The BCS GS

Old (usual) GS:

$$|FS\rangle\Big|_{T=0} = \begin{cases} |0\rangle_{\vec{k}}, & |\vec{k}| > k_F \\ |1\rangle_{\vec{k}}, & |\vec{k}| \le k_F \end{cases}$$ now add attraction

Main ideas

1) the Cooper pairs are macro-objects and we may expect them to „ignore" impurities if they form a coherent state

2) let's try to describe them statistically with fixed phase and fluctuating N (GCEns).

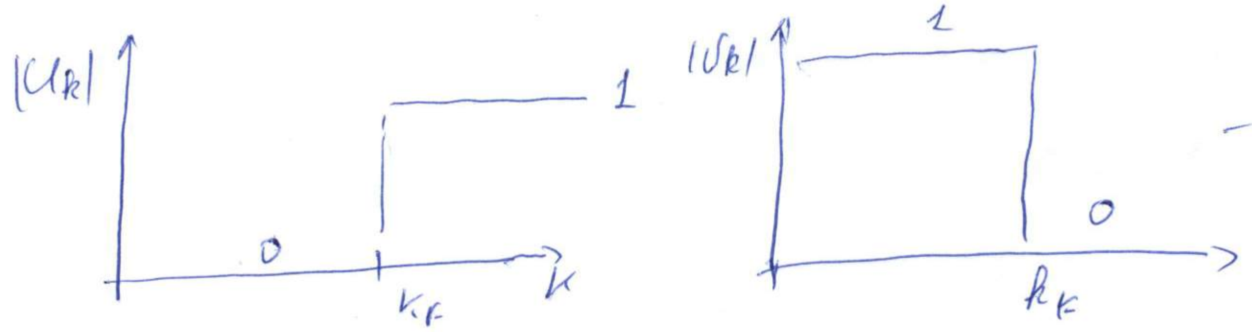
Now - write $|FS\rangle$ in terms of Cooper pairs

$$\underbrace{|1\rangle_{\vec{k},-\vec{k}}}_{\text{just } k} \equiv \hat{c}^+_{\vec{k},\uparrow}\,\hat{c}^+_{-\vec{k},\downarrow}\,|0\rangle_{\vec{k},-\vec{k}}$$

$$\Rightarrow |FS\rangle = \prod_{|\vec{k}|>k_F}|0\rangle_k + \prod_{|\vec{k}|\le k_F}|1\rangle_k = \prod_{\text{all}}\left\{u_k|0\rangle_k + v_k|1\rangle_k\right\}$$

where u_k & v_k are probabilities for states to be empty / occupied

$$|v_k|^2 + |u_k|^2 = 1 - \text{total probability}$$



- for $|FS\rangle$.

Without lost of generality

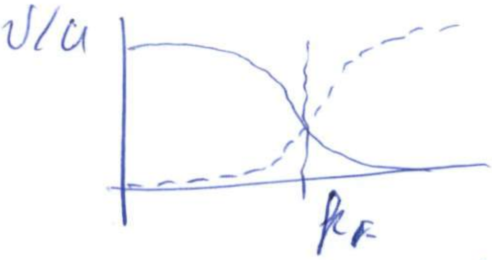
$$\mathrm{Im}\,u_k = 0, \quad v_k = |v_k|\,e^{i\alpha}$$

α - one and the same phase for the coherent state

Schrieffer: introduce

$$|BCS,\varphi\rangle = \prod_{\vec k} \left(u_k + |v_k| e^{i\varphi}\, \hat c^+_{k,\uparrow} \hat c^+_{-k,\downarrow}\right)|0\rangle_k$$

assuming that u & v are smeared due to formation of the new $|GS\rangle$



$|v_k|^2 + u_k^2 = 1$

but the total prob. is, of course, fixed to 1

$|BCS,\varphi\rangle$ must minimize the energy of

$$\hat H - \mu \underbrace{\hat N}_{\text{particle number operator}} \text{ — Hamiltonian for GCens.}$$

Mind that $\langle \hat N\rangle = \bar N$ - mean number of particles.

single particle contribution

$$\hat H_0 - \mu \hat N = \sum_{k,\sigma} \underbrace{\xi_k}_{\varepsilon_k - \mu} \overbrace{\hat n_{k,\sigma}}^{\hat c^+_{k\sigma}\hat c_{k\sigma}}$$

Energy of interactions is simplified

$$\hat H^{(red.)}_{int} \to \sum_{\vec k,\vec k'} V_{\vec k,\vec k'}\, \hat c^+_{k,\uparrow} \hat c^+_{-k,\downarrow} \hat c_{-k',\downarrow} \hat c_{k,\uparrow} \text{ —}$$

- only interaction of pairs is taking into account.

$$\hat H_0 - \mu \hat N + \hat H^{(red)}_{int} = \hat H_{red} \text{ - reduced (BCS) Hamiltonian.}$$

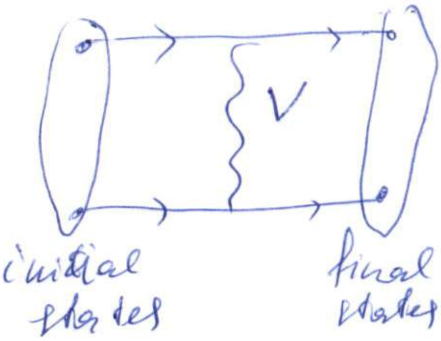
Task find $u_k, |v_k|$ from the minimization eq.

$$\delta_{u,v} \langle BCS,\varphi | \hat H_{red} | BCS,\varphi\rangle = 0$$

Direct calculations yield:

$$\underbrace{\langle BCS | \sum_{k,\sigma} \xi_k \hat{n}_k | BCS \rangle}_{H_0 - \mu \hat{N}} = \underbrace{2}_{\text{from spins}} \sum_{\vec{k}'} \xi_{\vec{k}'} \underbrace{|v_{\vec{k}'}|^2}_{\text{prob. that the state is occupied}}$$

$$\langle BCS | \hat{H}^{(red)}_{int} | BCS \rangle = \sum_{k,k'} V_{k,k'} (u_k v_k)^* (u_{k'} v_{k'})$$



	before	after
k	empt, u	occup., v
k'	occup, v	empt, u

$$\Rightarrow \quad \bar{E} = 2 \sum_{k'} \xi_{k'} |v_{k'}|^2 + \sum_{\vec{k},\vec{k}'} V_{\vec{k},\vec{k}'} \underbrace{u_{\vec{k}} v_{\vec{k}'} v^*_{\vec{k}} u_{\vec{k}'}}_{\text{phase drops out.}}$$

<u>Note</u> $\langle FS | \hat{H}^{(red)}_{int} | FS \rangle = 0$ because $u, |v|$ are not-overlapping θ functions $\Rightarrow$ the smearing is indeed needed.

$u_k^2 + |v_k|^2 = 1 \Rightarrow$ convenient parametrization

$$\begin{cases} u_k = \sin\theta_k \\ v_k = e^{i\varphi} \cos\theta_k \end{cases}$$

The minimization equation

$$\partial_{\theta_k} \bar{E}(\{\theta\}) = 0$$

yield for the BCS model of attraction

$$|v_k|^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\Delta_k^2 + \xi_k^2}}\right)$$

$$\Delta_k = \begin{cases} \Delta, & \text{inside the layer} \\ 0, & \text{otherwise} \end{cases} \quad \text{– the gap which obeys}$$

$$\frac{2}{g} = \sum_{l \in \text{layer}} \frac{1}{\sqrt{\Delta^2 + \xi_k^2}}$$

changing to $\int d\xi$:

$$\frac{2}{g} = \frac{\nu}{2}\, 2 \int_0^{\hbar\omega_D} \frac{d\xi}{\sqrt{\Delta^2 + \xi^2}} = \nu \sinh^{-1} \frac{\hbar\omega_D}{\Delta} -$$

different from the simplified Cooper model but in the weak coupling limit

$$\Delta = \left.\frac{\hbar\omega_D}{\sinh\left(\frac{2}{g\nu}\right)}\right|_{g\nu \ll 1} \simeq 2\hbar\omega_D\, e^{-\frac{2}{g\nu}} -$$

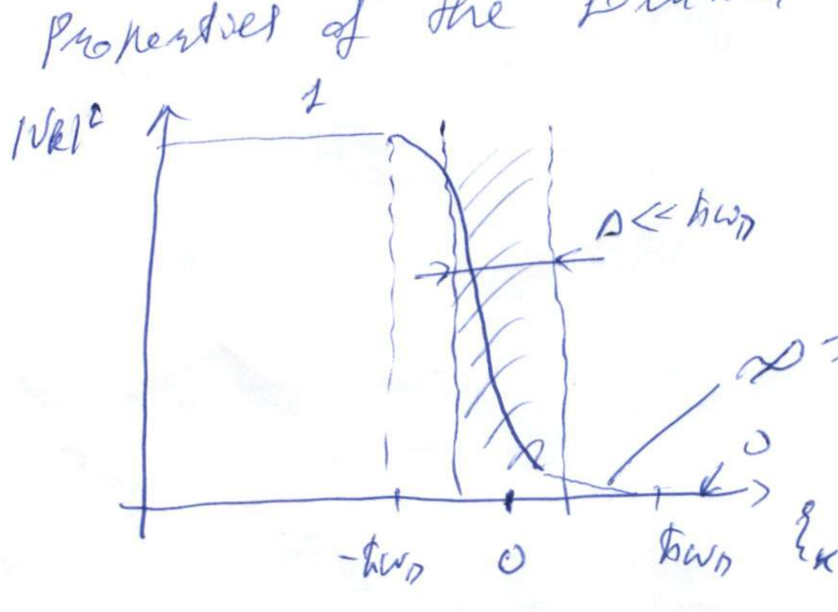
very similar to the Cooper result.

The energy of the new GS (check yourselves)

$$\underbrace{E(\Delta)}_{\text{Sup Cond}} - \underbrace{E(\Delta=0)}_{N-Me} = -\frac{1}{4}\nu\Delta^2 < 0 \,!$$

It's negative condensation energy - pairing is OK and the new GS is the coherent condensate of the Cooper pairs.

- Properties of the solution



Substantially changed only for $|\xi_k| \sim \Delta \ll \hbar\omega_D$ - this verifies the initial assumptions.