Problem Set 8: Diagramatics for disordered systems.

The basic building block of diagrammatic calculations for disordered systems are the so called "ladder diagrams". The ladder diagrams arise, for example, after averaging of response functions over disorder. You can find examples of the ladder diagrams, for instance, in the books "*Mesoscopic physics of electrons and photons*" by E. Akkermans and G. Montambaux (see Sect. 4.4-4.6 and figures therein) or "*Diagrammatics: lectures on selected problems in condensed matter theory*" by M.V. Sadovskii (see Sect. 4.4-4.5, 4.6.1 and figures therein).

In this Problem Set we consider non-interacting electrons in a disordered metal. Let impurities be static point-like isotropic scatterers. This means that the energy is not changes at an impurity vertex while each impurity line in disorder averaged diagrams is just a momentum independent constant $1/2\pi\nu_0\tau$. Here ν_0 is the DoS at the Fermi level and τ is the characteristic time of scattering of the electrons by the impurities.

You know already that the impurity scattering results in a finite imaginary part of the self energy, i.e. quasi-particles have a finite life time. The disorder averaged Green's functions are given by:

$$\bar{G}^{R/A}(\omega, k) = \frac{1}{\omega - \xi_k \pm i/2\tau}\,, \tag{1}$$

see DIS8-DIS9 in notes by Jan von Delft. We will use Eq.(1) to calculate the ladder diagrams in two examples: 1) disorder averaged density-density correlation function and 2) quantum correction to the classical Drude conductivity.

1. Disorder averaged density-density correlation function
The density-density correlation function is proportional to the polarization operator. Thus, we have to average the bubble diagram from the PS-7 over disorder:

$$\mathcal{K}(i\omega_n, \vec{q}) = \langle \chi(i\omega_n, q) \rangle\,, \tag{2}$$

$$\chi(i\omega_n, \vec{q}) = T\sum_{\omega_m} \int \frac{\mathrm{d}^3k}{(2\pi)^3} G(i\omega_m, \vec{k}) G(i(\omega_m + \omega_n, \vec{q} + \vec{k})\,. \tag{3}$$

Here $\langle \ldots \rangle$ means averaging over disorder. The leading contribution to $\mathcal{K}(i\omega_n, \vec{q})$ results from diagrams where impurity lines do not intersect with each other. We will consider the diffusive regime:

$$ql \ll 1 \quad \omega_n\tau \ll 1\,, \tag{4}$$

where l is the mean free path. First steps of the calculations are:

(a) Use the relation

$$G(i\omega_m) = \theta(\omega_m)G^R(i\omega_m) + \theta(-\omega_m)G^A(i\omega_m) \tag{5}$$

to single out 4 different contributions:

$$\langle G^R G^R\rangle,\ \langle G^A G^A\rangle, \tag{6}$$

$$\langle G^R G^A\rangle,\ \langle G^A G^R\rangle. \tag{7}$$

(b) Draw an example of the diagrams with several impurity lines which do not intersect with each other. Note, that the self energy corrections are taken into account by using Eq.(1) as a starting point. Thus, you have to analyze diagrams which are beyond the self energy and where the impurity lines connect two Green's function with different energies (vertex corrections).

(c) Carefully put all momentum and energy variables in the diagram from (b) and analyze analytical properties of products of the Green's function which result from Eq.(6,7). Find typical values of the energy variable which govern answers in different cases.

The static part of $\mathcal{K}$

$$\mathcal{K}_0 = \mathcal{K}(\omega_n = 0) \tag{8}$$

results from combinations which have poles in the the same half-plane, Eq.(6). Consider these contributions, use conditions $\omega_n \ll \epsilon_F,\ q \ll k_F$ and show that

$$\mathcal{K}_0 \simeq -\partial_\mu T \sum_{\omega_m} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \langle G(i\omega_m, \vec{k})\rangle = -\partial_\mu n = -\nu_0\,. \tag{9}$$

Here μ is the chemical potential, n is the mean density.

The most important part of the calculations, where the ladder diagrams appear, is related to Eq.(7) which governs the kinetic part of $\mathcal{K}$. Follow the next steps:

(d) Analyze diagrams with a different number of the non-intersecting impurity lines and find the block which is repeated in each diagram several time. Show that the analytic expression for this block reads:

$$B_{RA} = \frac{1}{2\pi\nu\tau} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} G^A(i[\omega_m + \omega_n], \vec{q} + \vec{k}) G^R(i\omega_m, \vec{k}) \tag{10}$$

(B_{AR} can be found similar by swapping G^R and G^A). *Hint*: mind restrictions for the energy variables which are necessary to obtain a given analytical structure.

(e) Integrate over the momentum using the diffusive approximation (4) and show that the building blocks which are discussed in (d) can be written at arbitrary ω_n as follows

$$B(i\omega_n, \vec{q}) \simeq 1 - \tau[|\omega_n| + Dq^2]\,, \tag{11}$$

where $D = v_F^2\tau/3$ is the diffusion constant.

The last step of the calculations involves a summation of the ladder diagrams which contain a different powers of $B(i\omega_n, \vec{q})$. Note that this summation is reduced to the summation of the geometric progression. Write down the expression for the kinetic part of the density-density correlation function, $\mathcal{K}_1$, calculate the sum over the Matsubara frequency ω_m and the sum of the geometric progression and prove that

$$\mathcal{K}_1 \simeq \nu_0 \frac{|\omega_n|}{Dq^2 + |\omega_n|}\,. \tag{12}$$

Finally, we find the answer:

$$\mathcal{K} = \mathcal{K}_0 + \mathcal{K}_1 = -\nu_0 \frac{Dq^2}{Dq^2 + |\omega_n|}\,. \tag{13}$$

Advanced questions:

- The correlation function $\mathcal{K}$ has an important property:

$$\mathcal{K}(q = 0) = 0\,. \tag{14}$$

  Explain it by analyzing a response of the electrons on a spatially homogeneous time-dependent potential. Do not forget that we consider electroneutral samples where the number of the electrons is fixed.

- One can study the above considered ladder diagrams regardless of the external vertices. Summing the ladders with a different number of "crossbars" yields an object which is called in mesoscopics *diffuson*:

$$\mathcal{D} = \frac{1}{Dq^2 - i\omega}\,. \tag{15}$$

  A real time representation is used here. Prove that the diffuson obeys the diffusion equation. Try to explain this result.

2. Quantum correction to the classical Drude conductivity

Let us now consider the conductivity of disordered metals, see DIS26-DIS32 of lecture notes by Jan von Delft. You know that the leading diagrams (with non-intersecting impurity lines) yield the classical Drude conductivity. Note that the ladder which is discussed in the previous problem does not appear in the equation for the conductivity in the case of isotropic scatterers (*Task*: calculate the simplest vertex correction to the conductivity and show that it is 0).

The ladders contribute to the conductivity in sub-leading terms where there is only one intersection of the propagators. Try to construct the ladder (which is called *Cooperon*):

(a) Draw the polarization operator with several *maximally crossed* impurity lines.

(b) Deform (twist!) this diagram is such a way that all impurity lines become parallel. Do you see a single intersection of the fermionic Green's functions?

(c) Carefully put all momentum and energy variables and compare them with the counterparts from the diffuson diagram.

(d) The diffuson and Cooperon are often referred to as "particle-hole"– and "particle-particle" channels respectively. Try you explain these names.

(f) *Advanced question*: It is known that the Cooperon describes a return probability for the electron in the disordered potential. Can you explain this property?

Once the Cooperon diagrams are constructed, calculations are similar to those from the 1st problem (mind vector external vertices which are present in the current-current correlation function!). Details of calculations, including an equation for the Cooperon, will be explained during the next class. The Cooperon diagrams yield the leading quantum mechanical correction to the classical conductivity:

$$\sigma_{WL}(\omega) = -\frac{\sigma_{Drude}}{\pi\nu_0}\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{Dk^2 - i\omega}\,, \tag{16}$$

which is negative and is called "the weak localization correction".