QFT for CM

Bosonization (functional)

Path integral (over traj.) $\xleftarrow{\text{Bosons}}$

Func. integral (over coh. st.) $\xrightarrow{\text{Fermions}}$

Non-linear tran. (Jacobian)

Gaussian theories (some)/generic rules

Stochastic quantization

Stochastic (classical) problems

1. Gaussian process (variables), Gaussian integrals
2. Pair and n-point corr. functions
3. Wick's theorem.
4. Fokker – Planck <=> functional integral (stochastic quantization)

Classical particle with random force

$$\cancel{m}\ddot{x} + \dot{x} = F(x) + \xi(t)$$

inertialess => Langevine $\dot{x} = F(x) + \xi(t)$

$\xi(t)$ – white noise:
1. Gaussian process
2. δ-correlated
$\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$

Gaussian random variables (discrete version)

$\xi^T = (\xi_1 \dots \xi_N)$ with joint PDF

$$P(\xi) = \frac{1}{Z} e^{-\frac{1}{2}\xi^T \hat{S} \xi}$$

$$\hat{S}^T = \hat{S}$$

$$Z = \int d^N\xi \, e^{-\frac{1}{2}\xi^T \hat{S} \xi}$$

Normalization (partition function)

$$\hat{S} = O^T \lambda O \quad , \quad O^T O = 1$$

$$\xi^T S \xi = \xi'^T \lambda \xi' \quad , \quad \xi' = O \xi$$

$$J = \left| \det \frac{\partial \xi'}{\partial \xi} \right| = \det \hat{O} = 1$$

$$\int d^N\xi \, e^{-\frac{1}{2}\xi^T \hat{S} \xi} = \int d^N \xi' e^{-\frac{1}{2}\sum \lambda_i \xi_i^2} = \left[\prod_{i=1}^{N} \frac{2\pi}{\lambda_i} \right]^{1/2}$$

$$Z = \det^{-1/2} S \qquad \int \mathcal{D}\xi \equiv \int \prod_{i=1}^{N} \frac{d\xi_i}{\sqrt{2\pi}}$$

Corr. func. $\langle \xi_i \xi_j \rangle = \int \mathcal{D}\xi \, \xi_i \xi_j \, P(\xi_1 \ldots \xi_N)$

n-point corr. func. $\langle \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n} \rangle$

Generating func. $= \langle e^{J^T \xi} \rangle = e^{-\sum_{n=1}^{\infty} \frac{(-1)^n}{n} K^{(n)}_{i_1 \ldots i_n} J_{i_1} \ldots J_{i_n}}$

J – source field $\qquad K^{(n)}_{i_1 \ldots i_n} \equiv$ n-th order cumulant

$$\boxed{\langle e^{J^T \xi} \rangle = e^{\frac{1}{2} J^T \hat{G} J} \quad , \quad \hat{G} = \hat{S}^{-1}}$$

All cumulants $= 0$ but $\hat{K}^{(2)} = \hat{S}^{-1} = \hat{G}$

$$\frac{1}{2}\xi^T \hat{S} \xi - J^T \xi = \frac{1}{2}(\xi^T - J^T \hat{S}^{-1}) \hat{S} (\xi - \hat{S} J) - \frac{1}{2} J^T \hat{S}^{-1} J$$

$\xi' = \xi - S J \Rightarrow$ Jacobian $= \left| \frac{\partial \xi'}{\partial \xi} \right| = 1$

Corr. functions

$$\langle e^{J^T \xi} \rangle = e^{\frac{1}{2} J^T \hat{G} J}$$

$\langle \xi_i \rangle = 0$ $\boxed{\langle \xi_i \xi_j \rangle = G_{ij}}$ 1) $P = \frac{1}{Z} e^{-\frac{1}{2} \xi^T G^{-1} \xi}$

2) $\langle e^{J^T \xi} \rangle = e^{\frac{1}{2} \langle (J^T \xi)^2 \rangle}$
because of $K^{(n>2)} = 0$

$(2n+1)$-points corr. func's $= 0$

$2n$-points corr. func.

$$\langle \xi_{i_1} \dots \xi_{i_{2n}} \rangle = ?$$

4-points $\frac{1}{4!} \sum J_1 J_2 J_3 J_4 \langle \xi_1 \xi_2 \xi_3 \xi_4 \rangle = \frac{1}{2! \, 2^2} \sum J_1 J_2 J_3 J_4 \, G_{12} G_{34}$
$(i_1 \to 1, i_2 \to 2 \dots)$

$$\sum J_1 \dots J_4 \langle \xi_1 \dots \xi_4 \rangle = 3 \sum J_1 \dots J_4 \, G_{12} G_{34}$$
$$= \sum J_1 \dots J_4 \left[G_{12} G_{34} + G_{13} G_{24} + G_{14} G_{23} \right]$$
now it's symmetric
wrt permutations

$$\langle \xi_1 \xi_2 \xi_3 \xi_4 \rangle = \langle \xi_1 \xi_2 \rangle \langle \xi_3 \xi_4 \rangle + \langle \xi_1 \xi_3 \rangle \langle \xi_2 \xi_4 \rangle + \langle \xi_1 \xi_4 \rangle \langle \xi_2 \xi_3 \rangle$$
Wick's theorem

$$\sum J_{i_1} \dots J_{i_{2n}} G^{(2n)}_{\dots} = \frac{(2n)!}{2^n n!} \sum J_{i_1} \dots J_{i_{2n}} \, G_{i_1 i_2} \dots G_{i_{2n-1} i_{2n}}$$

$\frac{(2n)!}{2^n n!} = (2n-1)!! = \#$ contractions $2n-1$, $2n-3$ …

$$\boxed{G^{(2n)}_{i_1 \dots i_{2n}} = \sum_{\alpha} G_{\alpha_1} \dots G_{\alpha_n}}$$ – Wick's theorem

$\alpha = \{\alpha_1 \dots \alpha_n\}$, $\alpha_1 = i_i i_j$ –
all possible contractions (pairs)

Complex Gaussian RV

$$P(z_1 \dots z_N) = \frac{1}{Z} e^{-z^+ \hat{H} z} \; , \quad \hat{H}^+ = \hat{H}$$

$$\hat{H} = U^+ \lambda U \Rightarrow Z = \int d^N z \, d^N \bar{z} \; e^{-\sum \lambda_i |z_i|^2} = \det{}^{-1} \hat{H}$$

$$\text{Jac.} = |\det U| = 1$$

Generating function $= \left\langle e^{J^+ z + z^+ J} \right\rangle$

$$z^+ H z - J^+ z - z^+ J = (z^+ - J^+ \hat{H}^{-1}) \hat{H} (z - \hat{H} J) - J^+ \hat{H}^{-1} J$$

$$\left\langle e^{J^+ z + z^+ J} \right\rangle = e^{\hat{J}^+ \hat{G} J} \; , \quad \hat{G} = \hat{H}^{-1}$$

$$\langle z_{i_1} \dots z_{i_n} \rangle = 0$$

$$\langle z_i \bar{z}_j \rangle = G_{ij} \Rightarrow \begin{cases} 1. \; P = \frac{1}{Z} e^{-z^+ \hat{G}^{-1} z} \\ 2. \; \left\langle e^{L[z]} \right\rangle = e^{\frac{1}{2} \langle L^2[z] \rangle} \end{cases}$$

2n-points corr. functions

$$\langle z_{i_1} \dots z_{i_n} \bar{z}_{j_1} \dots \bar{z}_{j_n} \rangle = \sum_{P(j)} \prod_{k=1}^{n} \langle z_{i_k} \bar{z}_{P_k(j)} \rangle$$

Wick's theorem again

$\equiv$ permanent $\hat{G}$

$$\boxed{G^{(2n)} = P_n G^{(2)}}$$

Anti-commuting variables (Grassmanns)

Joint PDF $P[\psi] = \frac{1}{Z} e^{-\psi^+ \hat{H} \psi}$

$\psi_i \psi_j = -\psi_j \psi_i \qquad \psi_i \bar{\psi}_j = -\bar{\psi}_j \psi_i$ (w/o comments)

Generating function (source fields also Grassmannian)

$$\left\langle e^{+J^+\psi + \psi^+ J} \right\rangle = e^{J^+ \hat{G} J}, \quad \hat{G} = \hat{H}^{-1}$$

Proof: the same shift $\psi \to \psi + \hat{G} J$

2-points corr. func. $\langle \psi_i \bar{\psi}_j \rangle = G_{ij}$

2n-points corr. func's

$$\sum_{\{i,j\}} \bar{J}_{i_1} \dots \bar{J}_{i_n} J_{j_1} \dots J_{j_n} \langle \psi_{i_1} \dots \psi_{i_n} \bar{\psi}_{j_1} \dots \bar{\psi}_{j_n} \rangle$$

$$= n! \sum_{\{i,j\}} \bar{J}_{i_1} \dots \bar{J}_{i_n} J_{j_1} \dots J_{j_n} \cdot G_{i_1 j_1} \dots G_{i_n j_n}$$

Anti-symmetrization:

$$\langle \psi_{i_1} \dots \psi_{i_n} \bar{\psi}_{j_1} \dots \bar{\psi}_{j_n} \rangle = \begin{vmatrix} G_{i_1 j_1} & G_{i_1 j_2} & \dots & G_{i_1 j_n} \\ G_{i_2 j_1} & G_{i_2 j_2} & \dots & G_{i_2 j_n} \\ - & - & - & - \\ G_{i_n j_1} & \dots & \dots & G_{i_n j_n} \end{vmatrix}$$

Good candidate for a description of Fermions.

Anti-commuting II

Corr. functions obtained assuming all standard rules of integrals (change of variables, shift...) are applicable.

$$\int \mathcal{D}\psi\, f(\psi) = ?$$

$$f(\psi) = f(0) + f'(0)\cdot\psi \qquad \text{since } \psi^2 = 0$$

Definition $\quad \int d\psi = 0 \qquad \int d\psi \cdot \psi = 1$

Many variables $\quad d\psi_i \cdot \psi_j = -\psi_j\, d\psi_i$

$$d\psi_i \cdot d\psi_j = -d\psi_j \cdot d\psi_i$$

These rules allow using standard rules for change of variables.

$$Z = \int \mathcal{D}\psi\, e^{-\psi^+ \hat{H} \psi} \qquad \mathcal{D}\psi \equiv \prod_{i=1}^{N} d\bar{\psi}_i\, d\psi_i$$

$$\hat{H} = U^+ \lambda U, \quad \psi \to U^+\psi, \quad \text{Jacobian} = 1$$

$$Z = \int \mathcal{D}\psi\, e^{-\sum \lambda_i \bar{\psi}_i \psi_i} = \prod \lambda_i = \det \hat{H}$$

$$\det \hat{H} = e^{\operatorname{tr} \ln \hat{H}}$$

Furutsu - Novikov

$$\left\langle \frac{\partial R[\xi]}{\partial \xi_i} \right\rangle = -\frac{1}{Z} \int \mathcal{D}\xi \; R[\xi] \frac{\partial}{\partial \xi_i} e^{-\frac{1}{2}\xi^T G^{-1} \xi}$$

$$= \sum_j G^{-1}_{ij} \langle \xi_j R[\xi] \rangle$$

$$\boxed{\langle \xi_i R[\xi] \rangle = \sum_j G_{ij} \left\langle \frac{\partial R[\xi]}{\partial \xi_j} \right\rangle}$$

Continuous limit = random process

$$\xi_1 \cdots \xi_N \to \xi(t) \,/\, \xi(x)$$

$$P[\xi] = \frac{1}{Z} e^{-\frac{1}{2}\xi^T \hat{G}^{-1} \xi}, \quad \xi^T \hat{G}^{-1} \xi = \int dt\,dt' \; \xi(t) H(t,t') \xi(t')$$

$$\langle \xi(t) R[\xi] \rangle = \int dt' \; G(t\,t') \left\langle \frac{\delta R[\xi]}{\delta \xi(t')} \right\rangle$$

Increment $\quad \delta R[\xi] = \sum \underbrace{A_i}_{\frac{\partial R}{\partial \xi_i}} \delta \xi_i + \frac{1}{2} B_{ij} \delta \xi_i \delta \xi_j + \ldots$

$$\delta R[\xi] = \int dt \; \underbrace{A(t)}_{\frac{\delta R[\xi]}{\delta \xi(t)}} \delta \xi(t) + \frac{1}{2} \int dt\,dt' \; B(t\,t') \delta \xi(t) \delta \xi(t')$$

White noise $\quad \langle \xi(t) \xi(t') \rangle = 2\mathcal{D}\, \delta(t-t')$

$$G = 2\mathcal{D}\,\delta(t-t'), \quad H(t,t') = \frac{1}{2\mathcal{D}} \delta(t-t')$$

$$\boxed{P_{WN}[\xi] = \frac{1}{Z} e^{-\frac{1}{4\mathcal{D}} \int dt \, \xi^2(t)}}$$

Langevine dynamics

$$\dot{x}(t) = \xi(t) \longleftarrow \text{white noise}$$

$$P(x,t) = \Big\langle \delta(x - x(t)) \Big\rangle_{\xi}$$

$$\partial_t P = -\partial_x \Big\langle \xi(t)\, \delta(x - x(t)) \Big\rangle$$

$$\parallel$$

$$\int dt'\, G(t-t') \Big\langle \frac{\delta}{\delta \xi(t')} \delta(x - x(t)) \Big\rangle$$

$$= -\partial_x \int dt'\, G(t-t') \Big\langle \frac{\delta x(t)}{\delta \xi(t')} \delta(x - x(t)) \Big\rangle$$

$$\frac{d}{dt} \frac{\delta x(t)}{\delta \xi(t')} = \delta(t-t') \Rightarrow \frac{\delta x(t)}{\delta \xi(t')} = \theta(t-t')$$

$$\partial_t P = \int^t dt'\, G(t-t')\, \partial_x^2 P(x,t')$$

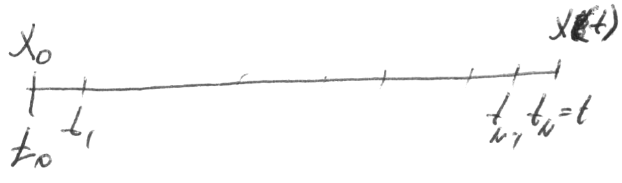
$$\boxed{\partial_t P(x\,t) = D\, \partial_x^2 P(x,t)}$$ – diffusion

Brownian motion

Stochastic "quantization" approach

$$\begin{cases} \dot{X}(t) = \xi(t) \\ X(t_0) = x_0 \end{cases}$$

$$X(t_{i+1}) = X(t_i) + \Delta\, \xi(t_i)$$



$$X(t) = \int_{X(t_0) = x_0} d^N x \prod_{i=1}^{N-1} \delta\left(\frac{x_{i+1} - x_i}{\Delta} - \xi(t_i)\right) \cdot x_N \cdot J$$

$$= \int dx\, x \cdot \int \mathcal{D}x\, \delta[\dot{x} - \xi]$$

$x_N \to x$

$\mathcal{D}x = J\, dx_1 \ldots dx_{N-1}$

$J = \frac{1}{\Delta^{N-1}}$

$x(t) = \int dx \cdot x\, P(xt, x_0 t_0)\,; \quad f(x(t)) = \int dx\, f(x)\, P(xt, x_0 t_0)$

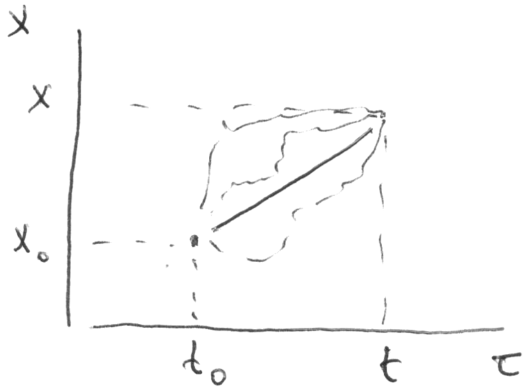
$\Rightarrow$ Probability to find particle at x for time t

$$P(xt, x_0 t_0) = \int_{x(t_0)=x_0}^{x(t)=x} \mathcal{D}x \;\; \delta[\dot{x} - \xi]$$

Averaging over white noise

$$P(xt, x_0 t_0) = \int \mathcal{D}\xi\, e^{-\frac{1}{4D}\int d\tau\, \xi^2(\tau)}\, P(xt, x_0 t_0)$$

$$= \int_{x(t_0)=x_0}^{x(t)=x} \mathcal{D}x \;\; e^{-\frac{1}{4D}\int_{t_0}^{t} d\tau\, \dot{x}^2(\tau)}$$



Extremum $\quad \ddot{x}_{min}(\tau) = 0$

$$x_{min}(\tau) = x_0 + v_0(\tau - t_0)$$

$$v_0 = \frac{x - x_0}{t - t_0}$$

$$\begin{cases} x_{min}(\tau = t_0) = x_0 \\ x_{min}(\tau = t) = x \end{cases}$$

Weighted integral
$\overset{\shortparallel}{e^{-\frac{1}{4D}\int d\tau\, \dot{x}^2}}$
over trajectories
$\Rightarrow$ path integral

$$\begin{cases} x(\tau) = x_{min}(\tau) + \delta x(\tau) \\ \delta x(t_0) = \delta x(t) = 0 \end{cases}$$

$$S = S_{min} + \delta S$$

$$P(xt, x_0 t_0) = const(t) \cdot e^{-\frac{v_0^2 (t - t_0)}{4D}} \sim C(t)\, e^{-\frac{(x - x_0)^2}{4D(t - t_0)}}$$

$$C(t) = \int_{\delta x(t_0)=0}^{\delta x(t)=0} \mathcal{D}\delta x\; e^{-\frac{1}{4D}\int_{t_0}^{t} d\tau\, \delta\dot{x}^2} = \frac{1}{\sqrt{4\pi D(t - t_0)}}$$

Trajectories
(stochastic quantization)

Diffusion : $\langle [x(t+\Delta t) - x(t)]^2 \rangle = \mathcal{D}\Delta t$

Equal-time correlations

$$a \to x(t)\, x(t+\Delta t) \qquad a-b \to x(t)[x(t+\Delta t) - x(t-\Delta t)]$$

$$b \to x(t)\, x(t-\Delta t) \qquad \underset{\Delta t \to 0}{\leadsto} \sqrt{\Delta t} \to 0$$

Limit is well defined when left-limit is equal to right-limit

$$A \Rightarrow \dot{x}(t) \cdot x(t+\Delta t)$$

$$B \to \dot{x}(t) \cdot x(t-\Delta t)$$

$$\dot{x}(t) = \lim_{\Delta \to 0} \frac{x(t+\Delta t) - x(t-\Delta t)}{2\Delta t}$$

$$A - B = \frac{1}{2\Delta t}[x(t+\Delta t) - x(t-\Delta t)]^2 \sim \frac{2\mathcal{D}\Delta t}{2\Delta t} = \mathcal{D}$$

Limit for $\langle \dot{x}(t)\, x(t) \rangle$ is ill defined.

Back to the original model

$$\langle x(t)\, \dot{x}(t') \rangle = \int^{t} dt_1 \langle \xi(t_1)\, \xi(t') \rangle = 2\mathcal{D}\,\theta(t-t')$$

$$x(t)\, \dot{x}(t) \to \mathcal{D} \quad \text{- Ito rule}$$