Gaussian process / Wick's theorem

N - random classical variables, Gaussian:

$$P(x) = \mathcal{N}^{-1} e^{-\frac{1}{2} x^T \hat{G}^{-1} x}, \quad x \equiv (x_1 \dots x_N)$$

$$\langle O(x) \rangle \equiv \int d^N x \, P(x) \, O(x)$$

Generating function

$$F = \langle e^{a^T x} \rangle = \langle e^{\sum a_i x_i} \rangle$$

can be calculated by completing the square

$$F = \langle e^{a^T x} \rangle = e^{\frac{1}{2} a^T \hat{G} a}$$

Pair correlation: $\langle x_i x_j \rangle = \frac{\partial^2 F}{\partial a_i \partial a_j}\Big|_{a=0} = G_{ij}$

Therefore $\boxed{F = e^{\frac{1}{2} \langle (a^T x)^2 \rangle}}$

Another way ($S = a^T x$)

$$\langle e^S \rangle = \sum \frac{1}{n!} \langle S^n \rangle = \sum \frac{1}{(2n)!} \langle S^{2n} \rangle$$

$$= \sum \frac{1}{(2n)!} \underbrace{(2n-1)!!}_{\#\text{ of pairing}} \langle S^2 \rangle^n = \sum \frac{1}{2^n n!} \langle S^2 \rangle^n = e^{\frac{1}{2} \langle S^2 \rangle}$$

Drude (classical) conductivity

In 3D the Kubo formula for conductivity should be written for the conductivity tensor:

$$\sigma_{ik} = \frac{\pi\hbar}{V} \sum_{\mu\nu} j^i_{\mu\nu} j^k_{\nu\mu} \left[-f'(\varepsilon_\mu)\right] \delta(\varepsilon_\mu - \varepsilon_\nu)$$

Since $|\mu\rangle$ and $|\nu\rangle$ are eigenstates of the Hamiltonian

$$\sigma_{ik} = \frac{\pi\hbar}{V} \int d\varepsilon \left[-f'(\varepsilon)\right] \sum_{\mu\nu} \langle\mu|\hat{j}^i|\nu\rangle \delta(\varepsilon - \varepsilon_\nu) \langle\nu|\hat{j}^k|\mu\rangle \delta(\varepsilon - \varepsilon_\mu)$$

$$= \frac{\pi\hbar}{V} \int d\varepsilon \left[-f'(\varepsilon)\right] \sum_{\mu\nu} \langle\mu|\hat{j}^i \delta(\varepsilon - \hat{\mathcal{H}})|\nu\rangle\langle\nu|\hat{j}^k \delta(\varepsilon - \hat{\mathcal{H}})|\mu\rangle$$

$$= \frac{\pi\hbar}{V} \int d\varepsilon \left[-f'(\varepsilon)\right] \mathrm{tr}\, \hat{j}^i \delta(\varepsilon - \hat{\mathcal{H}}) \hat{j}^k \delta(\varepsilon - \hat{\mathcal{H}})$$

Introducing the Green functions (resolvents)

$$\hat{G}^\pm(\varepsilon) = \frac{1}{\varepsilon - \hat{\mathcal{H}} \pm i0} \quad \left(\begin{array}{l} \hat{G}^+ \equiv \hat{G}^R \text{ - retarded} \\ \hat{G}^- = \hat{G}^A \text{ - advanced} \end{array} \right)$$

we can write

$$\delta(\varepsilon - \hat{\mathcal{H}}) = -\frac{1}{2\pi i}\left(\hat{G}^R - \hat{G}^A\right)$$

Disorder

The Hamiltonian (in $\vec{r}$-representation)

$$\hat{H} = -\frac{\hbar}{2m}\Delta + \hat{V}(\vec{r})$$

The simplest and most uncorrelated (random): Gaussian white noise

$$P[V] = \mathcal{N}^{-1} e^{-\frac{1}{2}\int d^3r\, d^3r'\, V(r)\, \hat{W}^{-1}(r-r')\, V(r')} \quad \text{(Gaussian)}$$

$$\Rightarrow \mathcal{N}^{-1} e^{-\pi\nu\tau\int d^3r\, V^2(r)}$$

Meaning $\langle \ldots \rangle \equiv \int \mathcal{D}V\, P[V] \ldots$

$$\langle V(\vec{r})\rangle = 0\,, \quad \langle V(\vec{r})V(\vec{r}\,')\rangle = \frac{\hbar}{2\pi\nu\tau}\delta(\vec{r}-\vec{r}\,')$$

Higher Cumulants $\equiv 0$!

Averaging the Green function

$$G = G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + \ldots = \ldots$$

we get Dyson eq-n

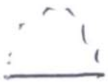
$$\langle G\rangle = G_0 + G_0 \Sigma \langle G\rangle = \ldots$$

$$\Sigma = \ldots$$

where $\ldots = \left[\frac{2\pi\nu\tau}{\hbar}\right]^{-1}\delta(\vec{r}-\vec{r}\,')$

Disorder - 2

Weak scattering: lowest order in disordered scattering potential for the mass-operator (self-energy)

$$\Sigma \approx$$  $$= \frac{\hbar}{2\pi\nu\tau}\,\delta(\vec r-\vec r\,')\,G_0(\vec r-\vec r\,')$$

$$\langle G\rangle = \frac{1}{G_0^{-1}-\Sigma}$$ – real part changes spectrum (not that important)

– imaginary part - new physics (scattering, finite life time)

$$\boxed{\operatorname{Im}\Sigma_p \approx -\frac{\hbar}{2\tau}}$$ since $\operatorname{Im} G_0(\vec r=0) = -i\pi\nu$

The averaged (mean) Green function in momentum representation

$$\boxed{\langle G\rangle(p) = \frac{1}{\varepsilon-\varepsilon_p + i\frac{\hbar}{2\tau}}}$$

Then $$\langle \delta(\varepsilon-\hat{H})\rangle(p) = \frac{1}{\pi}\,\frac{\hbar/2\tau}{(\varepsilon-\varepsilon_p)^2+(\hbar/2\tau)^2} = \delta_\tau(\varepsilon-\varepsilon_p)$$

Disorder-3

The simplest approximation

$$\langle \sigma_{ik} \rangle = \pi\hbar \int d\varepsilon \left(-f'\right) \operatorname{tr} \hat{j}_i \langle \delta(\varepsilon - \hat{H}) \rangle \hat{j}_j \langle \delta(\varepsilon - \hat{H}) \rangle$$

It will be shown later that this is the approximation valid for $\boxed{\varepsilon\tau \gg 1}$ where $\varepsilon \sim \varepsilon_F$. This is the same weak scattering parameter.

$$\langle \sigma_{ik} \rangle = \pi\hbar e^2 \int d\varepsilon \left(-f'\right) \int \frac{d^3p}{(2\pi\hbar)^3} v_i v_k \, \delta_\tau^2(\varepsilon - \varepsilon_p)$$

Exercise $\quad \delta_\gamma(x) = \frac{1}{\pi} \frac{\gamma}{x^2+\gamma^2}, \quad \lim_{\gamma \to 0} \delta_\gamma(x) = \delta(x)$

$$\int dx \, \delta_\gamma(x) = 1$$

$$\int dx \, \delta_\gamma^2(x) = \left(\frac{i}{2\pi}\right)^2 \int dx \left(\frac{1}{x+i\gamma} - \frac{1}{x-i\gamma}\right)^2 = \frac{1}{2\pi^2} \int \frac{dx}{x^2+\gamma^2} = \frac{1}{2\pi\gamma}$$

$$\Rightarrow \delta_\gamma^2(x) \xrightarrow[\gamma \to 0]{} \frac{1}{2\pi\gamma} \delta(x)$$

Disorder - 4

$$\langle \sigma_{ik} \rangle = \delta_{ik} \int d\varepsilon \left(-f'(\varepsilon)\right) \sigma(\varepsilon)$$

$$\sigma(\varepsilon) = \pi\hbar e^2 \int \frac{d^3p}{(2\pi\hbar)^3} \frac{v_p^2}{3} \frac{\tau}{\pi\hbar} \delta(\varepsilon - \varepsilon_p)$$

$$= e^2 \nu(\varepsilon) \frac{v^2\tau}{3} = e^2 \nu \mathcal{D}$$

diffusion coefficient

$$\boxed{\sigma = e^2 \nu \mathcal{D} = \frac{n e^2 \tau}{m}}$$ — Drude conductivity

since $\nu = \frac{mp}{2\pi^2\hbar^3}$

<u>This classical result</u>

can also be obtained from kin. eq-n in "τ-approximation"

$$\partial_t f + \vec{v}\, \partial_{\vec{r}} f + e\vec{E}\, \partial_{\vec{p}} f = -\frac{f - f_0}{\tau}$$

in the linear response regime

$f_p \approx f_0 + \delta f(\vec{p})$ $\qquad \delta f(\vec{p}) \approx -\tau e \vec{E} f'(\varepsilon_p) \vec{v}$

$$j_i = e \int \frac{d^3p}{(2\pi\hbar)^3} v_i \delta f(\vec{p}) = e^2 \tau \int \frac{d^3p}{(2\pi\hbar)^3} (-f') v_i v_k \cdot E_k$$

$$= \int d\varepsilon (-f') \underbrace{\boxed{e^2 \nu(\varepsilon) \mathcal{D}(\varepsilon)}}_{\sigma(\varepsilon)} \cdot \delta_{ik} E_k$$

Drude

Or even from Newton's Law with friction:

$$\dot{\vec{p}} = e\vec{E} - \vec{p}/\tau = 0 \Rightarrow \vec{v} = \frac{e\tau}{m}\vec{E}$$

$$\vec{j} = ne\vec{v} = \frac{ne^2\tau}{m}\vec{E}$$

$$\boxed{\sigma_{Drude} = \frac{ne^2\tau}{m} = e^2 \nu D}$$

The conductance for a system with length L and cross-section S:

$$G = \sigma_{Drude}\frac{S}{L} = \frac{e^2}{2\pi\hbar} N_\perp \frac{4l}{3L} = \frac{e^2}{2\pi\hbar} N_{open}$$

Number of "open" channels: $N_{open} \sim N_\perp \frac{l}{L}$, $\quad N_\perp = \frac{\pi p_F^3}{(2\pi\hbar)^2} S$

We have studied 3 regimes:

1. Ballistic
2. Localisation
3. Drude (diffusion)