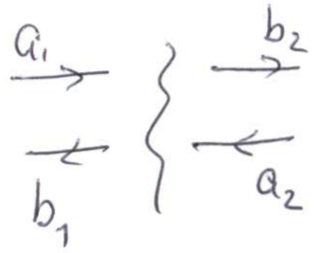
Transfer-matrix (in 1D)

$\hat{S}$-matrix (on-shell) definition



$$\phi_{out} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad \phi_{in} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\phi_{out} = \hat{S}\,\phi_{in}$$

$$\hat{S} = \begin{pmatrix} r & t \\ t & r' \end{pmatrix}$$

r — ref. amp. for incident from left wave

r' — ref. amp. for incident from right

Symmetries

1) Current conservation

$$I_{out} = I_{in} \Rightarrow \phi_{out}^{+}\phi_{out} = \phi_{in}^{+}\phi_{in} \Rightarrow \boxed{\hat{S}^{+}\hat{S} = 1}$$

2) Time reversal $\begin{cases} T\phi_{out} = \bar{\phi}_{in} \\ T\phi_{in} = \bar{\phi}_{out} \end{cases}$

$$\Rightarrow \boxed{T\hat{S}T^{-1} = \hat{S}^{-1}} \xrightarrow[\text{for TR-sym}]{} \boxed{\hat{S} = \hat{S}^{T}}$$

$$\phi_1 = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} b_2 \\ a_2 \end{pmatrix} : \quad \boxed{\phi_2 \overset{def}{=} \hat{M}\,\phi_1}$$

Symmetries?

Transfer-matrix (symmetries)

1) Current conservation $I_1 = I_2$

$$\phi_1^+ \sigma_z \phi_1 = \phi_2^+ \sigma_3 \phi_2 \Rightarrow \boxed{\hat{M}^+ \sigma_z \hat{M} = \sigma_z}$$

2) Time reversal $T\phi_i = \sigma_x \bar{\phi}_i$

Application of T to the definition of $\hat{M}$ gives

$$\boxed{T \hat{M} T^{-1} = \sigma_x \bar{M} \sigma_x}$$

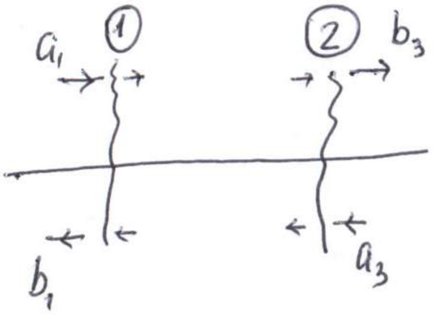
Generic tr.-matrix $\hat{M} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$, $|\alpha|^2 - |\beta|^2 = 1$

Relation to scattering amplitudes

$$\hat{M} = \begin{pmatrix} 1/\bar{t} & r'/t \\ -r/t & 1/t \end{pmatrix} \Rightarrow \alpha = \frac{1}{\bar{t}}, \; \beta = \underbrace{\frac{r'}{t} = -\frac{\bar{r}}{\bar{t}}}_{\text{Using symm. relations for S-matrix elements}}$$

Multiple scattering

Two scatterers separated by a distance d:



$$\phi_3 = \hat{M}_2 \phi(d) = \hat{M}_2 \hat{M}(d) \phi(0)$$
$$= \hat{M}_2 \hat{M}(d) \hat{M}_1 \phi_1$$

Total $\boxed{\hat{M} = \hat{M}_2 \hat{M}(d) \hat{M}_1}$

Free space tr. matrix $M(d) = \begin{pmatrix} e^{ipd} & \\ & e^{-ipd} \end{pmatrix}$

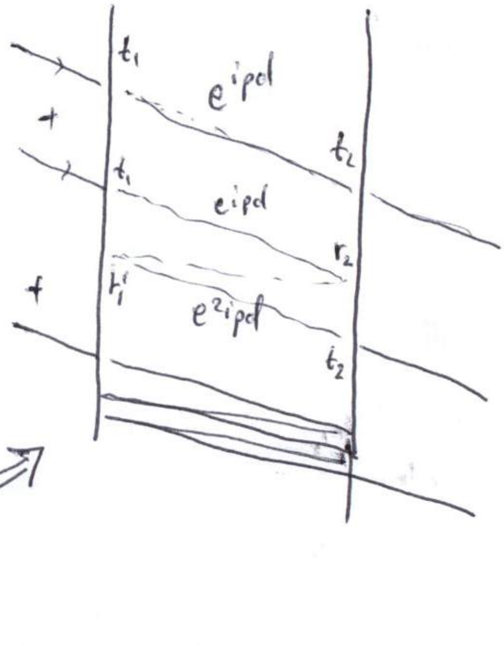
$$\hat{M} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha_2 & \beta_2 \\ \bar{\beta}_2 & \bar{\alpha}_2 \end{pmatrix} \begin{pmatrix} e^{ipd}\alpha_1 & e^{ipd}\beta_1 \\ e^{-ipd}\bar{\beta}_1 & e^{-ipd}\bar{\alpha}_1 \end{pmatrix}$$

$$\alpha = \frac{1}{\bar{t}}, \quad \alpha_i = \frac{1}{\bar{t}_i}, \quad \beta_i = \frac{r_i'}{t_i} = -\frac{\bar{r}_i}{\bar{t}_i}$$

Total transmission amplitude

$$t = \frac{t_1 t_2 e^{ipd}}{1 - r_1' r_2 e^{2ipd}}$$

Fabry-Perot resonator



$$t = t_1 \cdot \sum_{n=0}^{\infty} (r_1' r_2)^n e^{i(2n+1)pd} \cdot t_2$$

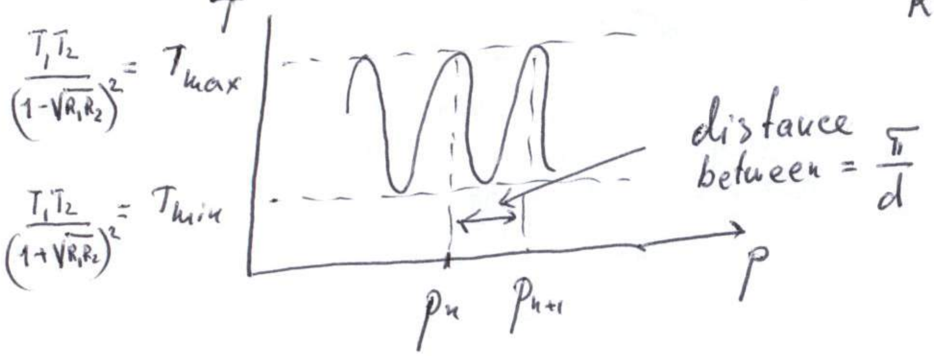
Double barrier

$$T = \frac{T_1 T_2}{\left(1-\sqrt{R_1R_2}\right)^2 + 4\sqrt{R_1R_2}\,\sin^2(pd+\phi)}, \qquad r_1' r_2 = \sqrt{R_1R_2}\,e^{i2\phi}$$

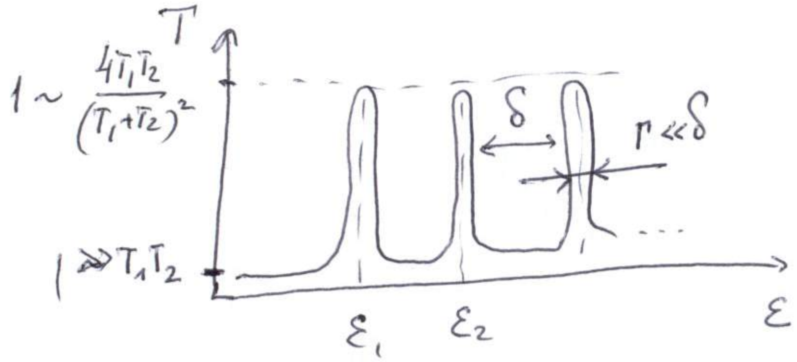
$T = T(\varepsilon)$ even if $T_i \neq T_i(\varepsilon)$

Res. at $p_n d + \phi = \pi n$

or $\varepsilon_n = \left(\frac{\pi n + \phi}{d}\right)^2 / 2m$



Extreme $T_1, T_2 \ll 1$



$\delta = v\Delta p = \frac{v\pi}{d}$

$\Gamma_i = \frac{\delta}{2\pi} T_i \ll \delta$

$$\boxed{T(\varepsilon) \approx \sum_n \frac{\Gamma_1\Gamma_2}{(\varepsilon-\varepsilon_n)^2 + (\Gamma/2)^2}}$$

$\varepsilon_n = \frac{p_n^2}{2m}$

$\Gamma = \Gamma_1 + \Gamma_2$

Breigh-Wigner formula

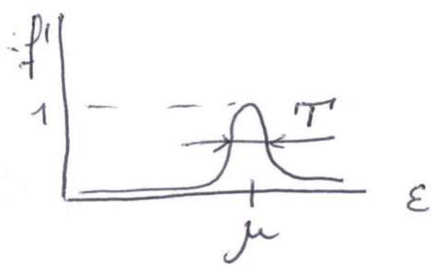
$T=0$ $\quad G = G_Q\, T(\varepsilon_F)$, $\quad \varepsilon_F = \varepsilon_F(n)$

or $\quad I(V) = \frac{G_Q}{e}\int_{\varepsilon_F}^{\varepsilon_F+eV} d\varepsilon\, T(\varepsilon)$

Coherent vs Incoherent

$$G = G_Q \int d\varepsilon \left(-\frac{\partial f}{\partial \varepsilon}\right) T(\varepsilon)$$

$$\frac{1}{4T\cosh^2\beta(\varepsilon-\mu)}$$



Averaging over energy/wave vector

$$T = T_1 T_2 \sum_{n,m} (r_1' r_2)^n e^{2inpd}\, \overline{(r_1' r_2)}\, e^{-2impd}$$

$$\langle T \rangle = \frac{1}{p_0}\int_{p-p_0/2}^{p+p_0/2} T(p)\,dp \underset{p_0 d \gg 1}{\approx} \frac{T_1 T_2}{1-R_1R_2}$$

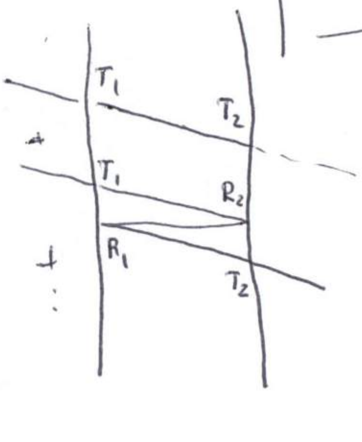
$\int d\varepsilon\, e^{i\varepsilon \frac{d}{v}} F\left(\frac{\varepsilon}{T}\right)$

$\frac{v}{d} \ll T$

$\delta = \frac{v}{d} \ll T$ or $\delta x \sim \frac{v}{T} \ll d$

classical

$$= T_1 T_2 \sum_{n=0}^{\infty} (R_1 R_2)^n$$



Incoherent sum
(of probabilities)

Wave packet with $\Delta p = p_0 \gg \frac{1}{d}$ $\left(\Delta\varepsilon \gg \frac{v}{d}\right)$ has spatial dimension $\Delta x \sim \frac{1}{\Delta p} \ll d$ and behaves classically! For conductance $\Delta\varepsilon \sim T \gg v/d \sim \delta$

$$\boxed{G_{clas} = G_Q \frac{T_1 T_2}{1-R_1R_2}}$$

Classical vs Quantum

Double barrier with strong individual scatterers $T_1, T_2 \ll 1$

$$G_{clas} \approx G_Q \frac{T_1 T_2}{T_1 + T_2} = \frac{G_1 G_2}{G_1 + G_2}$$

$\boxed{\text{Ohm's Law!}}$ $\Downarrow$

or $R_{clas} = R_1 + R_2$, $R_i = G_Q T_i$

resistance $= G_{clas}^{-1}$

$$R_{1..N} = R_1 + R_{2..N} = R_1 + R_2 + R_{3..N} = $$

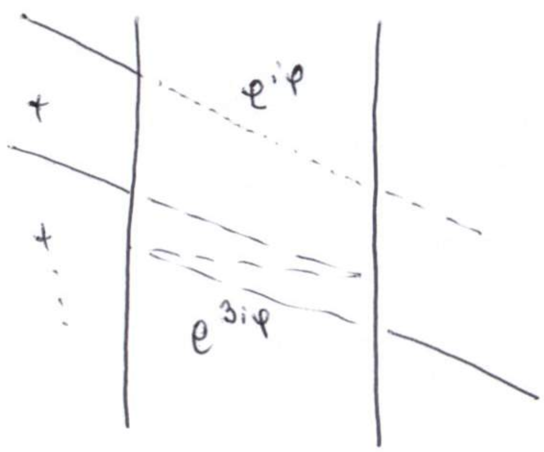
$$R = \sum_{i=1}^{N} R_i$$

$$R \sim N R_1 \Rightarrow G = \frac{R_1^{-1}}{N} = \frac{\Delta l}{R_1} \cdot \frac{1}{L}$$

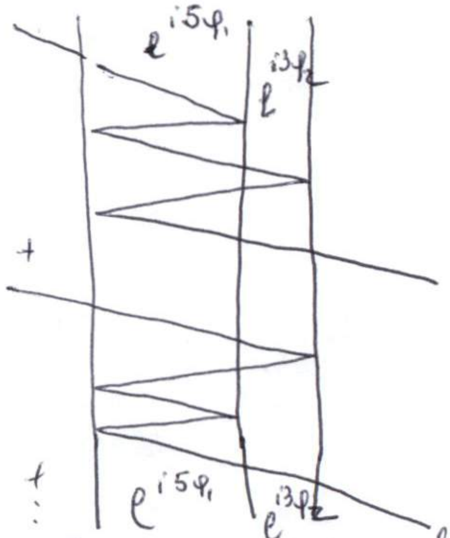
The same result from Breight Wigner

$$G(T \gg \delta) = G_Q \int d\varepsilon \left(-\frac{\partial f}{\partial \varepsilon}\right) \sum_n \frac{\Gamma_1 \Gamma_2}{(\varepsilon - \varepsilon_n)^2 + \Gamma^2/4} \approx G_Q 2\pi \frac{\Gamma_1 \Gamma_2}{\Gamma} \sum_n \left(-f'(\varepsilon_n)\right)$$

$$\approx G_Q 2\pi \frac{\Gamma_1 \Gamma_2}{\Gamma} \int \frac{d\varepsilon}{\delta} \left(-\frac{\partial f}{\partial \varepsilon}\right) = G_Q \frac{2\pi}{\delta} \frac{\Gamma_1 \Gamma_2}{\Gamma} = G_Q \frac{T_1 T_2}{T_1 + T_2}$$



No constructive interference for 2 scatterers



Always constructive between time-reversed trajectories