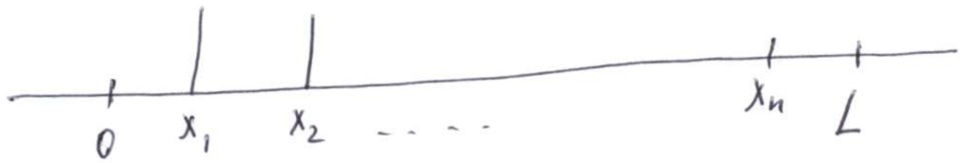
Anderson Localisation

$$M_{tot} = M^{(L-x_N)} M_N \, M(x_N - x_{N-1}) M_{N-1} \dots \; M(x_2 - x_1) M_1 \, M(x_1)$$



Consider i-th scatteres

$$\dots \begin{pmatrix} e^{ip(x_{i+1}-x_i)} & \\ & e^{-ip(x_{i+1}-x_i)} \end{pmatrix} M_i \begin{pmatrix} e^{ip(x_i - x_{i-1})} & \\ & e^{-ip(x_i - x_i)} \end{pmatrix} \dots$$

$$= \dots \begin{pmatrix} \alpha_i & \beta_i e^{-2ipx_i} \\ \bar{\beta}_i e^{2ipx_i} & \bar{\alpha}_i \end{pmatrix} \dots$$

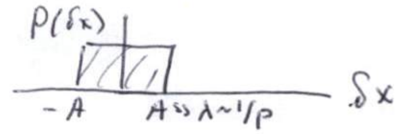
So we can write

$$\hat{M} = \prod_i^{\text{ordered}} \begin{pmatrix} \alpha_i & \beta_i e^{-2ipx_i} \\ \bar{\beta}_i e^{2ipx_i} & \bar{\alpha}_i \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$$

Since $\hat{M}\hat{M}^+ = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & 2\alpha\beta \\ 2\bar{\alpha}\bar{\beta} & |\alpha|^2 + |\beta|^2 \end{pmatrix}$

and $\tilde{M}_i \tilde{M}_i^+ = \begin{pmatrix} |\alpha_i|^2 + |\beta_i|^2 & 2\alpha_i\beta_i e^{-2ipx_i} \\ 2\bar{\alpha}_i\bar{\beta}_i e^{2ip_ix_i} & |\alpha_i|^2 + |\beta_i|^2 \end{pmatrix}$

we can average over $x_i = ia + \delta x_i$



AL-2

$$\langle\ \rangle_i = \frac{1}{\ell}\int_{ia-\ell}^{ia+\ell} dx_i \ldots \quad \text{and } \ell \gg \lambda \sim \frac{1}{p} \ (\text{i.e. } p\ell \gg 1)$$

Equivalently, throw away all oscillating terms

$$\langle \hat{M}\hat{M}^+ \rangle = \langle M_N \ldots M_2 \langle M_1 M_1^+ \rangle_1 M_2^+ \ldots M_N \rangle_{2\ldots N}$$

$$= \left(|\alpha_1|^2 + |\beta_1|^2\right) \langle M_N \ldots M_2 \cdot M_2^+ \ldots M_N^+ \rangle_{2\ldots N} = \ldots$$

$$= \prod_{i=1}^{N} \left(|\alpha_i|^2 + |\beta_i|^2\right)$$

meaning $\langle |\alpha|^2 + |\beta|^2 \rangle = \prod_{i=1}^{N} \left(|\alpha_i|^2 + |\beta_i|^2\right)$

or

$$\left\langle \frac{1+R}{T} \right\rangle = \prod_{i=1}^{N} \frac{1+R_i}{T_i}$$

Test $N=2$ we recover

$$T_{clas} = \frac{T_1 T_2}{1 - R_1 R_2}$$

AL-3

Weak scatterers limit $R_i \ll 1$

$$\left\langle \frac{1+R}{T} \right\rangle \approx \prod_{i=1}^{N} (1+2R_i) \xrightarrow[N\to\infty]{} e^{\sum_i \ln(1+2R_i)} \approx e^{2\sum_{i=1}^{N} R_i} \gg 1$$

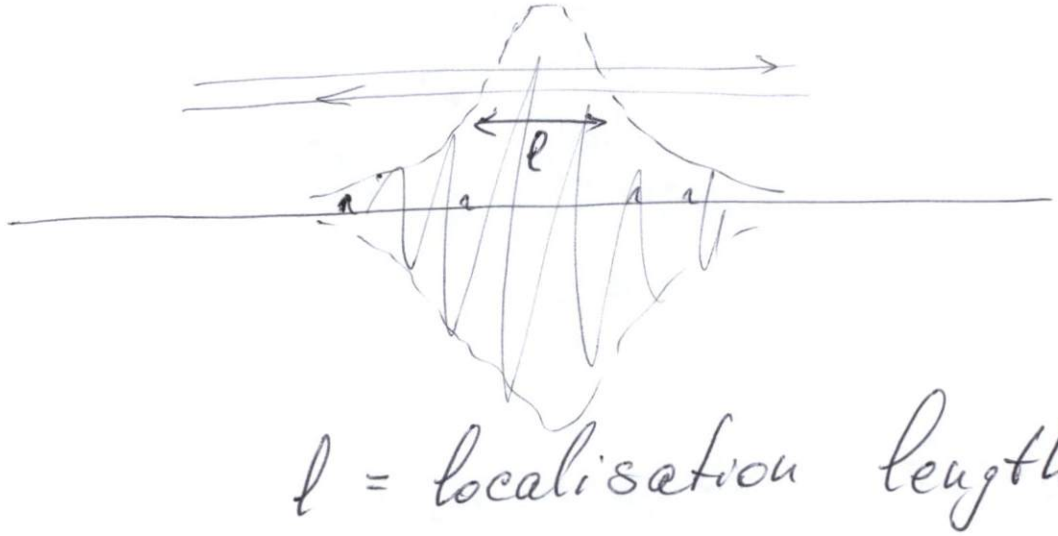
$$2\left\langle \frac{1}{T} \right\rangle - 1 \approx e^{2\sum R_i} \gg 1 \Rightarrow \left\langle \frac{1}{T} \right\rangle \approx \frac{1}{2} e^{2\sum R_i}$$

Average resistance $\sim e^{N \cdot 2R_{typ}}$, $R_{typ} \equiv \frac{1}{N}\sum_{i=1}^{N} R_i$

Average conductance $\sim e^{-2R_{typ} \cdot N}$

$$\boxed{\langle G \rangle \sim G_Q \, e^{-L/\ell}}\,, \quad \begin{array}{l} L = Na \\ \ell = a/R_{typ} \gg a \\ \langle T(L) \rangle \sim e^{-L/\ell} \end{array}$$

Localisation: all states are localised



ℓ = localisation length