Example: 1-part. scattering in 1D

$$\Big[\underbrace{-\frac{\hbar^2}{2m}\partial_x^2}_{\hat{H}_0} + \underbrace{V_0\delta(x)}_{\hat{V}}\Big]\psi = \varepsilon\psi \qquad p = \underbrace{\pm}_{\text{2 channels}}\sqrt{2m\varepsilon} = \pm p_0(\varepsilon)$$

$$\hat{T} = \begin{pmatrix} \langle p_0|\hat{T}|p_0\rangle & \langle p_0|\hat{T}|-p_0\rangle \\ \langle -p_0|T|p_0\rangle & \langle -p_0|\hat{T}|-p_0\rangle \end{pmatrix}$$

matrix in channel space.

$$\nu_\alpha(\varepsilon) = \int_0^\infty \frac{dp}{2\pi\hbar}\delta(\varepsilon-\varepsilon_p) = \int_{-\infty}^0 \frac{dp}{2\pi\hbar}\delta(\varepsilon-\varepsilon_p) = \nu/2$$

$$\hat{S}(\varepsilon) = 1 - i\pi\nu\,\hat{T}(\varepsilon)$$

For arbitrary p and p':

$$\langle p|\hat{T}|p'\rangle = \langle p|\hat{V}|p'\rangle + \int\frac{dp_1}{2\pi\hbar}\langle p|V|p_1\rangle G_{p_1}\langle p_1|V|p'\rangle + \dots$$

$$= V_0 + V_0(-i\pi\nu)V_0 + \dots = \frac{V_0}{1+i\pi\nu V_0}$$

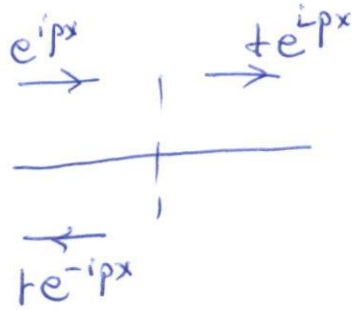
In channels $p = \pm p_0(\varepsilon)$

$$\hat{T}(\varepsilon) = \frac{V_0}{1+i\pi\nu(\varepsilon)V_0}\begin{pmatrix}1&1\\1&1\end{pmatrix} \Rightarrow \boxed{\hat{S}(\varepsilon) = \frac{1}{1+i\pi\nu(\varepsilon)V_0}\begin{pmatrix}1 & -i\pi\nu V_0 \\ -i\pi\nu V_0 & 1\end{pmatrix}}$$

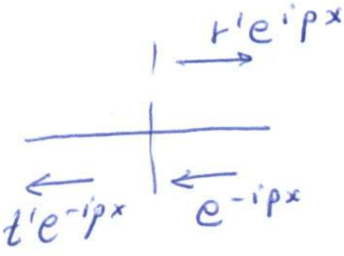
$$\hat{S}^+(\varepsilon)\hat{S}(\varepsilon) = 1$$

Example : 1D scattering states

$$\left[-\frac{\hbar^2}{2m}\partial_x^2 + V_0\,\delta(x)\right]\psi = \varepsilon\psi$$



$$\boxed{p>0!}$$



Matching $\left[\psi\right]_{-0}^{+0} = 0$, $\frac{\hbar^2}{2m}\left[\partial_x\psi\right]_{-0}^{+0} = V_0\psi(0)$

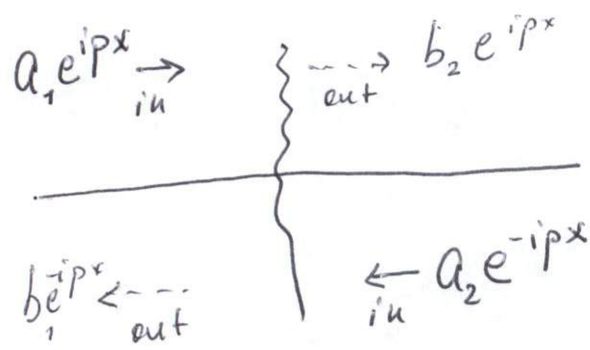
$$\begin{cases} 1 + r = t \\ i\frac{\hbar v}{2}(t-1+r) = V_0 t \end{cases} \Longrightarrow (r \to r',\ t \to t') \Rightarrow r' = r,\ t' = t$$

$$r = \frac{-i\pi\nu V_0}{1+i\pi\nu V_0}, \quad t = \frac{1}{1+i\pi\nu V_0}$$

scattering amplitudes.

Comparison : $\boxed{\hat{S}(\varepsilon) = \begin{pmatrix} t(\varepsilon) & r(\varepsilon) \\ r(\varepsilon) & t(\varepsilon) \end{pmatrix}}$

Generic 1D scattering



$$p = \sqrt{2mE} > 0$$

$$|\psi\rangle = (1 + G_0 T)|\psi_0\rangle$$

$$\psi_E^{(+)}(x) = a_1 e^{ipx} + a_2 e^{-ipx} - i\pi\nu \int dx'\,dx''\, e^{ip|x-x'|}\, T(x',x'')\left[a_1 e^{ipx''} + a_2 e^{-ipx''}\right]$$

$$\psi_E(x) = a_1 e^{ipx} + a_2 e^{-ipx} - i\pi\nu \int dx'\, e^{ip|x-x'|}\left[T(x',-p)\,a_1 + T(x',p)\,a_2\right]$$

$$\begin{cases} \psi(x\in 1) = a_1 e^{ipx} + e^{-ipx}\left[a_2 - i\pi\nu\left(\overset{T_{11}}{T(-p,-p)}\,a_1 + \overset{T_{12}}{T(-p,p)}\,a_2\right)\right] \\ \psi(x\in 2) = a_2 e^{-ipx} + e^{ipx}\left[a_1 - i\pi\nu\left(\underset{T_{21}}{T(+p,-p)}\,a_1 + \underset{T_{22}}{T(p,p)}\,a\right)\right] \end{cases}$$

$$\begin{cases} \psi_1 = a_1 e^{ipx} + b_1 e^{-ipx} \\ \psi_2 = a_2 e^{-ipx} + b_2 e^{ipx} \end{cases}$$

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \underbrace{\left(1 - i\pi\nu\,\hat{T}\right)}_{\boxed{\hat{S} = 1 - i\pi\nu\hat{T}}}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Take $a_2 = 0 \Rightarrow S_{11} = r,\ S_{21} = t$

$a_1 = 0 \Rightarrow S_{12} = t',\ S_{22} = r'$

$$\boxed{\hat{S} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}}$$

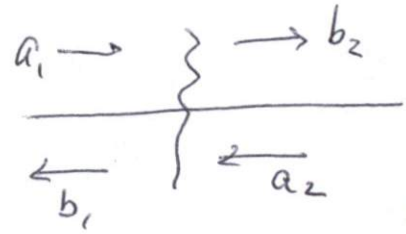
Symmetry? (restrictions)

Note: change of labeling!

1D current

If a particle in a state $|\psi\rangle$ the flux

$$j_\psi = \frac{\hbar}{2mi}\,\bar{\psi}\,\partial_x \psi + c.c.$$



Scat. state $|\psi\rangle$

Since $H\psi = \mathcal{E}\psi \Rightarrow \partial_x j_\psi = 0$ current conservation

$j_\psi(x) = \text{const} \Rightarrow$ calculate where it is convenient!

$$j_1 \equiv j(x\to-\infty) = v\left(|a_1|^2 - |b_1|^2\right)$$
$$j_2 \equiv j(x\to+\infty) = v\left(|b_2|^2 - |a_2|^2\right)$$

$\Rightarrow \sum_\alpha |a_\alpha|^2 = \sum |b_\alpha|^2$

$\bar{b} = \hat{S}\bar{a} \Rightarrow \boxed{\hat{S}^+\hat{S} = 1}$

current cons. unitarity

Time-reversal ψ and $\bar{\psi}$ are both sol-ns.

$\boxed{T\psi = \bar{\psi}}$ $\quad Ta = \bar{b}$, $\quad Tb = \bar{a}$ $\quad S' = TST^{-1} \underset{\text{if time-rev. symmetric}}{=} S$

$$Tb = S'Ta$$
$$\bar{a} = S'\bar{b} \Rightarrow S'\bar{S} = 1 \underset{\text{if t.-r.}}{\Rightarrow} \boxed{S\bar{S} = 1}$$

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1) $S^+S = 1$
2) $S = S^T$

$\Rightarrow$ scat. amp. $S = \begin{pmatrix} \overline{r} & \overline{t} \\ t & r' \end{pmatrix} \quad + \quad \begin{pmatrix} |r|^2 + |t|^2 & r\bar{t} + \bar{r}'t \\ \bar{r}t + r'\bar{t} & |r'|^2 + |t|^2 \end{pmatrix} = 1$

Scat. coeff. $R + T = 1$ $\qquad$ 1 mod. + 2 phases = 3 indep. var.