Problem Set 7: RPA; screening; decay of excitations.

1. The Random Phase Approximation
Consider 3d interacting fermions (ignore spin here) with the Hamiltonian

$$\begin{aligned}\mathcal{H} &= \int d^3 r \psi^\dagger(\vec{r},\tau)\left(-\frac{1}{2m}\nabla^2 - \mu + U(\vec{r},\tau)\right)\psi(\vec{r},\tau) \\ &+ \int d^3 r d^3 r' \psi^\dagger(\vec{r},\tau)\psi^\dagger(\vec{r}',\tau)V(\vec{r}-\vec{r}')\psi(\vec{r}',\tau)\psi(\vec{r},\tau)\,.\end{aligned} \tag{1}$$

Here U is a weak external potential and V is the Coulomb interaction.
Electrons in a metal screen any electrostatic potential. This leads to the onset of the effective external potential, U_{eff}, and the screened Coulomb interaction, Φ_{eff}, which in the random phase approximation are given by

$$U_{\text{eff}}(\vec{q}, i\omega_m) = \frac{U(\vec{q}, i\omega_m)}{\varepsilon(\vec{q}, i\omega_m)}\,, \tag{2}$$

$$\Phi_{\text{eff}}(\vec{q}, i\omega_m) = \frac{V(\vec{q})}{\varepsilon(\vec{q}, i\omega_m)}\,. \tag{3}$$

We have introduced the dielectric function $\varepsilon(\vec{q}, i\omega_m)$:

$$\varepsilon(\vec{q}, i\omega_m) = 1 - V(\vec{q})\chi(\vec{q}, i\omega_m)\,, \tag{4}$$

and the "polarization bubble"

$$\chi(\vec{q}, i\omega_m) = \frac{2}{\mathcal{V}\beta}\sum_{\vec{k},\xi}\frac{1}{i\xi - \epsilon_k}\,\frac{1}{i(\xi+\omega_m) - \epsilon_{k+q}} \tag{5}$$

($\mathcal{V}$ is the system volume). You have seen already $\chi(\vec{q}, i\omega_m)$ when the density-density (retarded) response function was calculated for 1d fermions, cf. Eq.(6) of the Problem Set 4.

Warm-up task: Draw diagrams for Eqs.(2,3). Compare their main building block with the diagram for the density-density response function.

In this problem, we use $\chi(\vec{q}, \omega)$ to study properties of excitation in the interacting system.

1a. Spectrum of collective excitation
Dispersion relation of collective excitations (plasma waves) can be found from poles of Eq.(3) after the analytic continuation to real frequencies. Let us, firstly, study the collective waves in the limit $|q| \ll p_F$, $\omega \ll E_F$ at $T \to 0$. This limit has been considered in details in the Problem Set 4 for the 1d system. Calculations of $\chi(\vec{q}, \omega)$ in 3d are very similar, the only difference results from angle integrations (spherical coordinates are convenient in this case):

i. Calculate the Matsubara sum in Eq.(5);

ii. Go to the continuum limit and use spherical coordinates for the momentum integration; (*Hint*: choose the angle between vectors $\vec{q}$ and $\vec{k}$ as the polar angle)

iii. Expand the numerator and denominator of the integrand in q;

iv. Integrate over a) azimuthal angle, b) modulus of $\vec{q}$, and c) polar angle to prove that

$$\chi \simeq 2\nu_F \left(\frac{s}{2} \log \left(\frac{s+1}{s-1} \right) - 1 \right) \tag{6}$$

where $s \equiv \omega / v_F q$ and ν_F is the density of states at the Fermi energy. *Hint*: do not forget that we consider the limit $T \to 0$.

The dispersion equation for the plasma waves reads:

$$V(\vec{q})\chi(\omega, \vec{q}) = 1\,, \quad V(\vec{q}) = \frac{4\pi e^2}{q^2}\,. \tag{7}$$

Note that the dispersion equation (7) can be obtained from the kinetic equation for the Fermi liquid. To understand some properties of the collective plasma waves:

i. Analyze Eqs.(6,7) in the limit of small and large momentum;

ii. Sketch the dispersion relation for the plasma waves, compare it with the dispersion relation of quasi-particle excitations, $\omega_{\mathrm{qp}} = v_F q$.

iii. Does the system (6,7) have a solution at arbitrary q? *Advanced question*: explain this property using the sketch from the previous item.

1b. Properties of the dielectric function and decay of excitation

Let us now study properties of the retarded polarization bubble and of the dielectric function in more details:

i. Evaluate both $\mathrm{Re}\chi(\vec{q}, \omega)$ and $\mathrm{Im}\chi(\vec{q}, \omega)$ in complete, explicit detail in 3d at $T = 0$ for arbitrary momentum. Show that

$$\mathrm{Re}\chi(\vec{q}, \omega) = \frac{2mp_F}{4\pi^2} \left\{ \frac{1}{2q'} \left[U(\nu/q' - q'/2) - U(\nu/q' + q'/2) \right] - 1 \right\}, \tag{8}$$

$$U(s) = \left(1 - s^2\right) \log \left| \frac{1+s}{1-s} \right| ; \quad \nu = \frac{\omega}{v_F p_F}\,, \; q' = \frac{q}{p_F}\,; \tag{9}$$

$$\mathrm{Im}\chi(\vec{q}, \omega) = -\frac{2mp_F\pi}{(2\pi)^3} \int \mathrm{d}^3k' \; \theta(|\vec{q}' + \vec{k}'| - 1)\theta(1 - k')\delta(\nu - \vec{q}'\vec{k}' - (q')^2/2)\,. \tag{10}$$

Hints: Perform the Matsubara sums first, then take the $T \to 0$ limit and consult Fetter and Walecka, □12 for hints on how to do the integrals. Note, that in □12, FW consider not the retarded but the time-ordered version of χ (since they are using the $T = 0$ formalism here, which is formulated in terms of time-ordered Green's functions). Thus their χ in □12 differs in its infinitesimals $\pm i\delta$ from the retarded χ that you are calculating, and that they discuss in □15.

ii. Make sure to state clearly the various conditions on ω and $\vec{q}$ for which $\mathrm{Im}\chi \neq 0$. Note that there are three separate cases to consider.

iii. Make a sketch, with ω on the vertical and $|\vec{q}|$ on the horizontal axis, on which you indicate in what region $\mathrm{Im}\chi \neq 0$. Show clearly which regions correspond to the three cases you had to distinguish in (ii.). Could you find the regions $\mathrm{Im}\chi \neq 0$ in the 1st part of the problem **1a**?

iv. *Advanced question*: Which property of the systems and physical effects are reflected by a non-zero imaginary part of χ (or by a a non-zero imaginary part of the dielectric function)?

Hint: In the present calculation, the effective potential is calculated by a summation of bubble diagrams to all orders. Thus, the effect of the external field on the medium is to create electron-hole pairs. If the energy and momentum $(\omega, \vec{q})$ that the external field imparts to the system is such that *real* (as opposed to virtual) electron-hole pairs can be created (i.e. without violating the Pauli-principle or energy conservation), one expects the system to readily absorb energy from the external field. Show that the conditions found in (ii.) above for $\mathrm{Im}\chi \neq 0$ are consistent with creation of the real electron-hole pairs.

1c. Screening

It has been shown during the lectures that the Thomas–Fermi equation (also called the Debye equation in some books) for the screened Coulomb interaction can be obtained in the static limit $\omega/v_F \ll q \ll p_F$ by summing the bubble diagrams to all orders, cf. PT72-PT74 of the notes by Jan von Delft.
Analyze behavior of $\mathrm{Re}\chi$ at $q \to 2p_F$. Can it change a decay of the potential at large distances? *Advanced question*: Discuss the physical reason for the peculiar behavior of $\mathrm{Re}\chi$. Does this property contradict to the Thomas–Fermi screening?

2. Decay of Momentum Pulse

Consider an interacting many-body system whose single-particle spectral function has the form

$$\mathcal{A}(\omega, \vec{p}) = \frac{a_{\vec{p}}\Gamma_{\vec{p}}/\pi}{(\omega - \tilde{\varepsilon}_{\vec{p}})^2 + \Gamma_{\vec{p}}^2}, \quad \Gamma_{\vec{p}} \ll \tilde{\varepsilon}_{\vec{p}} \,. \tag{11}$$

Consider the time-evolution of the correlator

$$\int d\vec{r}e^{-i\vec{p}\cdot\vec{r}}\langle\psi(t,\vec{r})\psi^{\dagger}(0,0)\rangle = iG^{>}(t,\vec{p}) \ , \tag{12}$$

which describes how, after the injection of a particle at time $t = 0$ at position $\vec{r} = 0$, a quasi-particle with momentum $\vec{p}$ decays with time. Show that in the limit $T \ll \tilde{\varepsilon}_{\vec{p}}$, the dominant behavior for $t > 1/\varepsilon_{\vec{p}}$ is given by

$$G^{>}(t,\vec{p}) = a_{\vec{p}}e^{-i\varepsilon_{\vec{p}}t}e^{-\Gamma_{\vec{p}}t} + \mathcal{O}\left(\frac{\Gamma_{\vec{p}}}{t\varepsilon_{\vec{p}}^{2}}\right) \ . \tag{13}$$

Explain the origin of the correction term.