The Bogolubov transform, BCS as MFA

Idea: $|BCS, \alpha\rangle$ describes a coherent macro-state with $N \gg 1$ particles $\Rightarrow$ let's use St Ph method assuming that the occupancy of each state $\vec{k}$ depends only on the average occupancy of other states – MFA.

The reduced $\hat{H}$ can be written as

$$\hat{H}_{red} = \sum_{k\sigma} \xi_k \hat{n}_{k\sigma} + \sum_{\vec{k},\vec{e}} V_{k,e} \underbrace{\hat{b}_k^{+} \hat{b}_e}_{\text{operators of the Cooper pairs (boson-like)}}$$

In N-Me $\langle \hat{b}_k \rangle (\equiv B_k) = \langle \hat{b}^{+}_k \rangle (\equiv B_k^{*}) = 0$ since such expect. values are prohibited by the gauge- (U(1)) invariance. One can show that U(1)-symmetry is broken in Sup Cond and, therefore,

$$B_k \neq 0 \quad \& \quad B_k^{*} \neq 0.$$

In other words: $B_k = B_k^{*} = 0$ in N-Me due to random phases and they are $\neq 0$ in Sup Cond due to the coherence.

Decompose

$$\hat{b}_k = B_k + \overbrace{(\hat{b}_k - B_k)}^{\text{def } \Delta B_k}$$

$(\hat{b}_k - B_k)$ describe fluctuations, which must be small in the BCS theory. Let's keep only $(\Delta B)^2$ in $\hat{H}_{red}$:

$$\hat{H}_{red.} \simeq \hat{H}_0 + \sum_{\vec{k},\vec{e}} V_{ke} \left(-B_k^{*} B_e + (\hat{b}_k^{+} - B_k^{*}) B_e + (\hat{b}_e - B_e) B_k \right)$$

Define

$$\Delta_m \overset{def}{\equiv} -\sum_p \underbrace{V_{mp}}_{\text{taken}} B_p \quad \text{from the BCS model}$$

$$\Rightarrow \hat{H}^{(MFA)}_{\cancel{red}} = \hat{H}_0 - \sum_k \left(\Delta_k \hat{b}_k^\dagger + h.c.\right) + \sum_k \Delta_k B_k^* \quad \text{– quadratic}$$

but not diagonal. Let's rotate basis to diagonalize:

$$\cancel{\hat{a}^\dagger_{k,\uparrow} = u_k^* \hat{\gamma}_{k\uparrow} + v_k^* \hat{\gamma}^\dagger_{-k\downarrow}} \qquad \cancel{\hat{a}_{k,\downarrow}}$$

$$\hat{a}_{k,\uparrow} = u_k^* \hat{\gamma}_{k,0} + v_k \hat{\gamma}^\dagger_{k,1}$$

$$\hat{a}^\dagger_{-k,\downarrow} = -v_k^* \hat{\gamma}_{k,0} + u_k \hat{\gamma}^\dagger_{k,1}$$

[Tinkham-book, Eq. (2.42)]

$$\{\hat{a}_p, \hat{a}^\dagger_{p'}\} = \delta_{p,p'} \; ; \quad \{\hat{a}_p, \hat{a}_{p'}\} = \{\hat{a}^\dagger_p, \hat{a}^\dagger_{p'}\} = 0 \quad \text{– fermionic}$$

the basis is rotated but ~~new~~ should remain fermionic

$$\{\hat{a}_{k\uparrow}, \hat{a}^\dagger_{k,\uparrow}\} = \{u_k^* \hat{\gamma}_{k,0} + v_k \hat{\gamma}^\dagger_{k,1},\; u_k \hat{\gamma}^\dagger_{k,0} + v_k^* \hat{\gamma}_{k,1}\} =$$

$$\underbrace{|u_k|^2 + |v_k|^2 = 1}_{\text{cf the variational approach}} \quad \text{since } \hat{a}^\dagger, a \text{ are fermionic}$$

The same parameters as previously

$$\operatorname{Im}(u_k) = 0, \quad v_k = \underbrace{|v_k|}_{\text{to be found}} e^{i\alpha} \quad \text{– one and the same}$$

If (to be checked by students)

$$\begin{Bmatrix} |v_k|^2 \\ u_k^2 \end{Bmatrix} = \frac{1}{2}\left(1 \mp \frac{\xi_k}{E_k}\right) \quad \text{with } E_k = \sqrt{\Delta^2 + \xi_k^2} \quad \text{– cf (BCS, a)}$$

then $\hat{H}^{(MFA)}$ becomes diagonal

$$\hat{H}^{(MFA)} = H_0^{(MFA)} + \sum_k E_k \left(\hat{n}^{(\gamma)}_{k,0} + \hat{n}^{(\gamma)}_{k,1}\right)$$

Where $H_0^{(MFA)} = -\frac{1}{4}\Delta^2 U$ – the same (GS) energy as before

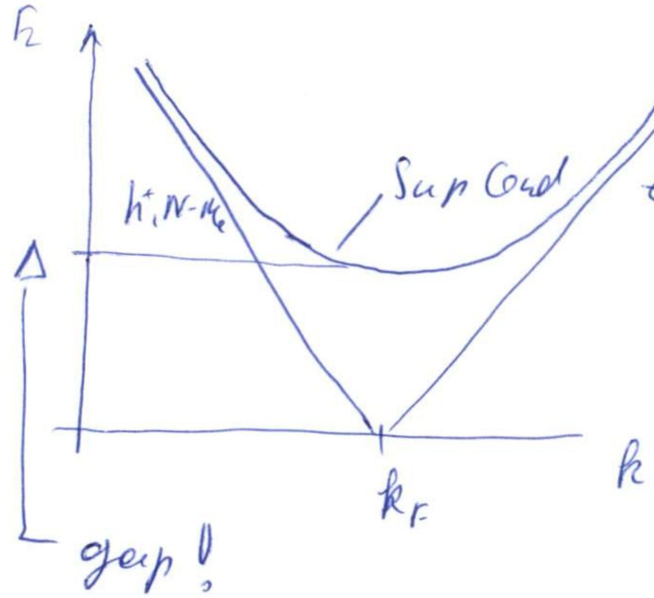
$\hat{n}_k^{(j)} = \hat{j}_k^+ \hat{j}_k$ – particle number operator in new basis

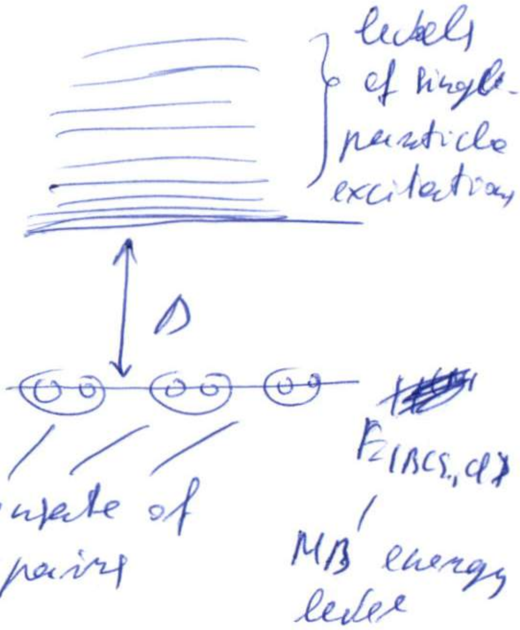
$E_k = \sqrt{\xi_k^2 + \Delta^2}$ – the disp. relation of new particles

$\Delta =$ const in the BCS model of attraction

Analysis

Cartoon





This answer removes the contradiction with the Landau criterion $\underline{V_c \sim \Delta/p_F > 0}$

Excitations are fermions $\Rightarrow$

$$\langle \hat{j}^+_{k\sigma} \hat{j}_{k,\sigma} \rangle = f_{Fermi}(E_k) = \frac{1}{e^{\beta E_k} + 1}$$

Note that $\mu_j = 0 \Rightarrow f(E_k)\big|_{T\to 0} \to 0$ as $e^{-\beta\Delta}$ for all k

The gap equation

Reminder

$$\Delta_k = -\sum_l V_{kl} \langle \hat{b}_l \rangle$$

where

$$\langle \hat{b}_l \rangle = \langle \hat{a}_{-l,\downarrow} \hat{a}_{l,\uparrow} \rangle =$$

$$= \langle (-v_k \hat{\gamma}^\dagger_{k,0} + u_k^* \hat{\gamma}_{k,1})(u_k^* \hat{\gamma}_{k0} + v_k \hat{\gamma}^\dagger_{k,1}) \rangle =$$

= || cross terms average out to zero ||

$$u_k^{\cancel{*}} v_k \left(1 - \langle \hat{n}_{k,1} \rangle - \langle \hat{n}_{k,0} \rangle \right) = u_k |v_k| e^{i\varphi} (1 - 2f(E_k))$$

$$1 - 2f(E) = 1 - \frac{2}{e^{\beta E}+1} = \frac{e^{\beta E}-1}{e^{\beta E}+1} = \tanh \frac{\beta E}{2}$$

$\Rightarrow$ the self consistency equation

$$\underbrace{\Delta}_{\text{for the BCS-model of attraction}} \cancel{e^{i\varphi}} = g \sum_{l \in \text{layer}} \tanh \frac{\beta E_l}{2} u_l |v_l| \cancel{e^{i\varphi}}$$

$$u_l |v_l| = \sqrt{\frac{1}{2}\left(1 - \frac{\xi_l}{E_l}\right) \frac{1}{2}\left(1 + \frac{\xi_l}{E_l}\right)} = \frac{1}{2}\sqrt{1 - \left(\frac{\xi_l}{E_l}\right)^2} =$$

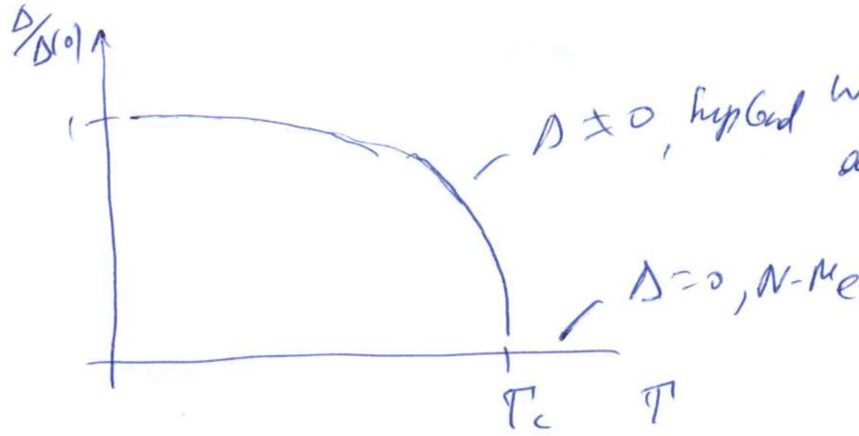
$$= \frac{1}{2} \frac{\Delta}{E_l}$$

$$\Rightarrow \cancel{\Delta} = \frac{g}{2} \sum_{l \in L} \tanh \frac{\beta E_l}{2} \frac{\cancel{\Delta}}{\underbrace{\sqrt{\Delta^2 + \xi_l^2}}_{E_l}}$$

change to the integral

$$\boxed{\frac{2}{gD} = \int_0^{\hbar\omega_D} \frac{\tanh\left(\frac{\beta E(\xi)}{2}\right)}{E(\xi)} d\xi, \quad E(\xi) = \sqrt{\Delta^2 + \xi^2}}$$

The solution looks like



where $\Delta(T=0)$ agrees with the BCS-answer

Eq. for T_c is obtained by inserting $\Delta \to 0$

$$\Rightarrow \frac{2}{g\nu} = \int_0^{\hbar\omega_D} \frac{\tanh\left(\frac{\beta_c \xi}{2}\right)}{\xi}\, d\xi = \int_0^{\frac{\beta_c \hbar\omega_D}{2}} \frac{\tanh x}{x}\, dx$$

In the weak-coupling limit $\beta_c \hbar\omega_D \gg 1$

$$\int_0^{a\gg 1} \frac{\tanh x}{x}\, dx = \log a \underbrace{\tanh a}_{\simeq 1} - \underbrace{\int_0^{a\to\infty} \frac{\log x}{\cosh^2 x}\, dx}_{\simeq -\gamma + \log\frac{\pi}{4}}$$

$a \to \infty$ since $\int dx$ converges

$\gamma \approx 0.577$

$$\simeq \log\left(\frac{a\,4}{\pi} e^{\gamma}\right)$$

$$\Rightarrow \frac{2}{g\nu} \simeq \log\left(2\, \frac{\hbar\omega_D}{k_B T_c}\, A\right) \quad \text{where } A = \frac{e^{\gamma}}{\pi}$$

which yields

$$k_B T_c \simeq \cancel{2} A\, \underbrace{2\hbar\omega_D\, e^{-2/g\nu}}_{} \ll \hbar\omega_D \text{ as expected}$$

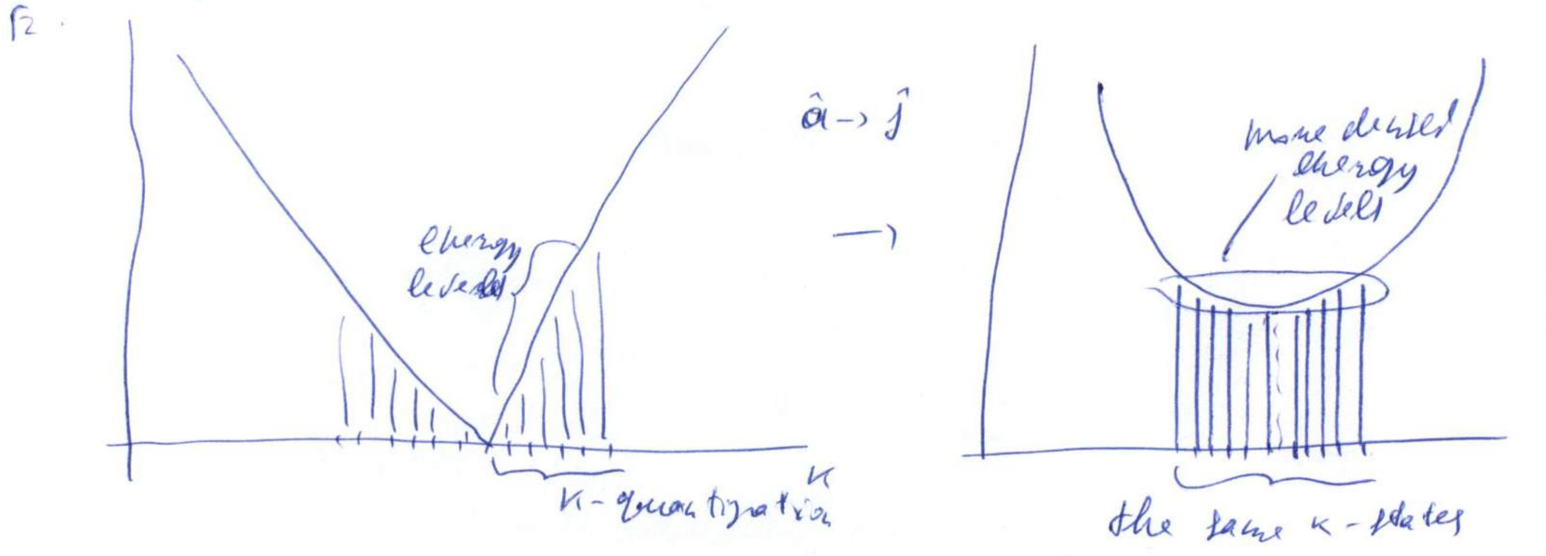
The ratio

$$\frac{k_B T_c}{\Delta(T=0)} = \frac{\cancel{2} A\, \cancel{\Delta(T=0)}}{\cancel{\Delta(T=0)}} = A$$

– universal in the BCS in the weak coupling limit.

DoS of the Bogolubov q-particles

N-Me

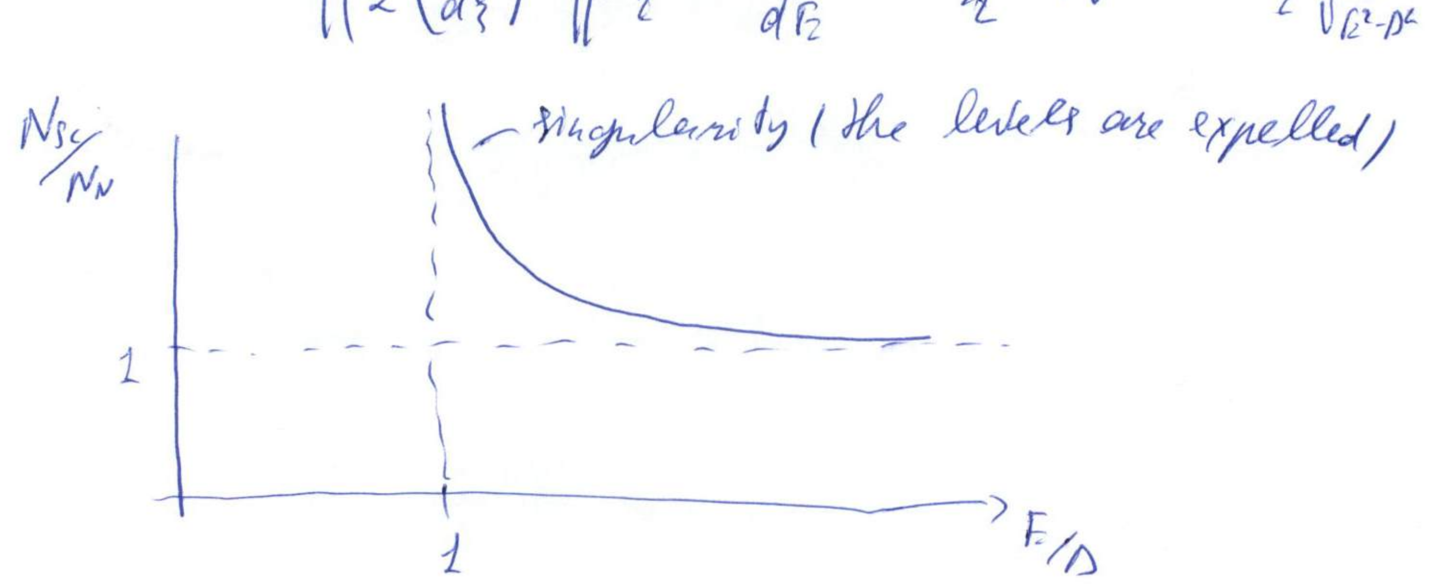


Due to correspondence of k-states

$$N_N(\xi)\,d\xi = \underbrace{N_{sc}(E)\,dE}_{\text{number of states in } \left[E-\frac{dE}{2};\, E+\frac{dE}{2}\right]}$$

In the small vicinity of ε_F $N_N(\xi) \simeq \nu = \text{const}$ per spin

$$\Rightarrow N_{sc}(E) = \left|\frac{\nu}{2}\left(\frac{dE}{d\xi}\right)^{-1}\right| = \frac{\nu}{2}\,\frac{d\xi(E)}{dE} = \frac{\nu}{2}\,\partial_E\sqrt{E^2-\Delta^2} = \frac{\nu}{2}\,\frac{E}{\sqrt{E^2-\Delta^2}}$$



N_{sc} can be studied in single-particle tunneling