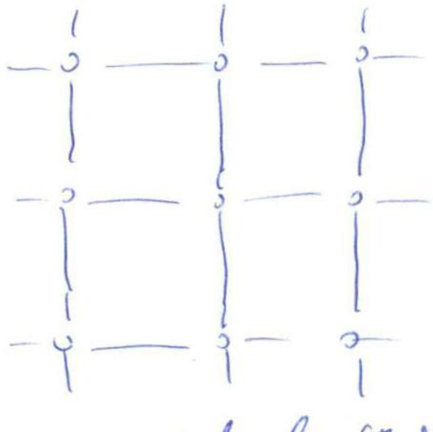
Models of disorder

They are needed to develop a „micro-theory":
Note - only the single-particle problem so far



ideal crystal

two ways of introducing the disorder → TBA + fluctuating lattice parameters - lattice models

→ almost free $\bar{e}$'s + $\underbrace{V_{dis}(\vec{r})}_{\text{random}}$ - cont. models

$\boxed{\text{1) Lattice models}}$: keep the geometric order on the lattice but „spoil" parameters of the Hamiltonian

$$\hat{H} = \sum_{i=1}^{N} \varepsilon_i (\hat{a}^+\hat{a})_i + \sum_{i \neq j} (t_{ij} \hat{a}_i^+ \hat{a}_j + h.c.)$$

i - site numbers

ε_i - on-site energy
t_{ij} - describes hopping } let them be random variables - usually uncorrelated

Use basis of atomic (localized) orbitals and write $\hat{H}$ as $N \times N$ matrix, $\hat{H} = \hat{H}^+$ - Hermitian

$$H = \begin{pmatrix} \varepsilon_1 & & H_{ij} \\ & \varepsilon_2 & \\ H_{ij}^* & & \ddots \end{pmatrix}$$

Random variables are described by the normalized prob. distr. (distr. function)

$$P[\{H_{ij}\}] \underbrace{\mathcal{D}\{H\}}_{\prod dH_{ij}} = \frac{1}{Z} \prod_{\substack{\text{all} \\ \text{independent}}} f(H_{ij}) \mathcal{D}\{H\}$$

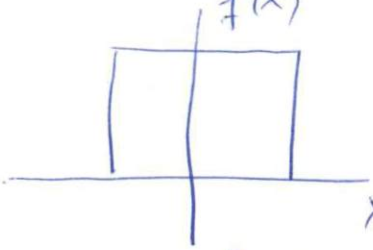
all independent

Averaging over different realizations of disorder

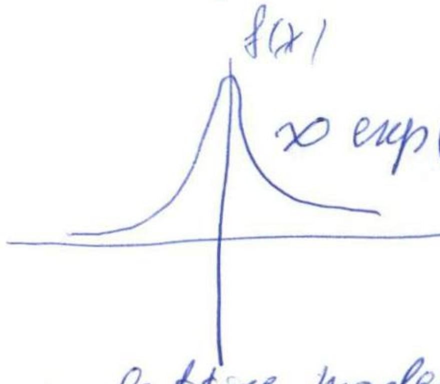
Dis-8

$$\overline{A} \ (\text{or } \langle A \rangle_{dis}) = \int A\{H\} P[\{H\}] \mathcal{D}\{H\}$$

Widely used examples

a) 

– box // or Poisson // distribution

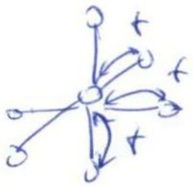
b) 

$\propto \exp(-\frac{x^2}{2\sigma})$ – Gaussian distribution

variance

Famous lattice models of the disorder

a) the Anderson model in d-dim space



– TBA : $t_{ij} \neq 0$ only for neighboring sites (NN)

Anderson, '58: $\begin{cases} t_{ij} = t = \text{const} & \text{for NN} \\ \varepsilon_i - \text{random}. \end{cases}$ – diagonal disorder

Do we still have delocalized wf ? (cf Bloch theory)

Naive expectation ;

$\varepsilon_i \xrightarrow{t} \varepsilon_i$ $\quad \Delta\varepsilon \sim (\varepsilon_i - \varepsilon_i)_{typ} = \sqrt{\langle(\varepsilon_i)^2\rangle_{dis}}$

$\langle \varepsilon_i \rangle_{dis} = 0$ - shift

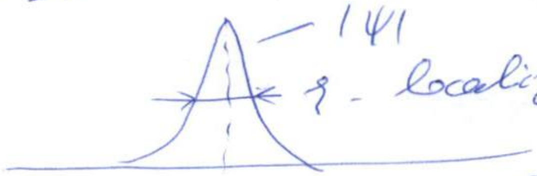
if $t \gg \frac{\Delta\varepsilon}{d}$ $(\Delta\varepsilon)_{eff}$ – hopping is possible, w/f are extended

BUT

if $t \ll \frac{\Delta\varepsilon}{d}$ – hopping becomes hindered, w/f ceased being extended

More rigorously:

- 1d - W/f are localized (c.f. the strength of WL)

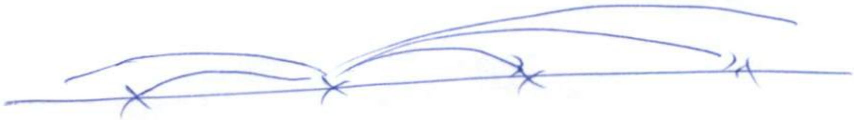


- $d \geq 3$ W/f are $\begin{cases} \text{extended at } \frac{\Delta\varepsilon}{t} < \left(\frac{\Delta\varepsilon}{t}\right)_c \\ \text{localized otherwise} \end{cases}$

$\left(\frac{\Delta\varepsilon}{t}\right)_c$ depends on d, distribution of ε_i, lattice type.
This is called the A-loc transit.

b) RMT-models (usually Gaussian)

Consider 1d chain + long range hopping



The Gaussian random variables are described by

$\langle H_{ij} \rangle_{dis} = 0$; $\langle H_{ii}^2 \rangle_{dis} = h_i \overset{\text{usually}}{=} \text{const}$; $\langle |H_{i\neq j}|^2 \rangle_{dis} = \mathcal{F}(|i-j|)$

Three famous cases

$\mathcal{F}(|i-j|) = \text{const}$ - extremely long-range hopping, called the WD-RMT. It describes the Q-dots, complex nuclei, Q-chaotic systems, etc.

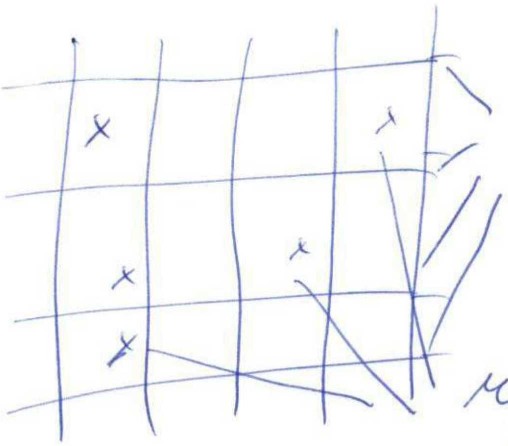
$\mathcal{F}(|i-j|) = \begin{cases} \text{const}, & |i-j| < \Lambda \\ \to 0, & |i-j| > \Lambda \end{cases}$ - banded RMT

$1 \ll \Lambda \ll N$ - good model to describe Q-wires.

$\mathcal{F}(-||-) = \begin{cases} \text{const}, & |i-j| < \Lambda \\ \propto \frac{1}{|i-j|^2}, & |i-j| > \Lambda \end{cases}$ - PL banded RMT - good model to describe the localization transition.

2) Continous models

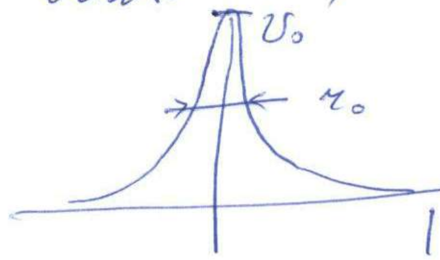
$$\hat{H} = \underbrace{\frac{\hat{p}^2}{2m^*}}_{\text{effective}} + \underbrace{V(\vec{r})}_{\text{random potential}}$$



lattice yield m^*

randomly placed impurities create $V(\vec{r})$

Potential of a single imp.



usually $r_0 \sim \lambda_F$ - small, i.e. δ-like impurities.

Let's add N_{imp} at random positions - the Edwards model.

The total potential

$$V(\vec{r}) = \sum_{j=1}^{N_{imp}} v(\vec{r} - \vec{r}_j)$$

where we will further take the T/D limit $Vol \to \infty$ at fixed $n_{imp} = N_{imp}/Vol = \text{const}$, i.e., N_{imp} also $\to \infty$

If further $n_{imp} \to \infty$ we reach the cont. limit since $d_{imp} \sim \frac{1}{\sqrt[d]{n_{imp}}} \to 0$ at $n_{imp} \to \infty$.

Formally, let's consider a homogeneous distr.

$$P(V)\mathcal{D}\{V\} = \prod_{j=1}^{N} \frac{d\vec{r}_j}{Vol.}$$

and introduce the generating functional

$$P[g] = \int P(V)\,dV \exp\Big(\sum_{i=1}^{N_{imp}} F_g(\vec r_i)\Big)$$

where $\quad F_g(\vec r) \equiv \int d\vec r\,'\, g(\vec r\,')\, v(\vec r - \vec r\,')$

<u>Reminder</u> $P[g]$ can generate correlation functions of the random potential

$$\langle V(\vec r_1)\, V(\vec r_2) \ldots V(\vec r_n)\rangle_{dis} = \frac{\delta^n P[g]}{\delta g(\vec r_1) \ldots \delta g(\vec r_n)}\Big|_{g=0}$$

and „connected" correlation functions or cumulants

$$\langle V(\vec r_1) \ldots V(\vec r_n)\rangle_{dis}^{(con)} = \frac{\delta^n \ln P[g]}{\delta g(\vec r_1) \ldots \delta g(\vec r_n)}\Big|_{g=0}$$

where $\langle V\rangle^c_{dis} = \langle V\rangle_{dis}$; $\langle V(1)\, V(2)\rangle^c_{dis} = \langle V(1)\, V(2)\rangle_{d} - \langle V(1)\rangle\langle V(2)\rangle$ etc.

Using the homog. distr.

$$P[g] = \int \prod \Big(\frac{dr_i}{\Omega}\Big) e^{\sum F(\vec r_i)} = \Big(\int \frac{d\vec r}{\Omega} \exp(F(\vec r))\Big)^{N_{imp}} \equiv$$

(Ω – volume)

$$\equiv \Big(1 + \underbrace{\frac{n_{imp}}{N_{imp}}}_{=\Omega^{-1}} \int d^d r\, \underbrace{(F(\vec r) - 1)}_{\text{yields Vol}}\Big)^{N_{imp}}$$

Now fix n_{imp} and put $N_{imp} \to \infty$

$$\lim_{N\to\infty} \Big(1 + \frac{A}{N}\Big)^N = \exp A$$

$$\Rightarrow \quad P[g]_{\substack{N_{imp}\to\infty \\ n_{imp}-\text{fixed}}} = \exp\Big(n_i \int d\vec r\, [e^{F(\vec r)} - 1]\Big)$$

Cumulants

check yourselves

$$\langle V(\vec r_1) \ldots V(\vec r_n)\rangle_{dis}^{(con)} = n_i \int d\vec r\, v(\vec r - \vec r_1) \ldots v(\vec r - \vec r_n)$$

Again shift energy so that

$$\langle V \rangle_{dis} = 0$$

and define finite 2nd cumulant at $n_{imp} \to \infty$

$$\langle V(\vec{r})\, V(\vec{r}\,')\rangle_{dis}^{(con)} = \mathcal{F}(\vec{r} - \vec{r}\,') = n_i \int d\vec{r}_1\, U(\vec{r}_1 - \vec{r})\, U(\vec{r}_1 - \vec{r}\,')$$

– can be finite if $n_i \underbrace{U^2(0)}_{V_0}$ – finite, i.e.

$\begin{cases} n_i \to \infty \\ V_0 \to 0 \end{cases}$ at fixed $n_i V_0$ – the limit of weak disorder

which is already continuous limit.

Important $\langle VVV\rangle^{(c)} \sim \langle VV\rangle^{(c)} \cdot V_0 \to 0$, i.e., all higher cumulants $\to 0$ and we arrive at the model of the Gaussian disorder

$$\langle V\rangle_d = 0; \quad \langle V\,V\rangle_d^{(c)} = \mathcal{F}(\vec{r} - \vec{r}\,'); \quad \underbrace{\langle V\,V \ldots V\rangle_d^{(c)}}_{n>2} = 0$$

due to the Weak theorem

The extreme case: point-like impurity

$$U(\vec{r}) = V_0\, \delta(\vec{r})$$

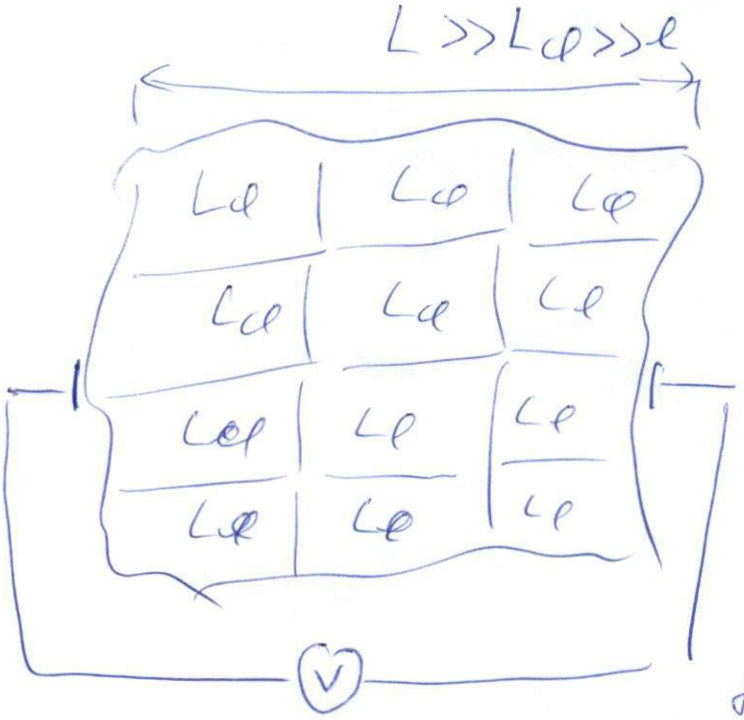
$$\Rightarrow \mathcal{F}(\vec{r} - \vec{r}\,') = n_i V_0^2 \int d\vec{r}_1\, \delta(\vec{r}_1 - \vec{r})\, \delta(\vec{r}_1 - \vec{r}\,') = n_i V_0^2\, \delta(\vec{r} - \vec{r}\,')$$

– the model of the white-noise disordered potential.

This model will be used further.

Why disorder averaging is needed?

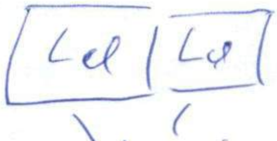
DLS-13



Consider coherent transport in a sample with

$l \ll L_\varphi \ll L$

Coherence length — the space-scale where the coherent transport is developed



two „independent" samples, i.e. they yield 2 independent contributions. If $L_\varphi \ll L$ - many independent regions act as if we have many disordered samples → self averaging, which is simulated by the ensemble (disorder) averaging.

The most natural / convenient apparatus for calculating $\langle \ldots \rangle_{dis}$ - the Green's functions.