How to cancell the diamagnetic term

Hamiltonian

$$\hat{H} = \int d^3r\, \hat{\psi}^\dagger(\vec{r})\, \varepsilon\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right) \hat{\psi}(\vec{r})$$

$\vec{A}$ - vector potential, ε - dispersion relation

$\hat{p} = -i\vec{\nabla}_r$

Charge current

$$\hat{j} = e\, \underbrace{\hat{\psi}^\dagger(\vec{r})\, \vec{\nabla}_{\vec{p}}\, \varepsilon\left(\vec{p} - \frac{e}{c}\vec{A}\right) \hat{\psi}(\vec{r})}_{\text{velocity}}$$

A is small $\Rightarrow$ Taylor expand $\hat{j}$:

$$\hat{j} = \underbrace{e\, \hat{\psi}^\dagger(\vec{r})\, \vec{\nabla}_{\vec{p}}\, \varepsilon(\vec{p})\, \hat{\psi}(\vec{r})}_{\text{gradient term}} - \underbrace{e\frac{e}{c}\, \hat{\psi}^\dagger(\vec{r})\, \vec{\nabla}_{\vec{p}} \left((\vec{A}\cdot\vec{\nabla}_{\vec{p}})\, \varepsilon(\vec{p}) \right) \hat{\psi}(\vec{r})}_{\text{diamagnetic term}}$$

Consider a static limit, $\omega = 0$, and small wave-vectors $\vec{k}$. Assume

$$\vec{j}^{\,(\alpha)}_{\vec{k}} = Q^{(\alpha\beta)}(k)\, \vec{A}^{(\beta)}(\vec{k}) \quad \text{//linear resp.//}$$

where $k \to 0$, $\alpha, \beta = x, y, z$ - indices in the real space.

Write decomposition

$$Q = Q_d + Q_g$$

„gradient contribution"

„diamagnetic term".

1) Expectation value of the diamagnetic term in the expression for $\hat{j}$ yields Q_d:

$$Q_d^{(\alpha\beta)}(k\to 0) = -\frac{e^2}{c}\left\langle \hat{\psi}^+ \frac{\partial^2 \varepsilon(\vec{p})}{\partial p_\alpha \partial p_\beta}\hat{\psi}\right\rangle =$$

$$= -2\frac{e^2}{c}\int \frac{d^3p}{(2\pi)^3}\frac{\partial^2 \varepsilon(\vec{p})}{\partial p_\alpha\, \partial p_\beta}\, n_F(\varepsilon(\vec{p}))$$

from spins

$$n_F(x) = \frac{1}{e^{\beta x}+1}$$

- cf RR/AA contributions on the PS-8 („Mesoscopics)

2) Q_g can be obtained from the Kubo formula

$$Q_g^{(\alpha\beta)}(i\Omega,\vec{k}) = -\frac{1}{2}\left(-\frac{1}{c}\right)\int_{-\beta}^{\beta} e^{i\Omega\tau}d\tau \int e^{-i\vec{k}\vec{z}}d^3z \times$$

$$\times \left\langle \hat{T}_\tau \hat{\psi}_\sigma^+(\vec{z},\tau)\, \hat{j}^{(\alpha)}\hat{\psi}_\sigma(\vec{z},\tau)\, \hat{\psi}_{\sigma'}^+(0)\, \hat{j}^{(\beta)}\psi_{\sigma'}(0)\right\rangle$$

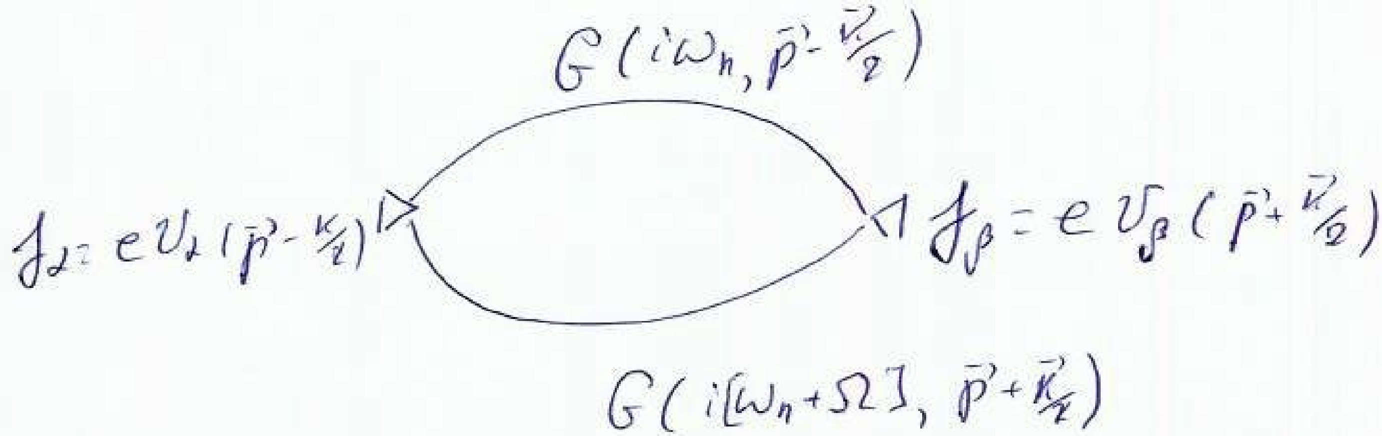
// with a sum over repeating spin indices σ,σ' //

Take into account only connected diagrams with normal Green's functions ||the case of a normal metall||

$$\Rightarrow Q_g^{(\alpha\beta)} = \underset{\text{spins}}{2}\,\frac{1}{2}\left(-\frac{1}{c}\right) 2\int_{\cancel{-\beta}}^{\beta} e^{i\Omega\tau}\, d\tau \int e^{-i\vec{k}\vec{r}}\, d^3r \,\times$$

$$\times \left[\, j_\alpha\, G(r,\tau)\, j_\beta\, G(-r,-\tau) \,\right]$$

Graphics for this expression (in Matsubara-momentum repr.||



$$\Rightarrow Q_g^{(\alpha\beta)} = -2\,\frac{e^2}{c}\, T \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3} \Big(v_\alpha(\vec{p}-\tfrac{\vec{k}}{2}) \cdot v_\beta(\vec{p}+\tfrac{\vec{k}}{2}) \,\times$$

$$\times\, G(i[\omega_n+\Omega], \vec{p}+\tfrac{\vec{k}}{2})\, G(i[\omega_n], \vec{p}-\tfrac{\vec{k}}{2}) \Big)$$

$$Q_g^{(\alpha,\beta)}(0) = -2\frac{e^2}{c}\, T \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3} \frac{v_\alpha(\vec{p})\, v_\beta(\vec{p})}{(i\omega_n - \varepsilon(\vec{p}))^2}$$

It's convenient to introduce a regularizing parameter $\delta t \to +0$ and to rewrite Q_g:

$$Q_g^{(\alpha,\beta)}(0) = -2\frac{e^2}{c} \lim_{\delta t \to +0} T \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3} \frac{v_\alpha(\vec{p})\, v_\beta(\vec{p})}{(i\omega_n - \varepsilon(\vec{p}))^2} e^{i\omega_n \delta t}$$

The role of δt can be understood from a comparison with the relation

$$n(x) = -i \lim_{\substack{\vec{r}' \to \vec{r} \\ t' \to t+\delta t \\ \boxed{\delta t \to +0}}} G(x, x')$$

where $x \equiv \{\vec{r}, t\}$ and

$$G = -i \langle \hat{T}_t\, \hat{\psi}(x)\, \hat{\psi}^\dagger(x') \rangle$$

Use the Ward identity:

$$v_\beta(\vec{p}) = \vec{\nabla}_\beta\, \varepsilon(\vec{p}) \Rightarrow v_\beta(\vec{p})\, G^2(i\omega_n, p) = \vec{\nabla}_\beta\, G(i\omega_n, p)$$

$$\Rightarrow Q_g^{(\alpha,\beta)}(0) = -2\frac{e^2}{c} \lim_{\delta t \to +0} T \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3}\, v_\alpha(\vec{p})\, \vec{\nabla}_\beta \frac{e^{i\omega_n \delta t}}{i\omega_n - \varepsilon(\vec{p})}$$

$$Q_g^{(\alpha,\beta)}(0) = 2\frac{e^2}{c}\int \frac{d^3p}{(2\pi)^3}\frac{\partial^2 \xi(\vec{p})}{\partial p_\alpha \partial p_\beta} \times$$

$$\times \lim_{\delta t \to +0} T \sum_{\omega_n} \frac{e^{i\omega_n \delta t}}{i\omega_n - \xi(\vec{p})}$$

Let's calculate Matsubara sum:

$$\lim_{\delta t \to +0} T \sum_{\omega_n} \frac{e^{i\omega_n \delta t}}{i\omega_n - \xi(\vec{p})} =$$

$$= \lim_{\delta t \to +0} T \sum_{\omega_n} e^{i\omega_n \delta t} \frac{-i\omega_n}{+\omega_n^2 + \xi^2} + \underbrace{T \sum_{\omega_n} \frac{-\xi}{+\omega_n^2 + \xi^2}}_{\text{converges fast} \Rightarrow \text{ we can put } \delta t = 0}$$

$$T \sum_{\omega_n} \frac{-\xi}{+\omega_n^2 + \xi^2} = -\frac{1}{2}\tanh\left(\frac{\xi}{2T}\right)$$

$$\lim_{\delta t \to +0} T \sum_{\omega_n} e^{i\omega_n \delta t} \frac{-i\omega_n}{\xi^2 + \omega_n^2}$$ - is governed by large $n \Rightarrow$ we can convert it to $\int d\omega$

$$\lim_{\delta t \to +0} T \sum_{\omega_n} e^{i\omega_n \delta t} \frac{-i\omega_n}{\xi^2 + \omega_n^2} = -i \lim_{\delta t \to +0} \int \frac{d\omega}{2\pi} e^{i\omega \delta t} \frac{\omega}{\omega^2 + \xi^2} = \frac{1}{2}$$

$$\Rightarrow \lim_{\delta t \to +0} T \sum_{\omega_n} \frac{e^{i\omega_n \delta t}}{i\omega_n - \xi(\vec{p})} = \frac{1}{2}\left(1 - \tanh\left[\frac{\xi(\vec{p})}{2T}\right]\right) = n_F(\xi(\vec{p}))$$

Thus we obtain

$$Q_g^{(\alpha\beta)}(0) = 2\frac{e^2}{c}\int\frac{d^3p}{(2\pi)^3}\frac{\partial^2 \varepsilon(\vec{p})}{\partial p_\alpha \partial p_\beta}\, n_F(\varepsilon(\vec{p}))$$

We have proven that

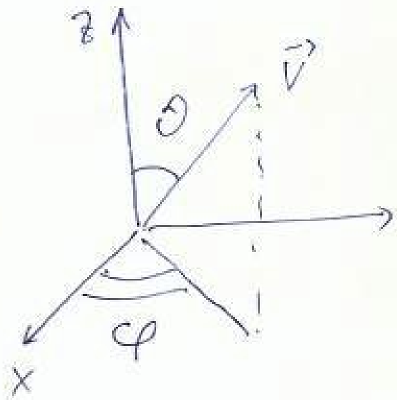
$$Q_g^{(\alpha\beta)}(0) = -Q_d^{(\alpha\beta)}(0)$$

for arbitrary $\varepsilon(p) \Rightarrow$

$$\boxed{Q^{(\alpha\beta)}(0) = Q_g^{(\alpha\beta)}(0) + Q_d^{(\alpha\beta)}(0) = 0}$$

Spherical coordinate - angle integral

AUX



$$V_x = V \sin\theta \cos\varphi$$

$$V_y = V \sin\theta \sin\varphi$$

$$V_z = V \cos\theta$$

$$\int \frac{d^3V}{(2\pi)^3} V_\alpha V_\beta f(V) =$$

$$= \int \frac{d\Omega_v}{4\pi} \int_0^\infty \frac{V^2 dV}{2\pi^2} f(V) V_\alpha V_\beta = \frac{1}{3} \delta_{\alpha\beta} \int_0^\infty \frac{V^4 dV}{2\pi^2} f(V)$$

Here $\int d\Omega_v = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta$