

Problem Set 3: 1d Fermi gas

1. Density fluctuations in a degenerate Fermi gas.

Consider spinless free 1d fermions (as in Problem Set 2) which are confined to the space interval $0 < x < L$ and are described by the Hamiltonian

$$\hat{H}_0 = \int_0^L \frac{1}{2m} \partial_x \psi^\dagger(x) \partial_x \psi(x), \quad \psi(x) = L^{-1/2} \sum_k \hat{c}_k e^{ikx}. \quad (1)$$

The operator of the total particle number for this system reads

$$\hat{N}_L = \int_0^L \hat{n}(x) dx, \quad \hat{n}(x) = \psi^\dagger(x) \psi(x). \quad (2)$$

Prove that

$$\langle \delta \hat{N}_L^2 \rangle \equiv \langle \hat{N}_L^2 \rangle - \langle \hat{N}_L \rangle^2 \simeq \log(p_F L) / \pi^2, \quad p_F^{-1} \ll L \ll L_T; \quad (3)$$

where p_F and v_F are the Fermi momentum and the Fermi velocity, respectively; $L_T \equiv v_F / T$ is the thermal length; $\hbar = 1$, $k_B = 1$.

Hint: L_T is the largest scale if $T \rightarrow 0$.

Useful identity:

$$\langle \hat{c}_{k_1}^\dagger \hat{c}_{k_2} \hat{c}_{k_3}^\dagger \hat{c}_{k_4} \rangle = \langle \hat{c}_{k_1}^\dagger \hat{c}_{k_2} \rangle \langle \hat{c}_{k_3}^\dagger \hat{c}_{k_4} \rangle + \langle \hat{c}_{k_1}^\dagger \hat{c}_{k_4} \rangle \langle \hat{c}_{k_2} \hat{c}_{k_3}^\dagger \rangle. \quad (4)$$

2. Resonant-level model.

Consider the fermionic resonant-level model: a localized level with energy ε_d is coupled with hybridization strength V to a flat band of fermions:

$$\hat{H} = \varepsilon_d \hat{c}_d^\dagger \hat{c}_d + \sum_k \varepsilon_k \hat{c}_k^\dagger \hat{c}_k + V \sum_k (\hat{c}_k^\dagger \hat{c}_d + \hat{c}_d^\dagger \hat{c}_k). \quad (5)$$

Here $-D \leq \varepsilon_k \leq D$ are the uniformly spaced energy levels of a band of half-width D and constant density of states,

$$\rho(\varepsilon) \equiv \sum_k \delta(\varepsilon - \varepsilon_k) = \begin{cases} \rho_0 & (|\varepsilon| < D), \\ 0 & (|\varepsilon| > D). \end{cases} \quad (6)$$

i. Consider the thermal Green's functions

$$\mathcal{G}_{nn'}(\tau) \equiv -\langle T_\tau \hat{c}_n(\tau) \hat{c}_{n'}^\dagger(0) \rangle, \quad n, n' \in \{d, k\}. \quad (7)$$

Derive motion equations for $\mathcal{G}_{dd}(\tau)$ and $\mathcal{G}_{kd}(\tau)$, Matsubara-transform these equations, and solve them for $\overline{\mathcal{G}}_{dd}(i\omega_l)$ and $\overline{\mathcal{G}}_{kd}(i\omega_l)$ by eliminating the latter. Result:

$$\overline{\mathcal{G}}_{dd}(i\omega_l) = \frac{1}{i\omega_l - \varepsilon_d - \overline{\Sigma}_d(i\omega_l)}, \quad \overline{\Sigma}_d(i\omega_l) = \sum_k \frac{v^2}{i\omega_l - \varepsilon_k}. \quad (8)$$

(The function $\overline{\Sigma}_d(i\omega_l)$ is called the “self-energy”, a nomenclature stemming from a perturbative treatment of this model.)

ii. Use analytic continuation to find the retarded/advanced Green’s functions $G_{dd}^{R/A}(\omega)$ of the d-level. Hint: first show that $\Sigma_d^{R/A}(\omega) = \mp i\Gamma$, with $\Gamma = \pi V^2 \rho_0$, by using Eq. (6) to perform the sum on k and taking the wide-band limit $D \rightarrow \infty$.

iii. Show that the *local density of states* of the d-level, given by its spectral function, $\mathcal{A}_d(\omega) = -\frac{1}{\pi} \text{Im} [G_{dd}^R(\omega)]$, has the following form in the wide-band limit:

$$\mathcal{A}_d(\omega) = \frac{1}{\pi} \frac{\Gamma}{(\omega - \varepsilon_d)^2 + \Gamma^2}. \quad (9)$$

iv. Establish the physical meaning of Γ by calculating the time-dependence of $G_{dd}^{R/A}(t)$.