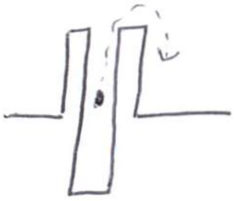
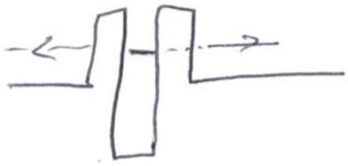
Relaxation and Decoherence





p_{in} / p_{out}

$p_{in}(0) = 1 \Rightarrow p_{in}(t) = 1$

Probability:
you can ask (measure)
and be sure.

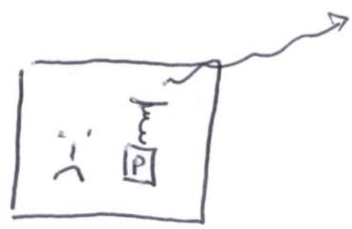
$p_{in}(t) = p_{in}(0)\, e^{-\gamma t}$

Relaxes to empty state.

$|state\rangle = e^{-\frac{1}{2}\gamma t}|in\rangle + \sqrt{1-e^{-\frac{\gamma}{2}t}}\,|out\rangle$

Never sure.
God plays dices.

Schrödinger cat



$$|state\rangle = |alive, in\rangle\, e^{-\frac{1}{2}\gamma t} + |dead, out\rangle \sqrt{1-e^{-\frac{\gamma}{2}t}}$$

Conclusion: cat (macro-object) in the superposition of dead-alive states due to coupling to the particle (micro-object).

Resolution: ...speculations... or decoherence.

Classical description

Discrete: $n = 1, 2, \dots, N$ states

$P_n(t)$ - probability to be in "n-state" at the moment t.

Markovian property (loss of memory)

$T(n,t/n',t')$ - transition probability depends on any moment in the past but not the whole history.

The Law of Total probability

$$P_n(t+\Delta t) = \sum_m T(n, t+\Delta t/m, t)\, P_m(t)$$

$\Delta t \to 0$

$$T(n, t+\Delta t/n, t) \approx 1 - W_n \Delta t$$

$$T(n, t+\Delta t/m, t) \underset{m\neq n}{\approx} W_{nm} \Delta t$$

$$\boxed{\dot{P}_n = -W_n P_n + \sum_{m\neq n} W_{nm} P_m}$$ - master eq-n

$$W_n = \sum_{m\neq n} W_{mn}$$

Markof:

$$\langle O_1 \rangle = \sum_{1,2,\dots N} O_1 P(1,2,\dots N) = \sum_1 O_1 P_1$$

$$P_1 = \sum_{2,\dots,N} P(1,2,\dots N)$$ - course graining loss of memory

Quantum mechanics

Pure state $|\psi\rangle$. Observable

$$O = \langle\psi|\hat{O}|\psi\rangle = \mathrm{tr}\, \hat{O}\hat{\rho}_\psi, \quad \hat{\rho}_\psi = |\psi\rangle\langle\psi|$$

N-quantum states

$$|\psi\rangle = \sum c_n |n\rangle$$

The density matrix

$$\hat{\rho} = \sum_{n,m} c_n \bar{c}_m |n\rangle\langle m|$$

Mixed state $\hat{\rho} = \sum_{nm} \rho_{nm} |n\rangle\langle m|$

$N\times N$ matrix

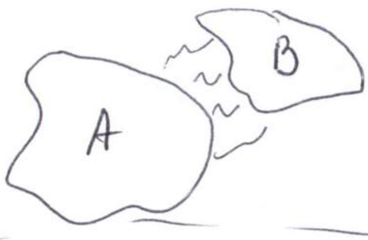
Unlike "vector" $P_n(t)$, $n=1,\dots,N$ for classical description

Probability to be in $|n\rangle$-state

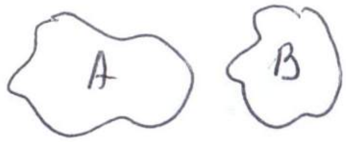
$$P_n(t) = \rho_{nn} \text{ - vector-like}$$

But in QM we need to know more, i.e. $N\times N$ density matrix.

Reduced density matrix



$\hat{\rho}_{A+B}$ - total d.m.



: no coupling $\hat{\rho}_{A+B} = \hat{\rho}_A \otimes \hat{\rho}_B$

$$\langle O_A \rangle = tr\, \hat{O}_A \hat{\rho}_A \otimes \hat{\rho}_B = \sum O^{ab} \rho_A^{ba} \rho_B^{\alpha\alpha} = tr_A \rho_A \hat{O}_A$$

$$\parallel$$

$$\sum_{a,\alpha} \langle a,\alpha | O_A \rho_A \otimes \rho_B | a,\alpha \rangle = \langle a | O_A \rho_A | a \rangle \langle \alpha | \rho_B | \alpha \rangle$$

$$\langle O_B \rangle = tr\, O_B \hat{\rho}_A \otimes \hat{\rho}_B = tr_B \hat{\rho}_B \hat{O}_B$$

If there is a coupling but we measure only system A:

$$\langle O_A \rangle = tr\, \hat{O}_A \hat{\rho}_{A+B} = \sum \langle a,\alpha | \hat{O} | b,\beta \rangle \langle b,\beta | \hat{\rho}_{A+B} | a,\alpha \rangle$$

$$= \sum O_{ab} \langle b,\alpha | \hat{\rho}_{A+B} | a,\alpha \rangle = \sum O_{ab} \rho_{ba}^{\alpha\alpha}$$

$$= \sum_{ab} O_{ab} \cancel{tr_B} \sum_{\alpha} \rho_{ba}^{\alpha\alpha} = tr_A \hat{O}_A \hat{\rho}_A$$

$$\boxed{\hat{\rho}_A \equiv tr_B \hat{\rho}_{A+B}}$$ - reduced density matrix

TLS / Qubit

Two states (spin) $|e\rangle$ - excited with energy ω_0

$|g\rangle$ - ground state with energy 0

$$\mathcal{H}_S = \omega_0 |e\rangle\langle e|$$

Environment (bath) of bosons

$$\mathcal{H}_B = \sum_k \omega_k \hat{b}_k^+ \hat{b}_k \,, \quad [\hat{b}_k, \hat{b}_{k'}^+] = \delta_{kk'}$$

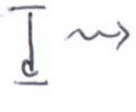
Interactions (minimal coupling)

$-\leftarrow \hat{B}$ = "fluct. energy shift"

$-\leftarrow 0$

"Diagonal" inter.

$$\mathcal{H}_{int} = |e\rangle\langle e| \cdot \hat{B}$$



$|e\rangle\langle g| \cdot \hat{B}$ Absorption

$|g\rangle\langle e| \cdot \hat{B}^+$ Radiation

"off-diagonal" inter.

$$\mathcal{H}_{int} = |e\rangle\langle g| \cdot \hat{B} + |g\rangle\langle e| \cdot \hat{B}^+$$

Mapping

$$|e\rangle\langle e| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \sigma_+ \sigma_-$$

$$|g\rangle\langle g| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \sigma_- \sigma_+$$

$$|e\rangle\langle g| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \sigma_+$$

$$|g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \sigma_-$$

Diagonal spin-boson model

$$\hat{\mathcal{H}} = \left(\omega_0 + \hat{B}\right)\sigma_+\sigma_- + \hat{\mathcal{H}}_B$$

The density matrix evolves as

$$\hat{\rho}_{S+B}(t) = e^{-i\hat{\mathcal{H}}t}\,\hat{\rho}_{S+B}(t=0)\,e^{i\hat{\mathcal{H}}t}$$

Initially system and bath are decoupled $\hat{\rho}_{S+B}(t=0) = \hat{\rho}_0 \otimes \hat{\rho}_B$, $\hat{\rho}_B = \mathcal{Z}^{-1} e^{-\beta\hat{\mathcal{H}}_B}$

The Hamiltonian is diagonal in "up/down" spin states

$$e^{-i\hat{\mathcal{H}}t} = \begin{pmatrix} e^{-i(\omega_0+\hat{B}+\hat{\mathcal{H}}_B)t} & 0 \\ 0 & e^{-i\hat{\mathcal{H}}_B t} \end{pmatrix}$$

Then

$$\hat{\rho}_{S+B}(t) = \begin{pmatrix} \rho_0^{11} e^{-i(\omega_0+\hat{B}+\hat{\mathcal{H}}_B)t}\,\hat{\rho}_B\,e^{i(\omega_0+\hat{B}+\hat{\mathcal{H}}_B)t} & \rho_0^{12} e^{-i(\omega_0+\hat{B}+\hat{\mathcal{H}}_B)t}\,\hat{\rho}_B\,e^{i\hat{\mathcal{H}}_B t} \\ \rho_0^{21} e^{-i\hat{\mathcal{H}}_B t}\,\hat{\rho}_B\,e^{i(\omega_0+\hat{B}+\hat{\mathcal{H}}_B)t} & \rho_0^{22} e^{-i\hat{\mathcal{H}}_B t}\,\hat{\rho}_B\,e^{i\hat{\mathcal{H}}_B t} \end{pmatrix}$$

Measuring on spin system only we need reduced density matrix

$$\hat{\rho}(t) = \mathrm{tr}_B\,\hat{\rho}_{S+B}(t)$$

DSB-2

$$\hat{\rho}(t) = \begin{pmatrix} \rho_0^{11} & \rho_0^{12} e^{-i\omega_0 t}\langle e^{iH_B t} e^{-i(H_B+B)t}\rangle \\ \rho_0^{21} e^{i\omega_0 t}\langle e^{i(H_B+B)t} e^{-iH_B t}\rangle & \rho_0^{22} \end{pmatrix}$$

$$\langle \ldots \rangle = tr_B \hat{\rho}_B \ldots$$

The overlap between states evolving with $\hat{H}_B$ and $\hat{H}_B + \hat{B}$:

$$D(t) = \langle e^{i\hat{H}_B t} e^{-i(\hat{H}_B+\hat{B})t}\rangle = \langle T e^{-i\int_0^t dt' \hat{B}(t')}\rangle$$

$$\hat{B}(t') = e^{i\hat{H}_B t}\hat{B} e^{-i\hat{H}_B t}$$

(recollect the wave operators!)

The minimal coupling:

$$\hat{B}(t) = \sum_k g_k \hat{b}_k(t) + h.c. = \sum_k g_k \hat{b}_k e^{-i\omega_k t} + h.c.$$

Notice that

$$[\hat{B}(t), \hat{B}(t')] = 2i \sum_k |g_k|^2 \sin\omega_k(t-t') = c\text{-number}$$

Campbell-Housdorff formula

$$e^{\hat{a}} e^{\hat{b}} = e^{\hat{a}+\hat{b}} e^{\frac{1}{2}[\hat{a},\hat{b}]}$$

if $\hat{c} = [\hat{a}, \hat{b}]$ and $[\hat{c}, \hat{a}] = [\hat{c}, \hat{b}] = 0$

DSB-3

$$T e^{-i\int_0^t dt' \hat{B}(t')} = \lim_{\substack{N\to\infty \\ \Delta t_i \to 0}} e^{-i\hat{B}_N \Delta t_N} e^{-i\hat{B}_{N-1}\Delta t_{N-1}} \dots e^{-i\hat{B}_1 \Delta t_1}$$

$$= \lim e^{-i(\hat{B}_N \Delta t_N + \hat{B}_{N-1}\Delta t_{N-1})} e^{-i\hat{B}_{N-2}\Delta t_2} \dots e^{-i\hat{B}_1 \Delta t_1} \cdot e^{-\frac{1}{2}[\hat{B}_N, \hat{B}_{N-1}]\Delta t_N \Delta t_{N-1}}$$

$$\dots = \lim e^{-i\sum_{i=1}^N \hat{B}_i \Delta t_i} \cdot e^{-\frac{1}{2}\sum_{i>j}[\hat{B}_i, \hat{B}_j]\Delta t_i \Delta t_j}$$

$$= e^{-i\int_0^t dt' \hat{B}(t')} e^{i\varphi(t)}$$

$$\varphi(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 \sum_k |g_k|^2 \sin\omega_k (t_1 - t_2)$$

So the reduced density matrix:

$$\hat{\rho}(t) = \begin{pmatrix} \rho_0^{11} & \rho_0^{12} e^{-i\omega t + i\varphi(t)} \left\langle e^{-i\int_0^t dt' \hat{B}(t')} \right\rangle \\ \times \xleftarrow{c.c.} & \rho_0^{22} \end{pmatrix}$$

The Wick's theorem

$$d(t) \equiv \left\langle e^{-i\int_0^t dt' \hat{B}(t')} \right\rangle = e^{-\frac{1}{2}\int_0^t dt_1 dt_2 \langle \hat{B}(t_1)\hat{B}(t_2)\rangle}$$

$$= e^{-\sum_k |g_k|^2 \frac{1-\cos\omega_k t}{\omega_k^2}(2N_k+1)}$$

$$N_k = \frac{1}{e^{\beta\omega_k} - 1}$$

Introducing spectral function

$$J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k)$$

decoherence function $d(t)$:

$$\ln d(t) = -\int_0^\infty d\omega \, J(\omega) \frac{1-\cos\omega t}{\omega^2} \underbrace{\coth\frac{\omega}{2T}}_{= 1+2N(\omega)}$$

The ohmic limit $J(\omega) = \alpha \omega e^{-\omega/\Omega}$.

Ω - UV cutoff.

$$-\ln d = \alpha \int_0^\infty d\omega \frac{1-\cos\omega t}{\omega} e^{-\omega/\Omega} + \alpha \int_0^\infty d\omega \frac{1-\cos\omega t}{\omega} 2N(\omega) e^{-\omega/\Omega}$$

Quantum Kinetic Eq-n

$$\mathcal{H}_A = \sum \Omega_n \hat{A}_n^+ \hat{A}_n \qquad \xleftarrow{\text{Coupling}} \qquad \mathcal{H}_B = \sum \omega_k \hat{b}_k^+ \hat{b}_k$$

$$\hat{V} = \sum_n \hat{A}_n^+ \hat{B}_n + h.c.$$

$$\hat{B}_n = \sum_k g_{nk} \hat{b}_k$$

System (A) Bath (B)

The dynamics of the total density matrix (DM) in interaction representation

$$\dot{\hat{\rho}}_{A+B} = -i\left[\hat{V}(t), \hat{\rho}_{A+B}(t)\right], \quad \hat{V}(t) = e^{i\hat{\mathcal{H}}_0 t}\hat{V}e^{-i\hat{\mathcal{H}}_0 t}$$

$$\hat{\mathcal{H}}_0 = \hat{\mathcal{H}}_A + \hat{\mathcal{H}}_B$$

Iterating once

$$\dot{\rho}_{A+B}(t) = -i\left[V(t), \rho_{A+B}(0)\right] - \int_0^t dt' \left[V(t), \left[V(t'), \rho_{A+B}(t')\right]\right]$$

The reduced DM: $\rho(t) = tr_B\, \rho_{A+B}(t)$

$$\dot{\rho}(t) = -\int_0^t dt'\, tr_B \left[V(t), \left[V(t'), \rho_{A+B}(t')\right]\right]$$

We used that $\rho_{A+B}(t=0) = \rho_A(0)\rho_B$ and

$$\langle V \rangle \equiv tr_B\, \rho_B V = 0$$

QKE-2

$$\dot{\rho}(t) = -\int_{-\infty}^{t} dt'\, tr_B\left[V(t), \left[V(t'), \rho_B \otimes \rho(t)\right]\right]$$

Markov approximation: bath correlation functions decay much faster than the time scale of changes of $\rho(t)$.

$$\rho_{A+B}(t') \underset{\substack{\text{Born}\\ \text{approximation}\\ \text{(weak coupling)}}}{\longrightarrow} \rho(t')\otimes\rho_B \underset{\substack{\text{Markov}\\ \text{app.}}}{\longrightarrow} \rho(t)\otimes\rho_B$$

Inserting $V(t) = \sum_n A_n^+ B_n(t)\, e^{+i\Omega_n t}$

and neglecting highly-oscillating terms

like $e^{i(\Omega_n t + \Omega_m t')}$ and $e^{i(\Omega_n t - \Omega_m t')}$ for $n \neq m$

(long time approximation)

$$\dot{\rho} = -\sum_n \left[K_n^{>}\left(A_n^+ A_n \rho - A_n \rho A_n^+\right) + K_n^{<}\left(\rho A_n A_n^+ - A_n^+ \rho A_n\right)\right] + h.c.$$

$$K_n^{>} = \int_0^\infty dt\, e^{i\Omega_n t}\langle \hat{B}_n(t)\hat{B}_n^+\rangle$$

$$K_n^{<} = \int_0^\infty dt\, e^{i\Omega_n t}\langle \hat{B}_n^+\hat{B}_n(t)\rangle$$

QKE-3

Separating real and imaginary parts

$$K_n^{\gtrless} = \frac{1}{2}\left(\Gamma_n^{\gtrless} + i\,I_n^{\gtrless}\right)$$

$$\boxed{\dot{\hat{\rho}} = -i[\hat{h},\hat{\rho}] - \mathcal{D}\{\hat{\rho}\}} \longleftarrow \text{Dissipator}$$

Unitary evolution
(correction to Hamiltonian)

$$\hat{h} = \frac{1}{2}\sum_n\left(I_n^{>}\hat{A}_n^{+}\hat{A}_n - I_n^{<}\hat{A}_n\hat{A}_n^{+}\right), \quad \hat{h}^{+} = \hat{h}$$

$$\hat{\mathcal{D}}\{\hat{\rho}\} = \frac{1}{2}\sum_n\Big\{\Gamma_n^{>}\left(A_n^{+}A_n\rho + \rho A_n^{+}A_n - 2A_n\rho A_n^{+}\right)$$
$$+\Gamma_n^{<}\left(A_nA_n^{+}\rho + \rho A_nA_n^{+} - 2A_n^{+}\rho A_n\right)\Big\}$$

QKE in $\boxed{\text{Lindbladt}}$ form.

For qubit $\quad \hat{\mathcal{H}}_0 = \Omega\,\hat{\sigma}_+\hat{\sigma}_- + \sum \omega_k \hat{b}_k^{+}\hat{b}_k$

$$\hat{V} = \sigma_+ B + \sigma_- B^{+} + \sigma_z B_z$$

$$\begin{bmatrix} A_1 = \sigma_-, & \Omega_1 = 0, & B_1 = B \\ A_2 = \sigma_z, & \Omega_2 = 0, & B_2 = B_z \end{bmatrix}$$ – correspondence to generic model

QKE-4

Spin-boson model

$$\mathcal{D}\{\hat\rho\} = \frac{1}{2}\Gamma^{>}\left(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_- - 2\sigma_-\rho\sigma_+\right) \to \Gamma^{>}\begin{pmatrix}\rho_{11} & \rho_{12}/2 \\ \rho_{21}/2 & -\rho_{11}\end{pmatrix}$$

$$+\frac{1}{2}\Gamma^{<}\left(\sigma_-\sigma_+\rho + \rho\sigma_-\sigma_+ - 2\sigma_+\rho\sigma_-\right) \to \Gamma^{<}\begin{pmatrix}-\rho_{22} & \rho_{12}/2 \\ \rho_{21}/2 & \rho_{22}\end{pmatrix}$$

$$+\frac{1}{2}\Gamma_z\left(\rho - \sigma_z\rho\sigma_z\right) \to \Gamma_z\begin{pmatrix}0 & \rho_{12} \\ \rho_{21} & 0\end{pmatrix}$$

$$\boxed{\dot\rho = -i[h,\rho] - \mathcal{D}\{\hat\rho\}}$$ - Quantum Kin. Eq-n

Since RHS is Hermitian and traceless we have 2 independent eq-ns ($\rho_{21} = \bar\rho_{12}$, $\rho_{11} + \rho_{22} = 1$)

$$\begin{cases} \dot\rho_{11} = -\Gamma_1\rho_{11} + \Gamma^{<} \quad \left(= -\Gamma^{>}\rho_{11} + \Gamma^{<}\rho_{22}\right) \\ \dot\rho_{12} = -\Gamma_2\rho_{12} \end{cases}$$

"out"-rate emission

"in"-rate absorption

$\Gamma_1 = \Gamma^{>} + \Gamma^{<}$ - relaxation rate

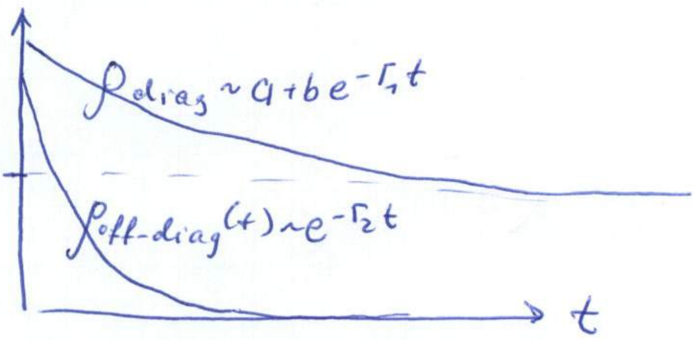
$\Gamma_2 = \frac{1}{2}\Gamma_1 + \Gamma_z$ - decoherence rate

QKE-5

$$\begin{cases} \rho_{11}(t) = [\rho_{11}(0) - f_{eq}]e^{-\Gamma_1 t} + f_{eq} \\ \rho_{12}(t) = \rho_{12}(0) e^{-\Gamma_2 t} \end{cases}$$

Energy relaxation: $E(t) = \mathrm{tr}\, \Omega \hat{\sigma}_+ \hat{\sigma}_- \hat{\rho}(t) = \Omega \rho_{11}(t)$

$$= \frac{\Gamma^<}{\Gamma^> + \Gamma^<} = f_{eq}$$



$$\begin{cases} \Gamma^> = \sum_k \pi |g_k|^2 (1 + N_k)\, \delta(\varepsilon_k - \Omega) = J(\Omega)[N(\Omega) + 1] \text{ - emission} \\ \Gamma^< = \sum_k \pi |g_k|^2 N_k\, \delta(\varepsilon_k - \Omega) = J(\Omega) N(\Omega) \text{ - absorption (induced)} \end{cases}$$

induced

spontaneous

$$\Gamma_1 = J(\Omega)[2N(\Omega) + 1] = J(\Omega) \coth \frac{\Omega}{2T}$$

We used $\Gamma^> = \int_0^\infty dt\, e^{i\Omega t} \langle B(t) B^+ \rangle$

$$\Gamma^< = \int_0^\infty dt\, e^{i\Omega t} \langle B^+ B(t) \rangle$$

Dephasing (part of Γ_2): $\Gamma_z = \mathrm{Re} \int_0^\infty dt \langle \{\hat{B}_z(t), \hat{B}^+\} \rangle$

$$\Gamma_z = \sum_k \pi |\tilde{g}_k|^2 (2N_k + 1) \cdot \delta(\varepsilon_k) = \lim_{\omega \to 0} J_z(\omega) \coth \frac{\omega}{2T} = \begin{cases} \infty / 0 \\ \alpha T, \text{ Ohmic} \end{cases}$$

QKE-6

If $\Gamma_2 \to \infty$ (or, at least, $\Gamma_2 \gg \Gamma_1$) then $\rho_{12}(t \gtrsim \Gamma_2^{-1}) \approx 0$ and we have eq-n for diagonal elements of DM only:

$$\begin{cases} \dot{\rho}_{11} = -\Gamma^{>}\rho_{11} + \Gamma^{<}\rho_{22} \\ \dot{\rho}_{22} = -\Gamma^{<}\rho_{22} + \Gamma^{>}\rho_{11} \end{cases}, \quad \rho_{11} + \rho_{22} = 1$$

Probability to be in state 1 or 2

$$p_1 = \rho_{11}, \; p_2 = \rho_{22} \quad (\text{normalization } p_1 + p_2 = 1)$$

$$\boxed{\frac{d}{dt}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} -\Gamma^{>} & \Gamma^{<} \\ \Gamma^{<} & -\Gamma^{>} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}}$$

Master eq-n: classical description (probabilities only)