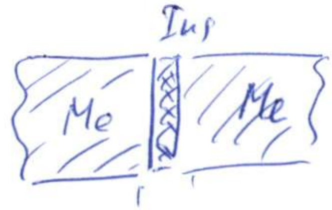
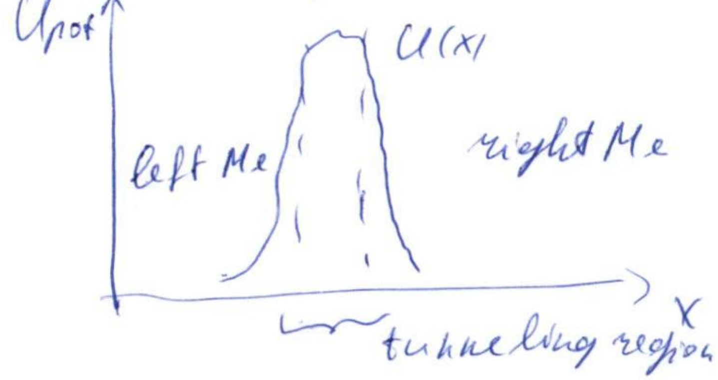
# Tunneling in NSS & SSS junctions






– 2 Me-samples separated by a potential barrier




– can be described by the tunneling $\hat{H}$:

$$\hat{H} = \hat{H}_{L-Me} + \hat{H}_{R-M} + \hat{H}_{tun}$$

where $\hat{H}_{tun} = \sum_{all} \{ T_{pq}\, \hat{a}^{+}_{p,\sigma}\, \hat{b}_{q,\sigma} + h.c. \}$

$\hat{a}^{+}, \hat{a}$ – for left-Me; $\hat{b}^{+}, b$ – for right-Me

Total current

$$J_{tot} = J_R - J_L$$

where currents in Right/Left-Me can be found as

$$J_{L,R} = e \left\langle \frac{d}{dt} \underbrace{\hat{N}_{L,R}}_{\text{particle number operator}} \right\rangle$$

$$\frac{d}{dt}\hat{N}_{L,R} = i\hbar [\hat{H}, \hat{N}_{L,R}] = i\hbar [\hat{H}_{tun}, \hat{N}_{L,R}]$$

This yields (see Jan's lecture „Single ē tunneling" or Abrikosov's book, sect. 22.3) at finite applied V

$$\underset{\text{per 1 cm}^2}{J_{tot}} = +\frac{4\pi e}{\hbar} \overbrace{|T|^2}^{\text{average (around } \varepsilon_F)} \int d\varepsilon\, \overbrace{\nu_L(\varepsilon - eV)\, \nu_R(\varepsilon)}^{\text{DoS of L/R-Me}} \times$$

$$\underbrace{\left[ f(\varepsilon - eV)[1 - f(\varepsilon)] - f(\varepsilon)[1 - f(\varepsilon - eV)] \right\}}_{\text{the Fermi-factors}}$$

$$= + \frac{4\pi e}{\hbar}|\bar{T}|^2 \int d\xi\, \nu_L(\xi - eV)\,\nu_R(\xi)\,\big(f(\xi - eV) - f(\xi)\big) \quad (6)$$

Case NIN $\Rightarrow$ at $\nu_L \simeq \nu_R = \nu(\xi_F) = \text{const}$

$$J_{tot} = + \frac{4\pi e}{\hbar}|\bar{T}|^2\, \nu^2(\xi_F) \underbrace{\int d\xi\, \big(f(\xi - eV) - f(\xi)\big)}_{\simeq eV}$$

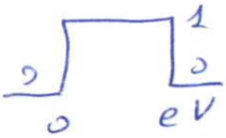
$$\Rightarrow \frac{J_{tot}}{S} = G V, \qquad G = \frac{4\pi e^2}{\hbar}\,\frac{|\bar{T}|^2}{S}\,\nu^2(\xi_F)$$

L Ohm's low

L square of the contact.

Case NIS take $T \to 0$

$$J_{tot} = \frac{4\pi e}{\hbar}|\bar{T}|^2\, \nu_L(\xi_F) \int d\xi\, \nu_R^{(SC)}(\xi) \underbrace{\left[f(\xi - eV) - f(\xi)\right]}_{\text{step function}}$$




$$\Rightarrow J_{tot} = \frac{4\pi e}{\hbar}|\bar{T}|^2\, \nu_L(\xi_F) \int_0^{eV} d\xi\, \nu_R^{(SC)}(\xi)$$

where $\nu_R^{(SC)}(E) = \nu_R \mathrm{Re}\left(\frac{E}{\sqrt{E^2 - \Delta^2}}\right) = \nu_R \mathrm{Re}\, \partial_E \sqrt{E^2 - \Delta^2}$

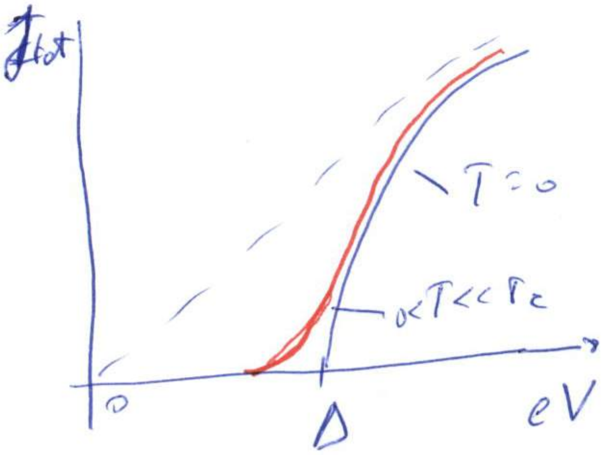
$\Rightarrow J_{tot} = 0$ at $eV < \Delta$

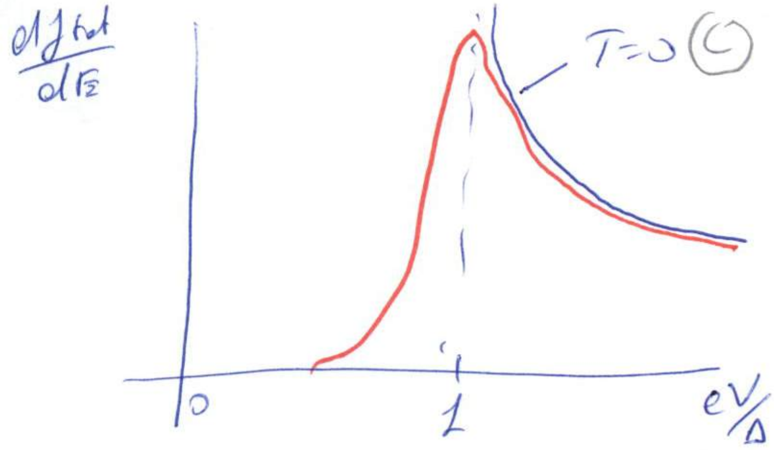
$$J_{tot} = \frac{4\pi e}{\hbar}|\bar{T}|^2\, \nu_L(\xi_F)\, \nu_R(\xi_F) \int_\Delta^{eV} d\xi\, \frac{d}{dE}\sqrt{E^2 - \Delta^2} =$$

$$= \frac{4\pi e}{\hbar}|\bar{T}|^2\, \nu_L(\xi_F)\, \nu_R(\xi_F) \sqrt{(eV)^2 - \Delta^2} \quad \text{at } eV > 0$$

Finite Temp. yields broadening.

Important: $\frac{dJ_{tot}}{dV} = \text{const}\ \nu_R^{(SC)}(eV)$

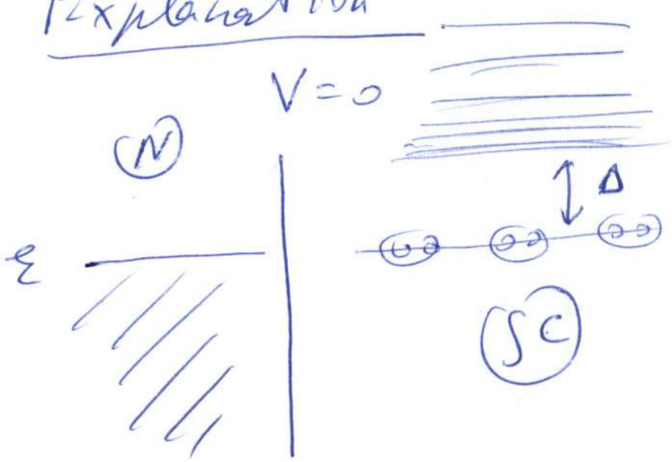


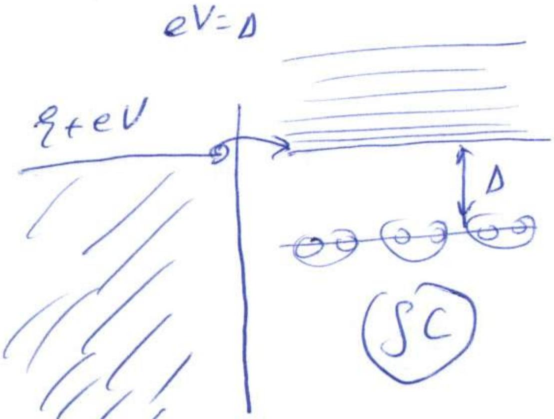




(C)

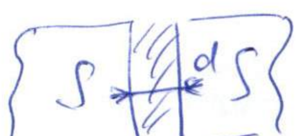
at $T \ll T_c$ tails below the gap is small as $e^{-\frac{\Delta}{k_B T}}$

Explanation




Most nontrivial case SIS or SNS at $V=0$




$d_{typ} \sim 1nm$ for SIS

$d_{typ} \sim 10^3 nm$ for SNS

Coherent tunneling of Cooper pairs is possible since $d$ is small - there can be super current at $V=0$, called the Josephson effect (weak sup cond).

Set up: let left-/right Sup Cond be made from the same material. Introduce the W/f for the bulks at $T_{in}=0$ (disconnected)

$$\Psi_{R/L} = \psi e^{i\varphi_{R/L}}$$ - describes the coherent condensate of the C-pairs

(d)

Normalization: $|\Psi_{R/L}|^2 = n_{sc}$ - density of particles in the condensate, which is the same if $T_L = T_R$

$\Rightarrow$ amplitudes are equal $\Psi_R = \Psi_L = \Psi = \sqrt{n_{sc}}$ but phases can be different

$$\begin{cases} \theta = \theta_R, & x > 0 \\ \theta = \theta_L, & x < 0 \end{cases}, \quad \varphi = \theta_R - \theta_L \neq 0$$

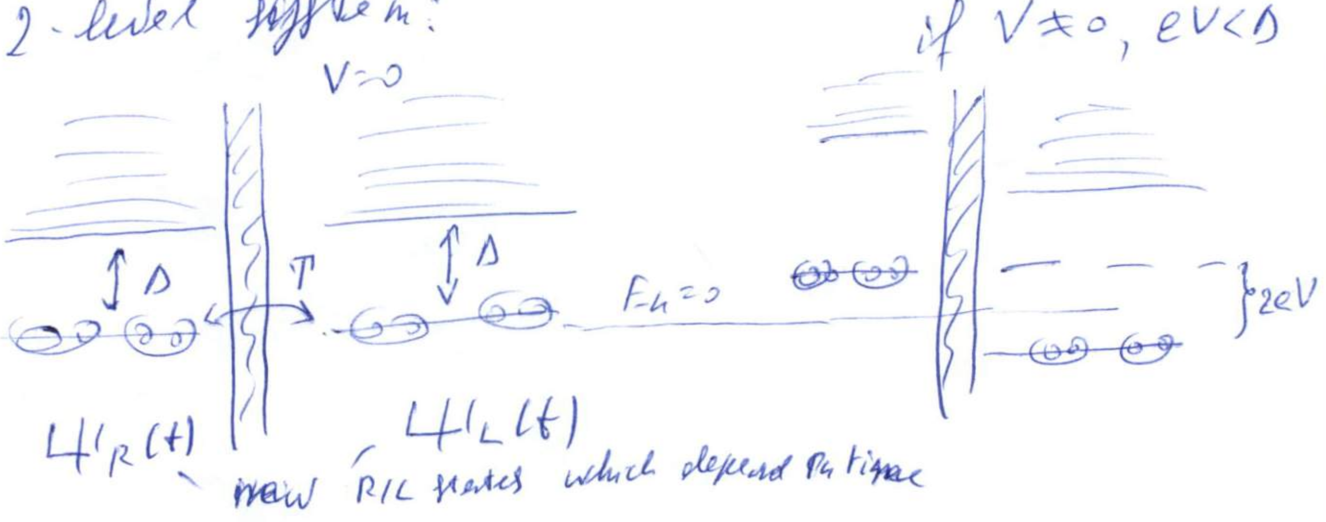
Main idea: $J_{Jos} = f(\varphi) < \begin{array}{l} f(0) = 0 \\ f(\varphi + 2\pi) = f(\varphi) \\ f(-\varphi) = -f(\varphi) \end{array}$

The simplest possible case:

$$J_{Jos}|_{V=0} = J_c \sin\varphi$$

where $J_c$ is max of the super-current through the tunneling contact

Calculation can be done starting from the tunneling Hamiltonian or using the Feynman's short-cut: let's model the Josephson junction with the help of 2-level system:




Schrödinger eq. $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$, where $\hat{H} = \underbrace{\begin{pmatrix} H_L & T \\ T & H_R \end{pmatrix}}_{\text{in the basis of initial states}}$

$T \Leftrightarrow$ transition probability, $H_{L/R} = \pm eV$

The vector $\Psi'(t) = \begin{pmatrix} \Psi'_L(t) \\ \Psi'_R(t) \end{pmatrix}$ is constructed from (e)

$$\Psi'_{L/R}(t) = \sqrt{n_{s,R/L}(t)}\, e^{i\theta_{L/R}(t)}$$

This yields 2 coupled differential equations

$$\begin{cases} i\hbar \frac{d}{dt}\Psi'_L = eV\Psi'_L + T\Psi'_R & \Big| \times(-i)\Psi'^*_L \\ i\hbar \frac{d}{dt}\Psi'_R = T\Psi'_L - eV\Psi'_R & \Big| \times(-i)\Psi'^*_R \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2}\frac{d}{dt} n_{s,L} + i\, n_{sL}\frac{d}{dt}\theta_L = \hbar^{-1}(-i)\left[eVn_{s,L} + T\sqrt{n_{s,L}n_{s,R}}\, e^{i\varphi}\right] \\ \frac{1}{2}\frac{d}{dt} n_{s,R} + i\, n_{s,R}\frac{d}{dt}\theta_R = \hbar^{-1}(-i)\left[T\sqrt{n_{s,L}n_{sR}}\, e^{-i\varphi} - eVn_{s,R}\right] \end{cases}$$

Take the Re:

$$\frac{1}{2}\frac{d}{dt} n_{s,L} = -\frac{1}{2}\frac{d}{dt} n_{s,R} = \frac{T}{\hbar}\underbrace{\sqrt{n_{s,L}n_{s,R}}}_{n_s^2}\sin\varphi \simeq \frac{Tn_s}{\hbar}\sin\varphi$$

in the leading order in $T$

$$\boxed{\frac{d}{dt} n_{s,L} = -\frac{d}{dt} n_{s,R} = \frac{2Tn_s}{\hbar}\sin\varphi}$$

Take the Im:

$$\frac{d}{dt}\theta_L = -\frac{eV}{\hbar} - \frac{T}{\hbar}\underbrace{\sqrt{\frac{n_{s,R}}{n_{s,L}}}}_{1+O(T)}\cos\varphi \simeq -\frac{eV}{\hbar} - \frac{T}{\hbar}\cos\varphi$$

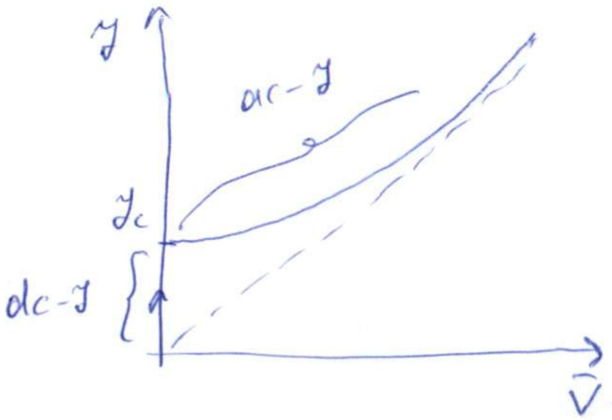
$$\frac{d}{dt}\theta_R = \frac{eV}{\hbar} - \frac{T}{\hbar}\underbrace{\sqrt{\frac{n_{s,L}}{n_{s,R}}}}_{1+O(T)}\cos\varphi \simeq \frac{eV}{\hbar} - \frac{T}{\hbar}\cos\varphi$$

$$\Rightarrow \boxed{\frac{d}{dt}\underbrace{(\theta_R - \theta_L)}_{\varphi} = \frac{2eV}{\hbar}}$$

Current $L \to R$ in the tunneling junction $\propto \frac{dn_{s,L}}{dt} \Rightarrow \boxed{J = J_c\sin\varphi}$

If the current source is applied



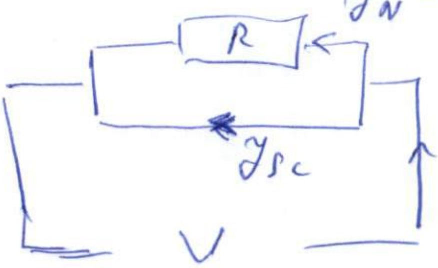



dc: $V=0$ at $|J|<J_c$

ac: $V(t)$ appears at $|J|>J_c$

Note $V(t)$ - periodic

$V(t)$ can be explained with the help of the resistive model




$J_N$ — appears since $J > \max J_{sc}$

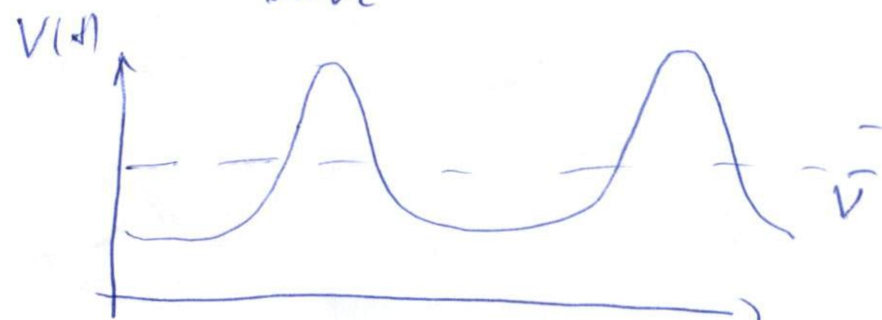
$$J = \underbrace{J_N}_{V/R} + \underbrace{J_{sc}}_{J_c \sin\varphi}$$

Insert $V = \frac{\hbar}{2e}\frac{d\varphi}{dt}$

$$\Rightarrow \quad J = J_c \sin\varphi + \frac{\hbar}{2eR}\frac{\partial\varphi}{\partial t}$$

Solving it and finding V yields

$$V(t)\Big|_{|J|>J_c} = R\,\frac{J^2 - J_c^2}{J + J_c\cos\omega t}\,, \quad \omega = \frac{2e}{\hbar} R\sqrt{J^2 - J_c^2}$$




— Josephson generation of ac $V(t)$

$$\bar{V} = \langle V(t)\rangle_t = \frac{\hbar\omega}{2e}$$

If the voltage source is applied

$V = const$ (easy for SIS) $\Rightarrow$

$$\frac{d}{dt}\varphi = 2\frac{eV}{\hbar} \Rightarrow \varphi = \frac{2eVt}{\hbar} + \varphi_0$$

and we come accross ac current

$$\boxed{J = J_c \sin\left(\varphi_0 + \frac{2eV}{\hbar}t\right)}$$