

Problem Set 2: Green's functions (basic properties and simple examples)

1. Time evolution for single-particle operators with quadratic Hamiltonian.

Consider spinless particles with a quadratic Hamiltonian:

$$\hat{H}_0 = \sum_k \varepsilon_k \hat{c}_k^\dagger \hat{c}_k. \quad (1)$$

Here ε_k is a dispersion relation (measured w.r.t. to the chemical potential μ , i.e. $\varepsilon_k = E_k - \mu$), k is a wave-vector label; $\hat{c}^\dagger, \hat{c}$ are creation/annihilation operators, with

$$[\hat{c}_k, \hat{c}_{k'}^\dagger]_\xi = \delta_{k,k'} \quad (2)$$

($\xi = \mp$ for fermions/bosons, respectively).

Use 3.ii from Set No.1 to show that the time evolution of the creation-/annihilation operators in the Heisenberg picture is given by ($\hbar = 1$)

$$\hat{c}_k(\tau) = e^{-\tau \varepsilon_k} \hat{c}_k, \quad \hat{c}_k^\dagger(\tau) = e^{\tau \varepsilon_k} \hat{c}_k^\dagger. \quad (3)$$

2. Single-particle temperature Green's functions.

Consider the free particles from problem 1. Their temperature Green's function is defined as follows:

$$\mathcal{G}_{kk'}(\tau) \equiv -\langle T_\tau [\hat{c}_k(\tau) \hat{c}_{k'}^\dagger(0)] \rangle. \quad (4)$$

i. Show that $G_{kk'}(\tau) = \delta_{kk'} G_k(\tau)$, with

$$\mathcal{G}_k(\tau > 0) = -e^{-\tau \varepsilon_k} (1 + \xi n_k); \quad (5)$$

$$\mathcal{G}_k(\tau < 0) = -\xi e^{-\tau \varepsilon_k} n_k. \quad (6)$$

Here

$$n_k \equiv \langle \hat{c}_k^\dagger \hat{c}_k \rangle = \frac{1}{e^{\beta \varepsilon_k} - \xi}; \quad \beta \equiv 1/T, \quad k_B = 1. \quad (7)$$

ii. Verify that Eqs. (6) and (5) satisfy the general identity:

$$\mathcal{G}(\tau < 0) = \xi \mathcal{G}(\tau + \beta). \quad (8)$$

3. Matsubara frequencies

The Green's function can be written in different representation, for example, one can use either imaginary time or (discrete) Matsubara frequencies which are related by the Fourier transform:

$$\tilde{\mathcal{G}}(i\omega_n) = \int_0^\beta e^{i\omega_n\tau} \mathcal{G}(\tau) d\tau; \quad (9)$$

$$\mathcal{G}(\tau) = \beta^{-1} \sum_n e^{-i\omega_n\tau} \tilde{\mathcal{G}}(i\omega_n); \quad (10)$$

where n is integer and ω_n are defined below for bosons/fermions.

- i. Show that Eq. (8) is satisfied provided that the Matsubara frequencies are defined as follows:

$$\omega_n = 2\pi nT \quad \text{for bosons,} \quad \omega_n = 2\pi \left(n + \frac{1}{2}\right) T \quad \text{for fermions.} \quad (11)$$

- ii. Consider the free particles from Problem 2 and show that

$$\tilde{\mathcal{G}}_k(i\omega_n) = \frac{1}{i\omega_n - \varepsilon_k}. \quad (12)$$

- iii. Evaluate $\mathcal{G}_k(\tau)$ in the limit $T = 0$ at $\varepsilon_k \neq 0$.

Hint: insert the result for ii. into Eq. (10), convert the sum into an integral, perform the latter explicitly and prove that

$$\mathcal{G}_k(\tau) = -\text{sign}(\tau) \theta(\tau \varepsilon_k) e^{-\tau \varepsilon_k}, \quad \varepsilon_k \neq 0.$$

4. Identities used for Matsubara transformations.

Show that the following identities hold:

- i. If ω_n and ω_m are both bosonic or both fermionic Matsubara frequencies, then:

$$\frac{1}{\beta} \int_0^\beta d\tau e^{i(\omega_n - \omega_m)\tau} = \delta_{nm} \quad (13)$$

- ii. If ω_n are all bosonic or all fermionic Matsubara frequencies, then the sum

$$\frac{1}{\beta} \sum_{n \in \mathbb{Z}} e^{i\omega_n\tau} = \sum_m \xi^m \delta(\tau - m\beta) \quad (14)$$

gives a periodic or anti-periodic δ -function, respectively. Hint: regularize the sum by introducing a damping factor $e^{-a|\omega_n|}$, and take the limit $a \rightarrow 0$ in the end.

5. **1D Green's functions** (partly optional / more advanced).

This problem is based on Appendix H of von Delft & Schoeller, Ann. Physik (Leipzig), **4**, 57-59 (1998). Please consult that Appendix for hints on how to solve it. (You may ignore the discrete index η used there, which distinguishes several different species of fermions or bosons.)

Consider free fermions/bosons in 1 dimension for a system of length L , with linear dispersion $\varepsilon_k = k$ or $\varepsilon_q = q$ (units are chosen such that energy is measured in units of inverse length) and discretized wave numbers (due to suitably chosen boundary conditions), such that the energy levels are quantized as

$$\varepsilon_q = q = \frac{2\pi}{L}(n_q), \quad \text{with } n_q \in N_+ \quad (\text{bosons}) \quad (15)$$

$$\varepsilon_k = k = \frac{2\pi}{L}(n_k - 1/2), \quad \text{with } n_k \in Z \quad (\text{fermions}). \quad (16)$$

Fermion and boson fields ψ and ϕ are defined as

$$\tilde{\psi}(x, \tau) = \left(\frac{2\pi}{L}\right)^{1/2} \sum_k e^{-k(\tau+ix)} c_k e^{-a|k|/2}, \quad (17)$$

$$\phi(x, \tau) = - \sum_{q>0} \frac{1}{\sqrt{n_q}} (e^{-q(ix+\tau)} b_q + e^{q(ix+\tau)} b_q^\dagger) e^{-aq/2}. \quad (18)$$

To regularize momentum sums in the limit of large momenta, a cutoff function $e^{-aq/2}$ or $e^{-a|k|/2}$ has been introduced for the boson or fermion fields, respectively, where a is a very short length, e.g. the lattice spacing.

The thermal Green's functions are defined as

$$\mathcal{G}(x, \tau) = -\langle T_\tau [\tilde{\phi}(x, \tau) \tilde{\phi}(0, 0)] \rangle, \quad \text{or} \quad \mathcal{G}(x, \tau) = -\langle T_\tau [\tilde{\psi}(x, \tau) \tilde{\psi}^\dagger(0, 0)] \rangle. \quad (19)$$

for both bosons and fermions, respectively. Evaluate these functions explicitly,

- i. for $T = 0$, $L \rightarrow \infty$;
- ii. (optional; advanced) for $T = 0$, $L \neq \infty$ (check that the limit $L \rightarrow \infty$ reproduces i.);
- iii. (optional; very advanced) for $T \neq 0$, $L \rightarrow \infty$ (check that the limit $T = 0$ reproduces i.).