D-Geometry

Using Quantum Field Theory to Define Quantum Space-Time

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In the twentieth century, physics underwent two major revolutions that deeply changed our understanding of the concepts of space and time: General Relativity and Quantum Theory.

General Relativity tells us that space-time is not merely a stage for physical processes but it is dynamical by itself. We perceive its curvature as gravity. If an object of mass M is very densely compressed the resulting curvature is so large that not even light can escape. A horizon forms at a radius $R = 2MG/c^4$ (where G is Newton's constant of gravity and c is the speed of light). The region behind the horizon is causally decoupled from the rest of the world and cannot influence the outside. In some sense it is "gone".

While originally General Relativity describes physics at large (interstellar to cosmological) scales, Quantum Theoretical concepts are needed to understand the world at very short sub-atomic distance scales. One of the main lessons is that at those scales there is no longer a fundamental distinction between particles and waves. As a consequence, it is no longer possible to simultaneously determine the position and the momentum of a particle. For example, a massless particle of energy $E = pc$ shows an uncertainty of

$$\Delta x \, \Delta E \geq \hbar/c.$$

As the position of the particle is not defined more precisely than Δx we cannot use a particle of energy ΔE as a probe for structures smaller than $\frac{\hbar}{c\Delta E}$. Thus higher and higher energies are required to resolve smaller and smaller structures.

But combining General Relativity and quantum theory leads us to a puzzle: Using $E = Mc^2$, we see that resolving a small Δx causes an uncertainty in mass $\Delta m = \frac{\hbar}{2c\Delta x}$ that creates a horizon of radius

$$R = \frac{\hbar G}{c^3} \frac{1}{\Delta x}.$$

For $\Delta x < \ell_p = \sqrt{\frac{\hbar G}{c^3}} = 10^{-35}$m this is bigger than the separation we were trying to resolve!

Thus, it is — in principle — impossible to resolve distances shorter than ℓ_p. It is said, space-time dissolves into some kind of space-time-foam.

A fundamental description of nature should take care of such an impossibility. We need a new structure to replace smooth manifolds that reveals the new short distance degrees of freedom.

Here, we will illustrate that **D-geometry** as suggested by string-theory can be a candidate for such a new structure.

We will demonstrate it in the most simple case: The M(atrix)-Model.

Its degrees of freedom are nine time-dependant hermitian matrices $X^i(t)$ that can be understood as functions

$$X : \mathbb{R} \to u(N)^9$$

(actually, one should rather consider the super-symmetrized version of this model, but here, for the sake of presentation, we will suppress the fermionic partners).

The Hamiltonian (energy) consists of two parts, the kinetic and the potential energy; it is given by

$$H = \frac{1}{2}\sum_i \mathrm{Tr}\left(\dot{X}^i\dot{X}^i\right) - \sum_{i<j}\mathrm{Tr}\left([X^i,X^j][X^i,X^j]\right).$$

It is minimized if for all i and j all matrices commute:

$$[X^i,X^j]=0$$

By a global $SU(N)$ rotation, we can thus assume the matrices to be diagonal:

$$X^i(t) = \begin{pmatrix} \lambda^i_1(t) & & \\ & \ddots & \\ & & \lambda^i_N(t) \end{pmatrix}$$

The equations of motion resulting from this ansatz are $\ddot{\lambda}^i_a(t)=0$. Trivially, they are solved by

$$\lambda^i_a(t) = b^i_a + v^i_a t.$$

It is now important to perform a formal reassembly of the eigenvalues into N nine-vectors as

$$\vec{\lambda}_a(t) = {}^t\left(\lambda^1_a(t),\ldots,\lambda^9_a(t)\right)$$

and to reinterpret them as the coordinates of N particles performing a free motion in $\mathbb{R}^{1,9}$. $\vec{b}_a$ and $\vec{v}_a$, the constants of integration from above, are seen to be the impact parameters and the velocities of the particles.

We can now perturb this ansatz by turning on some of the off-diagonal matrix entries, say at matrix position (a, b):

$$X^i(t) = \begin{pmatrix} \lambda_1^i(t) & & & & \\ & . & & \mu(t) & \\ & & . & & \\ & \mu(t) & & . & \\ & & & & \lambda_N^i(t) \end{pmatrix}$$

For this perturbed ansatz, the Hamiltonian is

$$\Delta H = \frac{1}{2}\dot{\mu}^2 + \frac{1}{2}\|\vec{\lambda}_a(t) - \vec{\lambda}_b(t)\|^2\mu^2 + O(\mu^4)$$

where we introduced the euclidian norm $\|\cdot\|$ in $\mathbb{R}^9$. If the $\vec{\lambda}_a$ were constant in time, this would be the Hamiltonian of a harmonic oscillator of frequency $\omega := \|\vec{\lambda}_a(t) - \vec{\lambda}_b(t)\|$. The oscillations would take place on a time-scale of $1/\omega$. So, if the separation of the particles is large compared to unity, this time scale is much shorter than the time scale of the motion of the particles. Therefore, the assumption of ω to be constant in time is approximately satisfied in this case.

We can proceed and quantize the system. This will result in a level spacing of ω for the oscillators. Thus, for macroscopic separations of the particles, they will all be frozen into their ground-states and will not participate in the dynamics. (One should think of this model to be written in "natural units" where $\hbar = G = c = 1$ in which all distances are measured in multiples of ℓ_p and all energies in multiples of the Planck mass.) Thus, for distances that can be obtained in "real world" physical processes, the level-spacing of the harmonic oscillators is dramatically large.

In this regime, the configuration of the system is described just in terms of the $\vec{\lambda}$'s. Realizing that our notion of space and time is only relative to positions of "particles" — we would conclude that a space is $\mathbb{R}^d$ if we find the configuration space of N particles to be $\mathbb{R}^{Nd}$ — we can conclude that in this generic situation of macroscopic separations the space of this model is perceived to be $\mathbb{R}^9$, a simple example of a smooth manifold.

The situation changes drastically if two of the particles approach each other and get closer than the Planck length ℓ_p: Now the level-spacing of the off-diagonal harmonic oscillators becomes small. As a result, the oscillatory off-diagonal degrees of freedom become dynamic. Energy is transferred back an forth between the particle motion and the oscillators. In fact, the distinction into diagonal and off-diagonal degrees of freedom is no longer possible.

The space of configurations is is now much more complicated and has grown more dimensions than just nine. But this is just the behavior we asked for for a quantum space-time: It should be manifold at generic points and at macroscopic scales but it should dissolve at very short scales; the meaning of coordinates is not valid any longer for such high energy processes as close encounters of particles.

Thus, our analysis suggests to use the matrix model as a new, indirect definition of a quantum space-time.

The model, we have presented has its origin in string theory. There, it fulfills further purposes: For example, studying quantum fluctuations around the diagonal solution more carefully than it is possible here, one finds induced interactions between the particles that can for specific processes be related to the eleven dimensional super-gravity, a theory that is believed to play an important rôle in a to be discovered "theory of everything".

To define quantum space-times with more sophisticated structure than $\mathbb{R}^9$ macroscopically, it is easily possible to generalize our approach: The idea is to start with a quantum field theory living on an auxiliary space, the real line in our simple example. The auxiliary space has no direct physical significance, generically it can be flat even if we are heading for a curved quantum space thus circumventing conceptual problems of quantum field theories on curved space-times. Then one figures out the space of low-energy configurations, the "moduli-space of vacua". In most cases, quantization of the model will "lift the valleys", meaning that quantum physical zero-point energies will destroy classical low-energy solutions. Here, super-symmetry is usually of help as it tends to protect classical low-energy solution during the process of quantization.

Subsequently, this quantum moduli space can be interpreted as configuration space of several objects (particles in our case but possibly also extended objects with inner structure). From this configuration space, the resulting quantum space-time is read off. The properties we found above are typical for "D-geometries" obtained in this way: At generic points the spaces will resemble manifolds but for objects that get very close, the spaces will have singularities that go beyond the usual notion of manifolds just as we would expect is for quantum spaces.