Problem Set 1: Useful Operator Identities

1. Baker-Hausdorff identity

   Define
$$\begin{cases} [A,B]_{n+1} & = & [[A,B]_n\,,B] \\ [A,B]_0 & = & A \end{cases}$$
   Show that
$$e^{-B}Ae^{B}=\sum_{n=0}^{\infty}\frac{1}{n!}\,[A,B]_n$$
   (Hint: Expand the function $\mathcal{A}(s)=e^{-sB}Ae^{sB}$ as a Taylor series in s, and evaluate this series at $s=1$)

2. Suppose $C=[A,B]$ satisfies $[A,C]=[B,C]=0$. Show that

   i. $e^{-B}Ae^{B}=A+C$
   (Hint: use Baker-Hausdorff)

   ii. $e^{A}e^{B}=e^{A+B}e^{C/2}$
   (Hint: Define $T(s)=e^{sA}e^{sB}$, calculate $\frac{dT(s)}{ds}$ and show that the solution of this differential equation is $T(s)=e^{s(A+B)}e^{s^2C/2}$)

   iii. $e^{-B}f(A)e^{B}=f(A+C)$
   (Hint: Taylor-expand $f(A)$, and find the n^{th}-term by induction starting from 2.i.)

   iv. $e^{A}e^{B}=e^{B}e^{A}e^{C}$

3. Suppose $[A, B] = DB$ and $[A, D] = [B, D] = 0$. Show that

   i. $f(A)B = Bf(D + A)$
      (Hint: Use a Taylor expansion and induction)

   ii. $e^A B = Be^{A+D}$
      (Hint: Use 3.i.)

   iii. $e^A B^n = (Be^D)^n e^A$
      (Hint: Use 3.ii. and induction)

   iv. $e^A e^B = e^{(Be^D)} e^A$
      (Hint: Expand e^B and use 3.iii.)