

Problem Set 4: Density-density response.

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Consider noninteracting 1d fermions, see the Problem Set 3 ($L \rightarrow \infty$), under the action of a weak time-dependent potential $\phi(x, t) = \int dk \int d\omega e^{i(kx - \omega t)} \phi_{k, \omega}$, which couples to the particle number operator. The Hamiltonian reads:

$$\hat{H} = \hat{H}_0 + \hat{H}_1; \quad \hat{H}_1 \equiv - \int \phi(x, t) \hat{n}(x, t) dx, \quad (1)$$

Use the Kubo formula to show that the linear response of the density at wave-number k and frequency ω ,

$$\langle \hat{n}_{k, \omega} \rangle = Q(k, \omega) \phi_{k, \omega}, \quad (2)$$

is governed by the response function $Q(k, \omega)$

$$Q(k, \omega) = i \int dx \int_0^{+\infty} \langle [\hat{n}(x, t), \hat{n}(0, 0)] \rangle e^{i(\omega t - kx)} dt. \quad (3)$$

(a) Prove that

$$Q(\omega, k) = \int dp \frac{n_{p+k} - n_p}{\omega - [\varepsilon_{p+k} - \varepsilon_p]}. \quad (4)$$

This step can be done by

- i. calculating the expectation value of operators in Eq.(3) directly with the help of Eq.(10); or
- ii. using the Matsubara response function $\bar{Q}(i\omega_n)$ and the method of analytical continuation.

Hints for i.:

Write $\hat{n}$ in $\{x, t\}$ representation (cf. the 1st problem from the Problem Set 3), calculate the expectation value following Lecture No.2 (by Oleg Yevtushenko), calculate the time integral and the sum over one momentum, go to the continuum limit.

Hints for ii.:

Start from the equation for $\bar{Q}(i\omega_n)$:

$$\bar{Q}(i\omega_n) = \int_0^\beta d\beta \int dx e^{i(\omega_n \tau - kx)} \langle T_\tau \hat{n}(x, \tau) \hat{n}(x, \tau) \rangle_T, \quad (5)$$

use Eq.(10) and prove that in the continuum limit

$$\bar{Q}(i\omega_n) = - \int \frac{dp}{2\pi} T \sum_m \mathcal{G}_p(i\omega_m) \mathcal{G}_{p+k}(i\omega_m + i\omega_n). \quad (6)$$

Here $\mathcal{G}$ are the thermal Green's functions, $\omega_m = 2\pi(m + \frac{1}{2})T$, $\omega_n = 2\pi nT$. The Matsubara sum in Eq.(6) can be calculated by converting the sum over m into the integral in a complex plane and calculating it with the help of the pole-theorem, see Eq.(GF66.5) in the lecture notes by Jan von Delft. Alternatively, the Matsubara sum can be calculated directly by using Eq.(11). Check that both methods give the same answer.

(b) Consider the limit $T \rightarrow 0$ where

$$n_q = \theta(|q| \leq p_F), \quad (7)$$

and show that at $|k| \ll p_F$ the Fermi statistics (i.e. the Pauli principle) leads to the following estimate for the integration variable in Eq.(4)

$$|p| \simeq p_F, \quad (8)$$

i.e. the integration phase space is small.

(c) Linearize the dispersion relation in the limit $|k| \ll |p| \simeq p_F$, calculate the momentum integral in Eq.(4) explicitly and show that $Q(k, \omega)$ is given by

$$Q(\omega, k) \simeq -\nu_1 \frac{(v_F k)^2}{\omega^2 - (v_F k)^2 + i0}, \quad \nu_1 \equiv \frac{1}{\pi v_F}; \quad (9)$$

in the case of small frequency $|\omega| \ll E_F$ (E_F is the Fermi energy) and small momentum $|k| \ll p_F$ at $T \rightarrow 0$.

Useful identities:

$$\langle \hat{c}_{k_1}^\dagger \hat{c}_{k_2} \hat{c}_{k_3}^\dagger \hat{c}_{k_4} \rangle = \langle \hat{c}_{k_1}^\dagger \hat{c}_{k_2} \rangle \langle \hat{c}_{k_3}^\dagger \hat{c}_{k_4} \rangle + \langle \hat{c}_{k_1}^\dagger \hat{c}_{k_4} \rangle \langle \hat{c}_{k_2} \hat{c}_{k_3}^\dagger \rangle \quad (10)$$

$$\sum_{m=-\infty}^{+\infty} \frac{1}{(2m+1)^2 + a^2} = \frac{\pi}{2a} \tanh\left(\frac{\pi a}{2}\right) \quad (11)$$