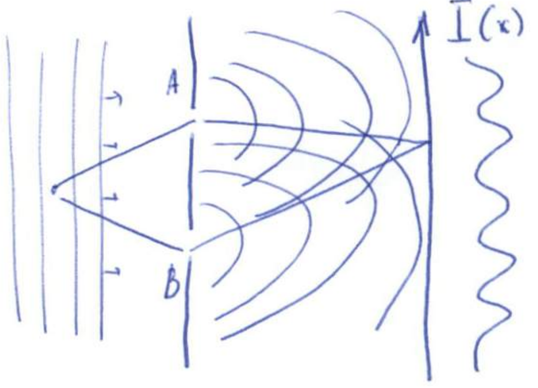
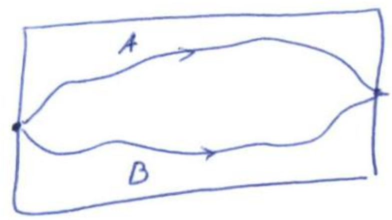
Lecture 1

Meso = middle (from Greek)

Characteristic scales:





$\leftarrow \psi = \psi_A + \psi_B$ (quantum mechanically)

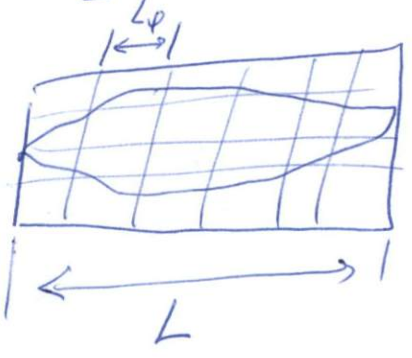
Famous double-slit experiment

$$P = \underbrace{P_A + P_B}_{\substack{\text{incoherent} \\ \text{classical}}} + \underbrace{2\,\mathrm{Re}\,\bar{\psi}_A \psi_B}_{\substack{\text{coherent} \\ \text{(interference)} \\ \text{quantum}}}$$

$$\text{Wavelength } \lambda \sim \frac{\hbar}{p} \sim \Delta r_{min} \underset{\text{"large"}}{\ll} L \underset{\text{"still quantum"}}{\ll} \text{Dephasing length } L_\varphi$$

[Micro/wave-like/ Coherent/quantum] Meso [Macro/particle-like/ incoherent/classical]

$L_\varphi = ?$



Classical

L1-2

States are vectors in Hilbert space:

$$|\psi\rangle,\ |\phi\rangle,\ \alpha|\psi\rangle+\beta|\phi\rangle \subset H$$

ket vectors

Conjugate to $|\psi\rangle$: $\langle\psi|$ - bra vector

Scalar product $\langle\psi|\phi\rangle$

Observables: Hermitian op. $O=\langle\psi|\hat{O}|\psi\rangle$

Basis: eigenvectors of an observable

Example: $\hat{\vec{r}}|\vec{r}\rangle=\vec{r}|\vec{r}\rangle \Rightarrow \{|\vec{r}\rangle\}$ - complete set

Projection $\langle\vec{r}|\psi\rangle\equiv\psi(\vec{r})$

Resolution of unity: $\hat{1\!\!1}=\int d^3r\,|\vec{r}\rangle\langle\vec{r}|$

Normalisation $\langle\vec{r}|\vec{r}\,'\rangle=\delta(\vec{r}-\vec{r}\,')$

Change of basis:

$$|\vec{r}\rangle=\int\frac{d^3p}{(2\pi\hbar)^3}\,|\vec{p}\rangle\cdot\langle\vec{p}|\vec{r}\rangle$$

$$\langle\vec{p}|\vec{p}\,'\rangle=(2\pi\hbar)^3\,\delta(\vec{p}-\vec{p}\,')$$

L1-3

States evolve (in Schrödinger picture)

$$i\hbar \partial_t |\psi_S(t)\rangle = \hat{\mathcal{H}} |\psi_S(t)\rangle$$

$$|\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{\mathcal{H}}(t-t')} |\psi_S(t')\rangle$$

Separate free particles Hamiltonian

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{V}$$

Change to interaction picture

$$|\psi_S(t)\rangle = e^{-\frac{i}{\hbar}\hat{\mathcal{H}}_0 t} |\psi(t)\rangle$$

to find the dynamics of $|\psi(t)\rangle$ is given by

$$i\hbar \partial_t |\psi(t)\rangle = \hat{V}(t) |\psi(t)\rangle , \quad \hat{V}(t) \equiv e^{\frac{i}{\hbar}\hat{\mathcal{H}}_0 t} \hat{V} e^{-\frac{i}{\hbar}\hat{\mathcal{H}}_0 t}$$

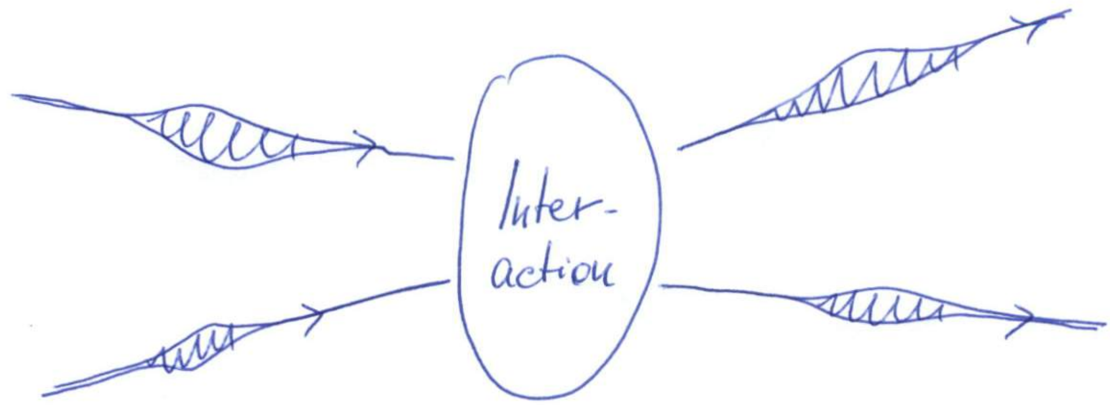
The evolution operator $\hat{U}(t,t')$ for the states in interaction picture ($\hbar \to 1$):

$$|\psi(t)\rangle = \hat{U}(t,t') |\psi(t')\rangle , \quad \hat{U}(t,t_1)\hat{U}(t_1,t') = \hat{U}(t,t')$$

$$\hat{U}(t,t') = e^{i\hat{\mathcal{H}}_0 t} e^{-i\hat{\mathcal{H}}(t-t')} e^{-i\hat{\mathcal{H}}_0 t'} = \hat{T} e^{-i\int_{t'}^{t} dt_1 \hat{V}(t_1)}$$

$\uparrow$ time (or chronological) ordering

L1-4



in (coming) $\qquad$ out (going)

$$|\psi_s(t\to\infty)\rangle = e^{-i\mathcal{H}_0 t}|\psi(t)\rangle = e^{-i\mathcal{H}_0 t}|in/out\rangle$$

because at $t=\infty$ no interaction ($\mathcal{H}\to\mathcal{H}_0$).

"Proper states" = wave packets (finite size)!

At an arbitrary moment $t=0$:

$$|\psi_s(t)\rangle = e^{-i\mathcal{H}t}|\psi_s(0)\rangle$$

$$\begin{cases} |\psi_s(0)\rangle = \lim\limits_{t\to-\infty} e^{i\hat{\mathcal{H}}t}e^{-i\hat{\mathcal{H}}_0 t}|in\rangle \\ |\psi_s(0)\rangle = \lim\limits_{t\to+\infty} e^{i\hat{\mathcal{H}}t}e^{-i\hat{\mathcal{H}}_0 t}|out\rangle \end{cases}$$

Møller wave operators:

$$\hat{\Omega}_\pm = \lim_{t\to\mp\infty} e^{i\hat{\mathcal{H}}t}e^{-i\hat{\mathcal{H}}_0 t} = \hat{U}(0,\mp\infty)$$

Evolution operator in interaction picture

Scattering Theory

$|out\rangle = \hat{S}\,|in\rangle$, $\hat{S} = \hat{U}(+\infty,-\infty) = \hat{T} e^{-i\int_{-\infty}^{+\infty} dt\, \hat{V}(t)}$

$$\begin{cases} |\psi(0)\rangle = \hat{\Omega}_+ |in\rangle \\ |\psi(0)\rangle = \hat{\Omega}_- |out\rangle \end{cases}$$

To relate $\hat{\Omega}_\pm$ to $\hat{S}$ we need to know properties of the wave operators.

1. $e^{i\hat{H}\tau} \lim_{t\to\pm\infty} e^{i\hat{H}t} e^{-i\hat{H}_0 t} = \lim_{t\to\pm\infty} e^{i\hat{H}t} e^{-i\hat{H}_0 t} \cdot e^{i\hat{H}_0\tau}$

$$e^{i\hat{H}\tau}\hat{\Omega}_\pm = \hat{\Omega}_\pm e^{-i\hat{H}_0\tau} \Rightarrow \boxed{\hat{H}\hat{\Omega}_\pm = \hat{\Omega}_\pm \hat{H}_0}$$

2. If $\hat{H}_0 |\psi_i\rangle = E_i |\psi_i\rangle$
   then $\hat{H}[\hat{\Omega}_\pm|\psi_i\rangle] = E_i[\hat{\Omega}_\pm|\psi_i\rangle]$

$\hat{\Omega}_\pm$ acting on an eigenvector of $\hat{H}_0$ generates the eigenvector of $\hat{H}$ <u>with the same</u> eigenenergy E_i.

$$\hat{H}|\psi_i^\pm\rangle = E_i|\psi_i^\pm\rangle, \quad |\psi_i^\pm\rangle = \hat{\Omega}_\pm|\psi_i\rangle = \hat{\Omega}_\pm(E_i)|\psi_i\rangle$$

$$\hat{\Omega}_\pm = \hat{\Omega}_\pm \sum_i |\psi_i\rangle\langle\psi_i| = \sum_i \hat{\Omega}_\pm(E_i)|\psi_i\rangle\langle\psi_i| = \sum_i |\psi_i^\pm\rangle\langle\psi_i|$$

Scattering states

$|\psi_i^{\pm}\rangle = \hat{\Omega}_{\pm}(E_i)|\psi_i\rangle$ – scattering state (*)

can be found from

$$(\hat{\mathcal{H}}_0 + \hat{V})|\psi_i^{\pm}\rangle = E_i|\psi_i^{\pm}\rangle \quad \text{or}$$

$$|\psi_i^{\pm}\rangle = |\psi_i\rangle + \frac{1}{E_i - \hat{\mathcal{H}}_0}\hat{V}|\psi_i^{\pm}\rangle \quad \text{Lippmann-Schwinger eq-n}$$

Formally $|\psi_i^{\pm}\rangle = \left[1 - \hat{G}_0(E_i)\hat{V}\right]^{-1}|\psi_i\rangle$, $\hat{G}_0(E_i) \equiv \dfrac{1}{E_i - \hat{\mathcal{H}}_0}$

[resolvent of $\hat{\mathcal{H}}_0$ (Green function)]

From (*) it means

$$\hat{\Omega}_{\pm}(E_i) = 1 + \sum_{n=1}^{\infty}\left[\hat{G}_0(E_i)\hat{V}\right]^n$$

Where does $\pm$ come from?

"+" wave packet should be the wave packet of free particles:

$$\int \frac{dE\, \nu(E)\, e^{-iEt}}{E - \hat{\mathcal{H}}_0 + i0}(\ldots)|\psi_i(E)\rangle \xrightarrow[t\to-\infty]{} 0$$!

There should be no poles in upper E-plane.

"Diagrammatics"

$$\hat{\Omega} = 1 + \hat{G}_0 \hat{V} + \hat{G}_0 \hat{V} \hat{G}_0 \hat{V} + \ldots = 1 + \underset{\substack{\uparrow \\ G_0}}{\text{—}}\!\!\!\!\wr + \text{—}\wr\text{—}\!\!\!\!\underset{\substack{\downarrow \\ V}}{\wr} + \text{—}\wr\text{—}\wr\text{—}\wr + \ldots$$

$$= 1 + \hat{G}_0 \hat{T} = 1 + \text{—}\left(\wr + \underset{\|}{\wr\text{—}\wr} + \wr\text{—}\wr\text{—}\wr + \ldots\right)$$

T-matrix: $\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{V} + \ldots$

Eq-n for T-matrix $\boxed{\hat{T} = \hat{V} + \hat{V}\hat{G}_0\hat{T}}$

Another form of $\hat{\Omega}$-wave operator

$$\hat{\Omega} = 1 + \underset{\|}{\left[\hat{G}_0 + \hat{G}_0\hat{V}\hat{G}_0 + \ldots\right]}\hat{V} = 1 + \left[\text{—} + \text{—}\wr\text{—} + \ldots\right]\wr$$

$\boxed{\hat{G} = \hat{G}_0 + \hat{G}_0\hat{V}\hat{G}}$ Dyson eq-n

$$\Downarrow$$

$\hat{G}_0^{-1} = \hat{G}^{-1} + \hat{V} \Rightarrow \boxed{\hat{G} = \frac{1}{E - \hat{\mathcal{H}}}}$ Resolvent of H (Green function)

Resumé:

$$\hat{\Omega}_{\pm}(E_i) = 1 + \frac{1}{E_i - \hat{\mathcal{H}}_0 \pm i0}\hat{T} = 1 + \frac{1}{E_i - \hat{\mathcal{H}} \pm i0}\hat{V}$$