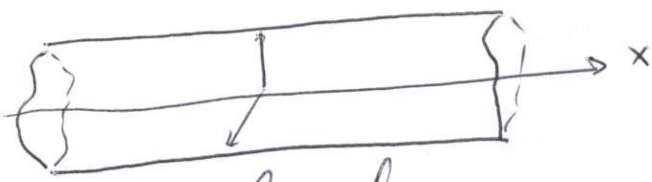
Multi-channel

"Waveguide" Schrödinger eq-n



$$-\frac{\hbar^2}{2m}\Delta\psi = \varepsilon\psi$$

or lead. Eigenstate $\psi_\varepsilon(\vec{r})$

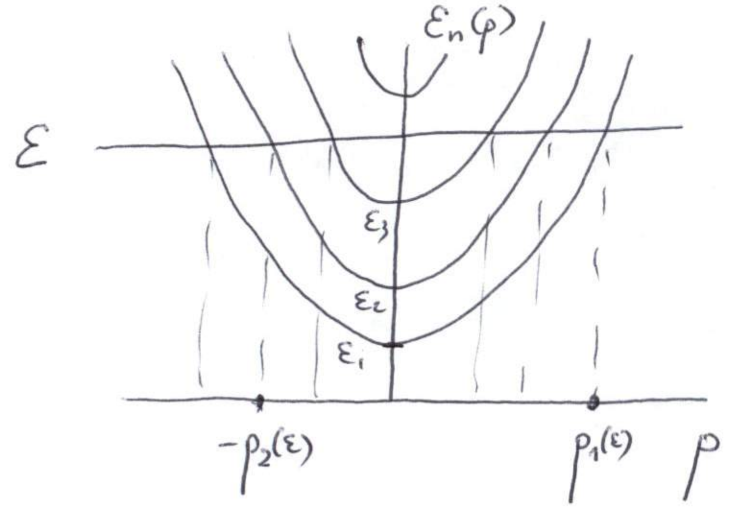
The variables can be separated

$$\psi(\vec{r}) = \chi(\vec{r}_\perp)\,\varphi(x), \quad \Delta = \Delta_\perp + \partial_x^2$$

$$-\frac{\hbar^2}{2m}\left[\frac{\Delta_\perp\chi}{\chi} + \frac{\partial_x^2\varphi}{\varphi}\right] = \varepsilon$$

$$-\frac{\hbar^2}{2m}\Delta_\perp\chi_n = \varepsilon_n\chi_n, \quad \varphi = e^{ipx}, \quad \varepsilon = \varepsilon_n + p^2/2m$$

Longitudinal momentum $= \pm p_n = \pm\sqrt{2m(\varepsilon - \varepsilon_n)}$



Number of "channels" (degeneracy of eigenstate with energy ε)

$N(\varepsilon) = 3$ for this graph.

General solution

$$\psi_\varepsilon(\vec{r}) = \sum_n \chi_n(\vec{r}_\perp)\left[a_n e^{ip_n x} + b_n e^{-ip_n x}\right]$$

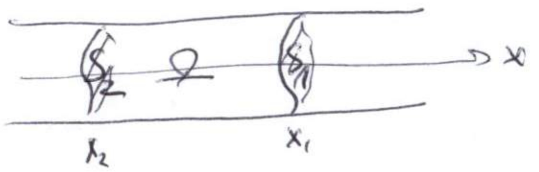
MC-2

Since $\psi_\varepsilon(x)$ is a solution of the Sch. eq-n

$$\nabla \vec{j} = 0, \quad \vec{j} = \frac{\hbar}{2mi}\, \bar{\psi} \nabla \psi + c.c.$$

Integrating over the volume between two cross-sections

$$\int_\Omega d^3r\, \nabla \vec{j} = \int_S d\vec{S}\cdot\vec{j} = \left(\int_{S_1} d^2r_\perp - \int_{S_2} d^2r_\perp\right) j(\vec{r}_\perp, x) = 0$$



assuming no current through waveguide surface.

current

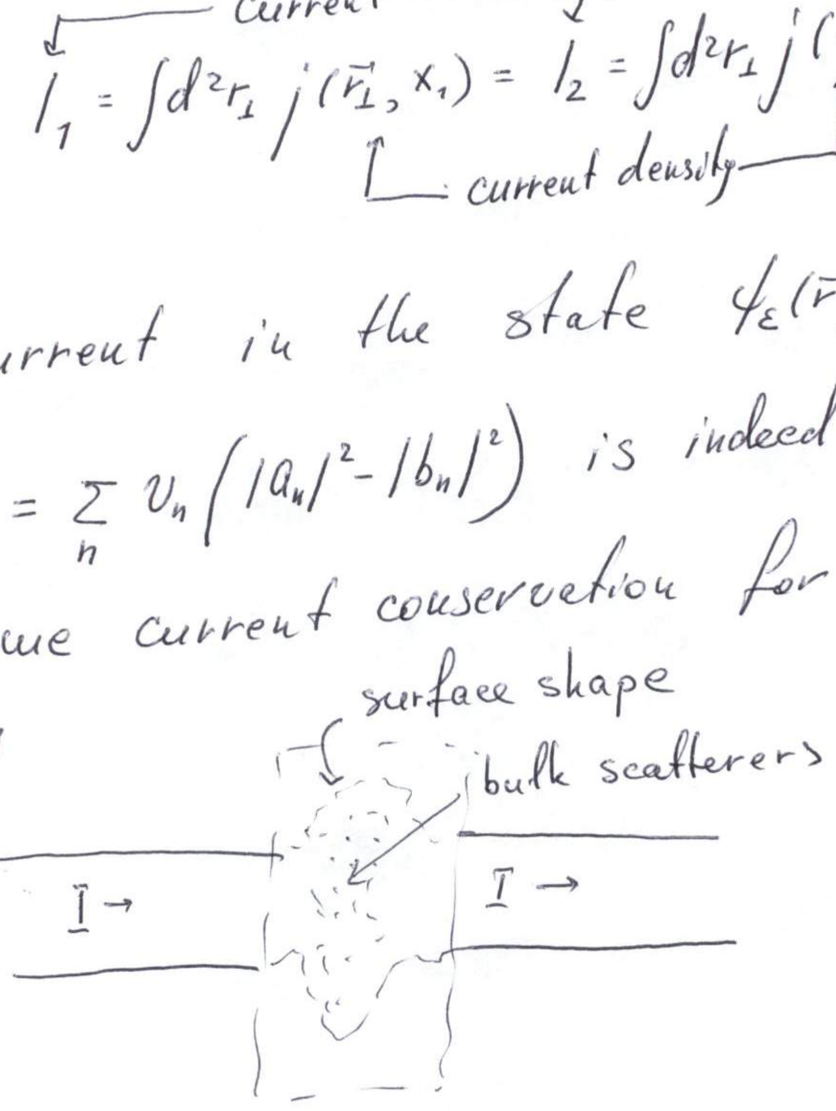
Then $$I_1 = \int d^2r_\perp\, j(\vec{r}_\perp, x_1) = I_2 = \int d^2r_\perp\, j(\vec{r}_\perp, x_2)$$

current density

The current in the state $\psi_\varepsilon(\vec{r})$ is indeed a const.

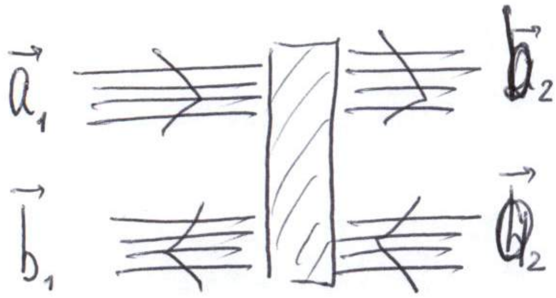
$$I = \sum_n v_n\left(|a_n|^2 - |b_n|^2\right)$$

The same current conservation for a generic geometry



MC-3

Visualization



Asymptotes:

$$\psi(x\to-\infty) = \sum_n \chi_n \left(a_{n1} e^{ip_n x} + b_{n1} e^{-ip_n x}\right)$$

$$\psi(x\to+\infty) = \sum_n \chi_n \left(a_{n2} e^{-ip_n x} + b_{n2} e^{ip_n x}\right)$$

Essentially $N\times N$ matrix eq-n of second order: $2N$ independet constants, not $4N$! Therefore

$$\vec{a}_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \end{pmatrix}; \quad \vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix} \text{ - } 2N\text{-column vector}$$

$$\boxed{\vec{b} = \hat{\tilde{S}}\,\vec{a}}$$

Current conservation

$$\sum_n v_n\left(|a_{n1}|^2 - |b_{n1}|^2\right) = \sum_n v_n\left(|b_{n2}|^2 - |a_{n2}|^2\right)$$

Since $\nu_n(\varepsilon) = \dfrac{1}{\pi\hbar v_n(\varepsilon)}$ we can rewrite it introducing diagonal matrix $\hat{\nu} = \mathrm{diag}(\nu_n)$

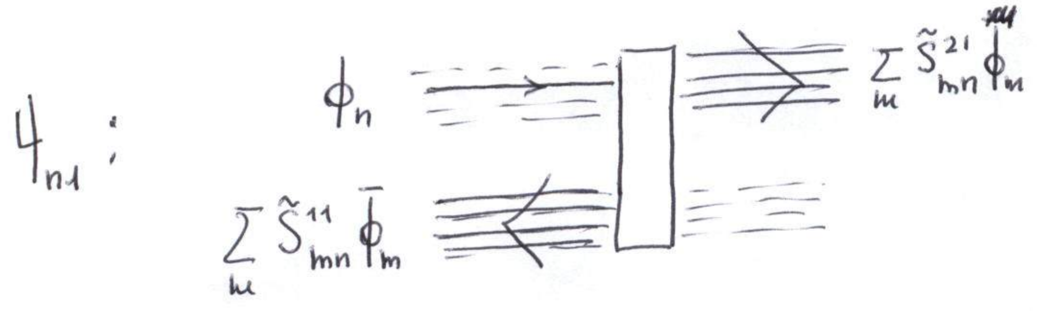
$$\boxed{\hat{\nu}^{-1} = \hat{\tilde{S}}^{+}\hat{\nu}^{-1}\tilde{S}}$$

Rescaling $\tilde{S} = \hat{\nu}^{1/2}\hat{S}\,\hat{\nu}^{-1/2}$ leads to $\hat{S}^{+}S$ and

$$\hat{S} = \begin{pmatrix} \hat{r} & t \\ \hat{t} & \hat{r}' \end{pmatrix} = \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix}$$

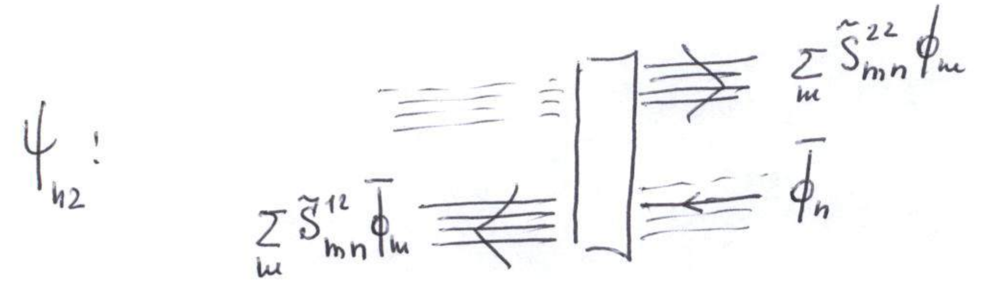
MC-4

2N scattering states we use as basis



$\phi_n \equiv \chi_n(\vec{r}_\perp) e^{ip_n x}$

$\bar{\chi}_n = \chi_n$



The field operator

$$\hat{\psi}(\vec{r}) = \sum_{n\alpha} \int d\varepsilon \, \nu_{n\alpha}(\varepsilon) \, \psi_{n\alpha}(\vec{r}) \, \hat{a}_{n\alpha}(\varepsilon)$$

$\nu_{n\alpha} = \nu_n / 2$

Normalization $\langle \varepsilon; n, \alpha | \varepsilon'; n', \alpha' \rangle = \nu^{-1}_{n\alpha}(\varepsilon) \delta_{nn'} \delta_{\alpha\alpha'} \delta(\varepsilon - \varepsilon')$

$$\langle 0 | \hat{a}^+_{n\alpha}(\varepsilon) \hat{a}_{n'\alpha'}(\varepsilon') | 0 \rangle$$

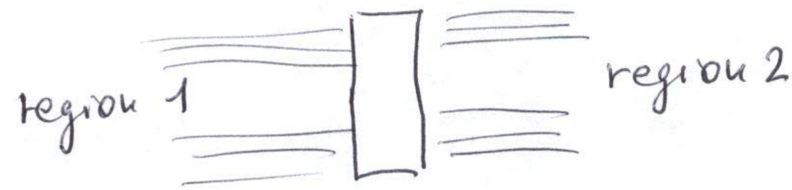
requires $\{ \hat{a}_{n\alpha}(\varepsilon), \hat{a}^+_{n'\alpha'}(\varepsilon') \} = \nu^{-1}_{n\alpha}(\varepsilon) \delta_{nn'} \delta_{\alpha\alpha'} \delta(\varepsilon - \varepsilon')$

Rescaling $\nu^{1/2}_{n\alpha} \hat{a}_{n\alpha} \to \hat{a}_{n\alpha}$ leads to (in new $\hat{a}_{n\alpha}$)

$$\boxed{\begin{gathered} \hat{\psi}(\vec{r}) = \sum_{n\alpha} \int d\varepsilon \, \nu^{1/2}_{n\alpha} \psi_{n\alpha}(\vec{r}) \, \hat{a}_{n\alpha}(\varepsilon) \\ \{ \hat{a}_{n\alpha}(\varepsilon), \hat{a}^+_{n'\alpha'}(\varepsilon') \} = \delta_{nn'} \delta_{\alpha\alpha'} \delta(\varepsilon - \varepsilon') \end{gathered}}$$

MC-5

$$\begin{cases} \psi(\vec r \subset 1) = \sum_n \int d\varepsilon\, \nu_n^{1/2} \left[\hat a_{n1} \left(\phi_n + \sum_m \tilde S^{11}_{mn} \bar\phi_m \right) + \hat a_{n2} \sum_m \tilde S^{12}_{mn} \bar\phi_m \right] \\ \psi(\vec r \subset 2) = \sum_n \int d\varepsilon\, \nu_n^{1/2} \left[\hat a_{n1} \sum_m \tilde S^{21}_{mn} \phi_m + \hat a_{n2} \left(\bar\phi_n + \sum_m \tilde S^{22}_{mn} \phi_m \right) \right] \end{cases}$$



$$\psi_1 = \sum_n \int d\varepsilon\, \nu_n^{1/2} \left[\hat a_{n1} \phi_n + \hat b_{n1} \bar\phi_n \right]$$

$$\hat b_{n1} = \sum_{m,\beta} \tilde S^{1\beta}_{nm} \nu_m^{1/2} \hat a_{m\beta} = \sum_{m,\beta} S^{1\beta}_{nm} \hat a_{m\beta}$$

$$\psi_2 = \sum_n \int d\varepsilon\, \nu_n^{1/2} \left[\hat a_{n2} \bar\phi_n + \hat b_{n2} \phi_n \right]$$

$$\hat b_{n2} = \sum_{m\beta} S^{2\beta}_{nm} \hat a_{m\beta}$$

$$\psi_\alpha(r) = \sum_n \int d\varepsilon\, \nu_n^{1/2}(\varepsilon) \chi_n(\vec r_\perp) \left[\hat a_{n\alpha}(\varepsilon) e^{i p_{n\alpha}(\varepsilon) x} + \hat b_{n\alpha}(\varepsilon) e^{-i p_{n\alpha}(\varepsilon) x} \right]$$

α ← region/lead

$$p_{n\alpha} = \begin{cases} p_n, & \alpha = 1 \\ -p_n, & \alpha = 2 \end{cases}$$

Hamiltonian $\quad \hat H = \int_{\Omega} d^3r\, \hat\psi^+ \left(-\frac{\hbar^2}{2m}\Delta + V(\vec r) \right) \hat\psi \quad$ in scat. states

$$\hat H = \int d\varepsilon\, d\varepsilon'\, \varepsilon \sum_{nn'} \left(\nu_n(\varepsilon) \nu_n(\varepsilon') \right)^{1/2} \hat a^+_{n\alpha}(\varepsilon) \hat a_{n'\alpha'}(\varepsilon') \underbrace{\int d^3r\, \psi_{n\alpha}(\vec r, \varepsilon) \psi_{n'\alpha'}(\vec r, \varepsilon')}_{\nu_n^{-1}(\varepsilon) \delta_{nn'} \delta_{\alpha\alpha'} \delta(\varepsilon - \varepsilon')}$$

$$\boxed{\hat H = \sum_n \int d\varepsilon\, \varepsilon\, \hat a^+_{n\alpha}(\varepsilon) \hat a_{n\alpha}(\varepsilon)}$$

MC-6

Assuming equilibrium for each "α-particles" group:

$$\langle \hat{a}^+_{n\alpha}(\varepsilon)\hat{a}_{n'\alpha'}(\varepsilon')\rangle = \delta_{nn'}\delta_{\alpha\alpha'}\delta(\varepsilon-\varepsilon')f_\alpha(\varepsilon)$$

α = index of scattering state, it means "from where" the particles came from, ~~f~~from which infinity/reservoir.

The current operator in the region 1:

$$\hat{I}_\alpha = e\int d\varepsilon\, d\varepsilon' \sum_{nn'} [\nu_n(\varepsilon)\nu_{n'}(\varepsilon')]^{1/2} \cdot \left[\hat{a}^+_{n\alpha}(\varepsilon)e^{-ip_n(\varepsilon)x} + \hat{b}^+_{n\alpha}(\varepsilon)e^{ip_n(\varepsilon)x}\right]$$

$$\cdot \frac{ev_{n'}(\varepsilon')}{2}\left[\hat{a}_{n'\alpha'}(\varepsilon')e^{ip_{n'}(\varepsilon')x} - \hat{b}_{n'\alpha'}(\varepsilon')e^{-ip_{n'}(\varepsilon')x}\right] + h.c.$$

This current in any region flowing into scattering region (black box).

Averaging with density matrix

$$I_\alpha = e\sum_n \int d\varepsilon\, \underbrace{\nu_n(\varepsilon)v_n(\varepsilon)}\left[f_\alpha(\varepsilon) - \sum_{m\beta}|S^{\alpha\beta}_{nm}|^2 f_\beta(\varepsilon)\right]$$

$$\nu_{n\alpha} = \frac{1}{2\pi\hbar v_n} \Rightarrow \nu_n \to \nu_n/2$$

$$I_\alpha = e\int\frac{d\varepsilon}{2\pi\hbar}\sum_n\left[f_\alpha - \sum_\beta (S^+_{\alpha\beta}S_{\beta\alpha})_{nn} f_\beta\right]$$

MC-7

$$I_\alpha = \frac{e}{2\pi\hbar}\int d\varepsilon\, tr\left[f_\alpha - \sum_\beta S^+_{\alpha\beta} S_{\beta\alpha} f_\beta\right]$$

trace is only over channels

Current conservation

$$\sum_\alpha I_\alpha = \frac{e}{2\pi\hbar}\int d\varepsilon\, Tr\left[\hat{f} - S^+\hat{f}S\right] = 0$$

$\hat{f} = diag(f_1, f_2)$

Trace over the whole space

$$I_\alpha = \frac{e}{2\pi\hbar}\int d\varepsilon\, tr\left[(1-r^+r) f_\alpha - t^+t\, f_{-\alpha}\right]$$

$$S = \begin{pmatrix}\hat{r} & \hat{t}\\ \hat{t} & \hat{r}'\end{pmatrix}, \quad \hat{S}^+\hat{S} = \begin{pmatrix}\hat{r}^+ & \hat{t}^+\\ \hat{t} & \hat{r}'^+\end{pmatrix}\begin{pmatrix}\hat{r} & \hat{t}\\ \hat{t} & \hat{r}\end{pmatrix} = \begin{pmatrix}\hat{r}^+\hat{r}+t^+t & \cdot\\ \cdot & \hat{r}'^+\hat{r}'+t'^+t\end{pmatrix} = 1$$

$$I_1 = \frac{e}{2\pi\hbar}\int d\varepsilon\, tr\,\hat{T}(\varepsilon)\cdot(f_1 - f_2), \quad \hat{T} \equiv \hat{t}^+\hat{t}$$

If $\hat{T}(\varepsilon) \approx \hat{T}$ then

$$I_1 = \frac{e^2}{2\pi\hbar}\, tr T \cdot V = G \cdot V$$

$$\boxed{G = G_Q \cdot tr\,\hat{T}}$$