Problem Set 5: Interaction of free particles with a scalar potential.

1. **Single point-like impurity.**

   Consider noninteracting 1d fermions, see the Problem Set 3 ($L \to \infty$) under the action of an external scalar potential:

$$\hat{H} = \hat{H}_0 + \hat{H}_p \,; \quad \hat{H}_p = \int \psi^\dagger(x) V(x,t) \psi(x) \, \mathrm{d}x \,. \tag{1}$$

   The simplest but, nevertheless, important example of the potential V is a static point-like impurity

$$V(x,t) = \gamma \delta(x) \,; \tag{2}$$

   here $\gamma = \mathrm{const} > 0$ is the impurity strength. The Green's function of the system Eqs.(1,2) can be calculated exactly and it helps to illustrate some interesting effects.

   Derive the thermal Green's function of the system Eqs.(1,2):

   (a) Start with Eq.(PT6-1) from notes of Jan von Delft. Note that the impurity breaks the translational symmetry and, therefore, the coordinate representation is more convenient than the momentum one. Insert Eq.(2) into Eq.(PT6-1) and Fourier transform it. Prove that

$$\begin{aligned} \mathcal{G}(i\omega_n; x_1, x_2) &= \mathcal{G}_0(i\omega_n; x_1, x_2) + \int \mathcal{G}_0(i\omega_n; x_1, x_3) V(x_3) \mathcal{G}(i\omega_n; x_3, x_2) \, \mathrm{d}x_3 \\ \mathcal{G}_0(i\omega_n; x_1, x_2) &= \mathcal{G}(i\omega_n; x_1, x_2) \Big|_{V=0} \end{aligned} \tag{3}$$

      i. Draw a diagram (graphic representation) of Eqs.(3).

      ii. Explain why all Green's functions in Eq.(3) have the same frequency.

   (b) Calculate the integral over x in Eq.(3) and prove that

$$\mathcal{G}(i\omega_n; x_1, x_2) = \mathcal{G}_0(i\omega_n; x_1, x_2) + \gamma \frac{\mathcal{G}_0(i\omega_n; x_1, 0) \mathcal{G}_0(i\omega_n; 0, x_2)}{1 - \gamma \mathcal{G}_0(i\omega_n; 0, 0)} \,. \tag{4}$$

   (c) Find $\mathcal{G}_0(i\omega_n; x_1, x_2)$ by calculating the Fourier transform of Eq.(12) from the Problem Set 2 with $\varepsilon_k = k^2/2m - E_F$. Show that in the limit $\omega_n \ll E_F$ the expression for $\mathcal{G}_0(i\omega_n; x_1, x_2)$ can be reduced to

$$\mathcal{G}_0(i\omega_n; x_1, x_2) \simeq -\frac{i}{v_F} \mathrm{sign}(\omega_n) e^{i|x_1 - x_2|(k_F + i\omega_n/v_F)\mathrm{sign}(\omega_n)} \,. \tag{5}$$

(d) Study the limit $\gamma \to \infty$ which corresponds to the perfect scatterer (transmission goes to 0). Insert Eq.(5) into Eq.(4) and prove that

$$\mathcal{G}(i\omega_n; x_1, x_2)\Big|_{\gamma\to\infty} = \mathcal{G}_0(i\omega_n; x_1, x_2) - \mathcal{G}_0(i\omega_n; x_1, -x_2)\,. \tag{6}$$

(e) *Optional task*: Try to obtain Eq.(6) by using the method of images (see any book on "Electrostatics").

2. **Friedel oscillations**

Density of particles can be found from the thermal Green's function using the following expression

$$n(x) = \lim_{\tau\to -0} \mathcal{G}(\tau; x, x)\,. \tag{7}$$

Prove Eq.(7) and show that a scatterer generates density oscillations with a period $1/2p_F$. This effect is called "the Friedel oscillations". Description of the Friedel oscillations is simple in the case of the perfect scatterer:

(a) Use results of 1.d to show that

$$n(x)\Big|_{\gamma\to\infty} = \lim_{\tau\to -0}\Big(\mathcal{G}_0(\tau; 0, 0) - \mathcal{G}_0(\tau; 2x, 0)\Big)\,. \tag{8}$$

(b) Use results of the Problem Set 2 or Problem Set 1 (Supplementary Materials) to find $\lim_{\tau\to-0}\mathcal{G}_0(\tau; x, 0)$:

$$\lim_{\tau\to -0} \mathcal{G}_0(\tau; x, 0) = \frac{T\sin(p_F x)}{v_F \sinh(\pi T x/v_F)} \tag{9}$$

[*Hint*: consider only momentums which are closed to $\pm p_F$]. Alternatively, Eq.(9) can be obtained from the Matsubara transform of Eq.(5).

(c) Insert Eq.(9) into Eq.(7) and obtain the following expression

$$\frac{n(x)}{n_0} = 1 - \frac{\pi T \sin(2p_F x)}{v_F p_F \sinh(2\pi T x/v_F)}\,. \tag{10}$$

Here

$$n_0 = \lim_{x\to\infty, T>0}\Big(n(x)\Big) = \frac{p_F}{\pi} \tag{11}$$

is the density far from the scatterer. Analyze the limit $T \to 0$.

(d) *More advanced*: Explain origin of the Friedel oscillations at $x \ll v_F/2\pi T$. Why are the oscillation suppressed at $x \gg v_F/2\pi T$?