Problem Set 10: Local linear electrodynamics of superconductors, Londons' equations.

Let's consider a 3d metal under the action of a weak spetially inhomogeneous vector potential $\vec{A}$ (we assume a gauge where there is no scalar potential ϕ). A linear relation beween electric curent and $\vec{A}$ in a momentum space can be written as follows:

$$\vec{j}_{\vec{k}} = Q(\vec{k})\vec{A}(\vec{k})\,, \tag{1}$$

where Q is some kernel. Let's consider the static limit, $\omega = 0$ and small wave-vectors. In the case of the normal metal, the kernel has a fundamental property:

$$\text{Normal metal, } \omega = 0: \quad \lim_{q \to 0} Q(\vec{k}) = 0. \tag{2}$$

1st warm-up task:
Consider a general expression for $\vec{j}_{\vec{k}}$ and prove Eq.(2) for arbitrary dispersion relation of normal electrons.

Eq.(2) can be explained as follows: current in the normal metals must be gauge invariant. Therefore, relations $\vec{j}_{\vec{k}} = \text{const}\,\vec{A}(\vec{k})$ are not allowed and Q must be zero in the static limit at $k \to 0$. This ensures a cancellation of the diamagnetic term while calculating the Drude conductivity, see DIS29–32 in lecture notes by Jan von Delft.

The situation in superconductors is different: the BCS theory is usually formulated in such a way that it is not entirely gauge invariant, see the lecture "Superconductivity-3" by Oleg Yevtushenko. One can say that it reflects a spontaneous breaking of the $U(1)$-symmetry at the point of "Normal metal – Superconductor" phase transition. As a result, the so called Londons' equation can be obtained

$$\text{Superconductor}: \; \vec{j}_{\vec{k}} = -\frac{e^2 n_s(T)}{mc}\vec{A}(\vec{k}) \tag{3}$$

for $\omega = 0$ and small k. We have introduced the temperature dependent density of superconducting electrons, $n_s(T)$, in Eq.(3). The Londons' equation is valid in the BCS gauge:

$$\operatorname{div}\vec{A} = 0\,, \tag{4}$$

and it is one of the basic equations of the theory of superconductors. In particular, Eqs.(3-4) allow one to describe the Meissner effect.

2nd warm-up task:
Derive Eqs.(3-4) in a real space by using the phenomenological 2-liquid model, classical equations of motion for the superconducting electrons and and the Maxwell equations:

$$\mathrm{curl}\vec{E} = -\frac{1}{c}\frac{\partial \vec{H}}{\partial t}\,, \ \mathrm{curl}\vec{H} = \frac{4\pi}{c}\vec{j}\,. \tag{5}$$

The basic idea of the 2-liquid model is to decompose the density of charge carriers into normal- (n_n) and superconducting components:

$$n = n_n + n_s\,. \tag{6}$$

It is assumed that the superconducting particles propagate in a sample without any dissipation and, therefore, they support supercurrent, $\vec{j}_s \equiv e n_s \vec{V}_s$ ($V_s \ll c$ is an average velocity of the superelectrons). Straightforward calculations yield the Londons' equations:

$$\vec{E} = \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\frac{m}{n_s e^2}\vec{j}_s\right), \quad \lambda^2 \mathrm{curl}\,\mathrm{curl}\,\vec{H} + \vec{H} = 0\,; \tag{7}$$

where we have introduced the penetration depth $\lambda \equiv \sqrt{mc^2/4\pi n_s e^2}$. Show that Eqs.(7) and Eqs.(3-4) are equivalent. To understand the meaning of λ, you can consider a system which consists of two semi-spaces (the superconductor at $x > 0$ and vacuum at $x < 0$) and solve the Londons' equation for the magnetic field with the boundary condition $\vec{H}(x = 0) = (0, 0, H_0)$. Coordinate dependence of the microscopic magnetic field at $x > 0$ reads $h_z(x) = H_0 e^{-x/\lambda}$.

Main task:
Derive the Londons' equation (3-4) by using the apparatus of the Green's functions. You have to write down the expression for $Q(i\Omega, \vec{k})$ in terms of the field operators $\psi^\dagger(\tau, \vec{k}), \psi(\tau, \vec{k})$ (cf. the derivation of conductivity for normal metals) and to apply the Wick theorem taking into account contributions from usual- and *anomalous* Green's functions. The latter are defined as (see the book by Abrikosov et. al., Section 34, or the lecture "Superconductivity-3" by Oleg Yevtushenko):

$$F_{\alpha,\beta}(\tau_1, \vec{r}_1; \tau_2, \vec{r}_2) \;=\; \langle \hat{T}_\tau \hat{\psi}(\tau_1, \vec{r}_1)\hat{\psi}(\tau_2, \vec{r}_2)\rangle, \tag{8}$$

$$F^\dagger_{\alpha,\beta}(\tau_1, \vec{r}_1; \tau_2, \vec{r}_2) \;=\; \langle \hat{T}_\tau \hat{\psi}^\dagger(\tau_1, \vec{r}_1)\hat{\psi}^\dagger(\tau_2, \vec{r}_2)\rangle\,, \tag{9}$$

The diamagnetic part of Q is the same as in the case of the normal metal:

$$Q_d(k \to 0) = -\frac{ne^2}{mc}\,. \tag{10}$$

The gradient part of Q is given by

$$Q_g(k \to 0) = -\frac{2e^2}{3m^2 c} T \sum_{\omega_n} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \vec{p}^{\,2} \left(G^2(i\omega_n, \vec{p}) + |F(i\omega_n, \vec{p})|^2\right), \ \omega_n = (2n+1)\pi T. \tag{11}$$

Eqs.(10–11) yield the Londons' equation (3) with the density of the superelectrons given by

$$n_s = -\frac{2\nu(\epsilon_F)p_F^2}{3m} T \sum_{\omega_n} \int \mathrm{d}\,\xi \left[\frac{\xi^2 - \omega_n^2}{(\xi^2 + \omega_n^2)^2} - \frac{\xi^2 - \omega_n^2 + \Delta^2}{(\xi^2 + \omega_n^2 + \Delta^2)^2} \right] . \tag{12}$$

Here Δ denotes the BCS gap

$$\Delta \equiv |\lambda|\, F\big(\vec{r}_1 = \vec{r}_2; \tau = +0\big) . \tag{13}$$

which is taken to be independent of $\vec{A}$ in our calculations.