robert Helling's MQM lecture of October 17, 2008

Let A be a symmetric operator with domain $\mathcal{D}(A)$ dense in a Hilbert space $\mathcal{H}$. Then the domain of the adjoint is $\mathcal{D}(A^*) = \{w \in \mathcal{H}\colon v \mapsto \langle w, Av\rangle \text{ bounded for all } v \in \mathcal{D}(A)\}$. Symmetry implies $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ but eventually we want to obtain a self-adjoint operator, that is $\mathcal{D}(A) = \mathcal{D}(A^*)$. To this end, we have to enlarge $\mathcal{D}(A)$ in $\mathcal{D}(A^*)$. The value of A on the additional vectors is then determined by the requirement of symmetry.

When we include more vectors in $\mathcal{D}(A)$, the condition of boundedness in the definition of $\mathcal{D}(A^*)$ becomes stricter, therefore we expect $\mathcal{D}(A^*)$ to shrink at the same time.

We want to discuss this procedure in the case of the momentum operator on the interval. Thus we take $\mathcal{H} = L_2([0,2\pi])$ and $p = i\frac{d}{dx}$ the usual momentum operator in position representation. For its domain of definition, our discussion of the uncertainty principle had let us to consider only functions that vanish at the ends of the interval.

Here, we don't want to be concerned with the problem that the derivative might not exist. In the lecture, I therefore suggested that p should only act on differentiable functions. Unfortunately, this set is too small and we should better generalise the derivative a bit. So we better define $C_a := \{f \in \mathcal{H} | f \text{absolutely continious with derivative in } \mathcal{H}\} = 0$ (see `http://en.wikipedia.org/wiki/Absolutely_continuous`) and take $\mathcal{D}(p) = \{f \in C_a | f(0) = f(2\pi) = 0\}$. But the question of differentiability is not really our concern, it is the boundary condition.

Integration by parts shows that p is symmetric:

$$\langle pf, g\rangle = \int_0^{2\pi} dx\, \overline{if'(x)}g(x) = \int_0^{2\pi} dx\, \overline{f(x)} ig'(x) - i\overline{f}(x)g(x)\Big|_0^{2\pi} = \langle f, pg\rangle - i\overline{f}(x)g(x)\Big|_0^{2\pi},$$

as for $g \in \mathcal{D}(p)$ the last term vanishes. This happens without any assumption on the boundary values of f. Therefore, we find that for the adjoint we have at least $C_a \subset \mathcal{D}(p^*)$. For the choice of C_a above we have here in fact equality.

Our next observation is that on C_a, the operator p^* has eigenvalues $\pm i$, that is the differential equation $(p^* f)(x) = if'(x) = \pm i f(x)$ has solutions $f(x) = ce^{\pm x}$ for $c \in \mathbb{C}$. Self-adjoint operators have a real spectrum. Thus these imaginary eigenvalues are at odds with self-adjointness. It is not hard to see that here in fact $\mathcal{D}(p^*)/\mathcal{D}(p)$ is two dimensional and thus the two imaginary eigen-spaces span the difference.

This is true in general: For symmetric A one finds $\mathcal{D}(A^*) = \mathcal{D}(A) \oplus \mathcal{D}_+ \oplus \mathcal{D}_-$ where $\mathcal{D}_\pm = \ker(A^* \mp i)$ are the eigenspaces of eigenvalues $\pm i$. The dimensions $n_\pm = \dim \mathcal{D}_\pm$ are called "deficiency indices" and there exists a self-adjoint extension of A if and only if $n_+ = n_-$.

Our aim is to find a subspace $\mathcal{D}_e \subset \mathcal{D}_+ \oplus \mathcal{D}_-$ such that A is self-adjoint on $\mathcal{D}(A) \oplus \mathcal{D}_e$. Since there should be no imaginary numbers in the spectrum we need $\mathcal{D}_e \cap D_\pm = \{0\}$.

Hence, the non-zero elements of $\mathcal{D}_e$ are of the form $w_+ + w_-$ with non-zero $w_\pm \in \mathcal{D}_\pm$. For each w_+ there can only be one w_- with $w_+ + w_- \in \mathcal{D}_e$ since otherwise the difference would also both be in $\mathcal{D}_e$ and in $\mathcal{D}_-$. Thus there is a linear function $S\colon \mathcal{D}_+ \to \mathcal{D}_-$ such that only elements $w_+ + Sw_+$ are in $\mathcal{D}_e$. A similar argument shows that S has to be injective. Indeed, one finds that S has to be bijective for A self-adjoint on $\mathcal{D}(A) \oplus \mathcal{D}_e$. Let us check the symmetry of A^* on $\mathcal{D}_e$:

$$\begin{aligned}&\langle A^*(v_+ + Sv_+), w_+ + Sw_+\rangle - \langle v_+ + Sv_+, A^*(w_+ + Sw_+)\rangle \\ &\quad = \langle (iv_+ - iSv_+, w_+ + Sw_+\rangle - \langle v_+ + Sv_+, iw_+ - iSw_+\rangle \\ &\quad = -2i(\langle v_+, w_+\rangle - \langle Sv_+, Sv_+\rangle)\end{aligned}$$

Therefore, S has to be an isometry in order for A to have a chance to be self-adjoint. This condition is in fact also sufficient: For $n_+ = n_-$, each choice of an isometry $S : \mathcal{D}_+ \to \mathcal{D}_-$ determines a domain $\mathcal{D}(A) \oplus \{w_+ + Sw_+ | w_+ \in \mathcal{D}_+\}$ on which $A = A^*$ is a self-adjoint operator.

Let's see how this works in our example. We had $\mathcal{D}_\pm = \{x \mapsto ce^{\pm x} | c \in \mathbb{C}\}$ both one dimensional. Therefore, the freedom in the choice of S is a phase $e^{i\alpha}$:

$$S(x \mapsto e^x) = x \mapsto e^{i\alpha} e^{2\pi - x}$$

for $\alpha \in [0, 2\pi)$. Hence all $f \in \mathcal{D}_e$ are multiples of $x \mapsto e^x + e^{i\alpha} e^{2\pi - x}$. All these have

$$\left|\frac{f(0)}{f(2\pi)}\right| = \frac{1 + e^{2\pi + i\alpha}}{e^{2\pi} + e^{i\alpha}} \cdot \frac{1 + e^{2\pi - i\alpha}}{e^{2\pi} + e^{-i\alpha}} = \frac{1 + e^{2\pi + i\alpha} + e^{2\pi - i\alpha} + e^{4\pi^2}}{e^{4\pi^2} + e^{2\pi - i\alpha} + e^{2\pi + i\alpha} + 1} = 1$$

and thus $f(0) = e^{i\beta} f(2\pi)$ for some phase $e^{i\beta}$: They are quasi-periodic.

We have found that p is self-adjoint on quasi-periodic functions with fixed twist $e^{i\beta}$. Note well that this choice of phase has observable consequences since this p has eigenvalues $\mathbb{Z} + \beta/2\pi$ which are the possible outcomes of measurements of the observable p.

We will later see that only self-adjoint operators A can be integrated to unitary operators $U(t)$ that informally have the form $U(t) = e^{itA}$. We have seen that if A is symmetric but not unitary the adjoint A^* has eigenvectors to the eigenvalues $\pm i$. Acting on those such a $U(t)$ would not be unitary and would in fact change the norm. For the momentum operator p, $U(t)$ is the translation operator $(U(t)f)(x) = f(x - t)$. This obviously does not make sense unitarily for functions on the interval $[0, 2\pi]$. Only if we glue together the ends via $f(0) = e^{i\beta} f(2\pi)$ such a shift operator can make sense (taking into account the twist as $(U(t)f)(x) = e^{ik\beta} f(x - t + 2k\pi)$ for the integer k such that $x - t + 2k\pi \in [0, 2\pi]$).

A physical interpretation of the twist $e^{i\beta}$ can be given in terms of the Aharonov-Bohm effect: The interval with quasi-periodic boundary conditions can be identified with a circle S^1. Assume there is now a magnetic field B with flux $\beta = \int_\Sigma B$ through a surface bounded by the circle $\partial\Sigma = S^1$ such that B vanishes in a neighbourhood of S^1. Then locally, the vector potential $\vec{A}$ can be gauged away as $\vec{A} + \vec{\nabla}\Lambda = 0$. Globally, however, going around the circle we have (in a somewhat symbolic notation)

$$\Lambda(2\pi) - \Lambda(0) = \int_0^{2\pi} \vec{\nabla}\Lambda \cdot d\vec{s} = -\int_0^{2\pi} \vec{A} \cdot d\vec{x} = -\int_{S^1} A = \int_\Sigma B = -\beta$$

However, under such a gauge transformation, the wave function $f(x)$ is transformed to $e^{i\Lambda(x)} f(x)$. Thus, we see that the twist $e^{i\beta}$ can be viewed as the global obstruction to gauge away the effect of the magnetic field.

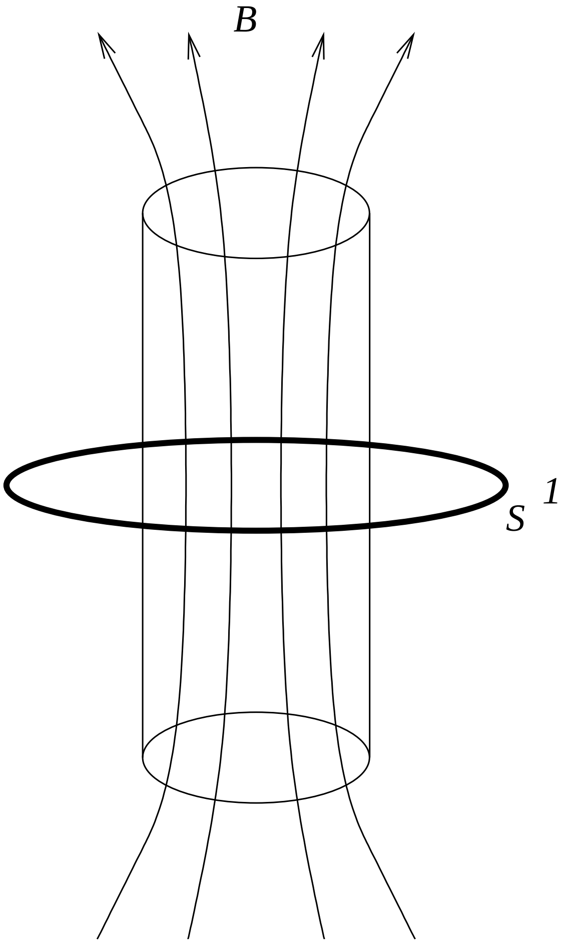

Fig. 1: A particle on a circe S^1 around a solenoid