Quantum Mechanics

Schrodinger eq-n $\quad i\partial_t|\psi\rangle = \hat{\mathcal{H}}|\psi\rangle$

State $|\psi\rangle$ lives in Hilbert space $\mathcal{H}_N$

1. Linear $\quad |\psi_1\rangle + |\psi_2\rangle \subset \mathcal{H}_N$
2. Inner (scalar) product

$$\langle\phi|\psi\rangle = \langle\bar{\psi}|\phi\rangle$$

$$\langle\psi|\psi\rangle \geq 0$$

$$\langle\phi|\lambda_1\psi_1 + \lambda_2\psi_2\rangle = \lambda_1\langle\phi|\psi_1\rangle + \lambda_2\langle\phi|\psi_2\rangle$$

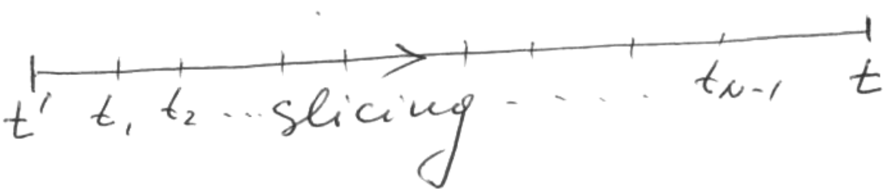
Evolution operator $\quad |\psi(t)\rangle = \hat{U}(t,t')|\psi(t')\rangle$

$$\begin{cases} U(t,t') = e^{-i\hat{\mathcal{H}}(t-t')}, & \hat{\mathcal{H}} \neq \hat{\mathcal{H}}(t) \\ U(t,t') = \hat{T} e^{-i\int_{t'}^{t} dt_1 \hat{\mathcal{H}}(t_1)}, & \hat{\mathcal{H}} = \hat{\mathcal{H}}(t) \end{cases}$$

$$\hat{T} e^{-i\int_{t'}^{t} dt_1 \hat{\mathcal{H}}(t_1)} \equiv \lim_{\substack{N\to\infty \\ \Delta_i \to 0}} e^{-i\mathcal{H}(t_N)\Delta_N}\, e^{-i\mathcal{H}(t_{N-1})\Delta_{N-1}} \cdots e^{-i\mathcal{H}(t_1)}$$

$$\sum_{i=1}^{N} \Delta_i = \Delta$$

$$i\partial_t \hat{U} = \hat{\mathcal{H}}\hat{U}$$



Example: <u>1-particle QM</u>

From abstract Hilbert space $\mathcal{H}_1$ to presentations (projections) onto

$$\hat{x}|x\rangle = x|x\rangle$$
$$\hat{p}|p\rangle = p|p\rangle$$
$$\vdots$$

For any basis (orthonormal)

$$\langle x|x'\rangle = \delta(x-x')$$
$$\langle p|p'\rangle = \delta(p-p')$$

Transformation matrix

$$\langle x|p\rangle = e^{ipx}$$
$$\langle p|x\rangle = e^{-ipx}$$

Completness (resolution of unity)

$$\int dx\, |x\rangle\langle x| = \int \frac{dp}{2\pi} |p\rangle\langle p| = \ldots = 1$$ (sum of all projectors)

$$\psi(x) \equiv \langle x|\psi\rangle$$
$$\psi(p) \equiv \langle p|\psi\rangle$$

$$\psi(x) = \int \frac{dp}{2\pi} \langle x|p\rangle\langle p|\psi\rangle = \int \frac{dp}{2\pi} e^{ipx}\psi(p)$$

(Fourier transform)

Sch. eq-n

$$i\partial_t \langle x|\psi\rangle = \int dx' \langle x|\hat{\mathcal{H}}|x'\rangle\langle x'|\psi\rangle$$

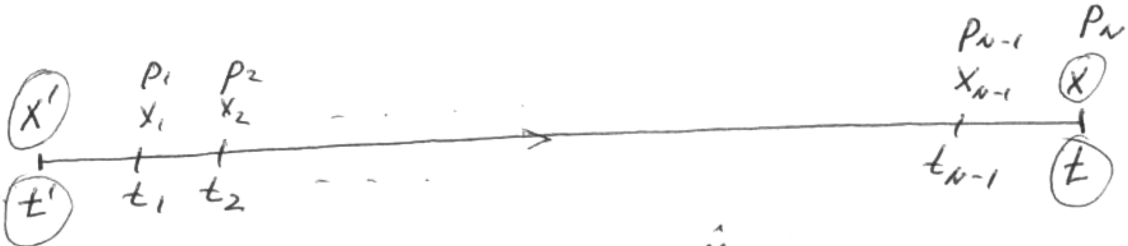
$$\hat{\mathcal{H}} = \varepsilon(\hat{p}) + V(\hat{x})$$

$$i\partial_t \psi(x) = \left[\varepsilon(-i\hbar\partial_x) + V(x)\right]\psi(x)$$

Evolution operator (path int.)

$$\psi(xt) = \int dx' \, U_{tt'}(xx') \, \psi(x't')$$

$$U_{tt'}(xx') = \langle x | e^{-i\hat{\mathcal{H}}(t-t')} | x' \rangle$$



$$U = \int_{x(t')=x'}^{x(t)=x} \prod_{i=1}^{N-1} dx_i \, \langle x_{i+1} | e^{-i\hat{\mathcal{H}}\Delta_i} | x_i \rangle$$

$$\hat{\mathcal{H}} = \mathcal{E}(\hat{p}) + V(\hat{x})$$

$$\langle x_{i+1} | e^{-i\hat{\mathcal{H}}\Delta_i} | x_i \rangle \approx \int \frac{dp_{i+1}}{2\pi} \, e^{i p_{i+1}(x_{i+1}-x_i) - i\mathcal{H}(p_{i+1}, x_i)\Delta}$$

$$\boxed{U_{tt'}(xx') = \int_{x(t')=x'}^{x(t)=x} \mathcal{D}x \, \mathcal{D}p \; e^{iS}}$$

$$\mathcal{D}x \equiv \prod_{i=1}^{N-1} dx_i, \quad \mathcal{D}p = \prod_{i=1}^{N} \frac{dp_i}{2\pi}$$

$$S = \sum p_{i+1}(x_{i+1}-x_i) - \mathcal{H}(p_{i+1}, x_i) \to \int_{t'}^{t} dt \left[p\dot{x} - \mathcal{H}(p,x)\right]$$

Restoring $\hbar$: $\quad e^{\frac{i}{\hbar}S}$

$\hbar \to 0$: extremum (minimum action or least action principle) $\delta L = 0$

$$\begin{cases} \dot{x} = \dfrac{\partial \mathcal{H}}{\partial p} \\ \dot{p} = -\dfrac{\partial \mathcal{H}}{\partial x} \end{cases} \quad \text{classical eq-ns}$$

If $S_{cl} \gg \hbar \Rightarrow$ quasi-classics (stationary phase method)

Evolution operator for $\mathcal{E}(p) = p^2/2m$

$$e^{-\frac{i}{\hbar}\int dt \frac{p^2}{2m}} \xrightarrow[\text{Gaussian}]{} \langle p(t) p(t') \rangle = -im\hbar\,\delta(t-t')$$

$$\left\langle e^{\frac{i}{\hbar}\int p\dot{x}} \right\rangle_{\text{"}p\text{"}} = e^{-\frac{1}{2\hbar^2}\langle [\int p\dot{x}]^2 \rangle_p} = e^{\frac{i}{\hbar}\int dt \frac{m\dot{x}^2}{2}}$$

$$U_{tt'}(x x') = \int_{x(t')=x'}^{x(t)=x} \mathcal{D}x \; e^{\frac{i}{\hbar}\int_{t'}^{t} dt_1 \frac{m\dot{x}^2(t_1)}{2}}$$

Diffusion in imaginary time

Same problems with equal-time corr.

Back to Lagrangian $L = p\dot{x} - \mathcal{H}(p, x)$

It corresponds to

$$\hat{\mathcal{H}} = \mathcal{H}(\hat{p}, \hat{x}) = \mathcal{E}(\hat{p}) + V(\hat{x})$$

with quantization rule

$$[\hat{p}, \hat{q}] = -i$$

And this equal-time correlation.

What does actually path integral gives for observables?

Keldysh contour I

Heizenberg picture: $|\psi_0\rangle$ does not evolve, operators do

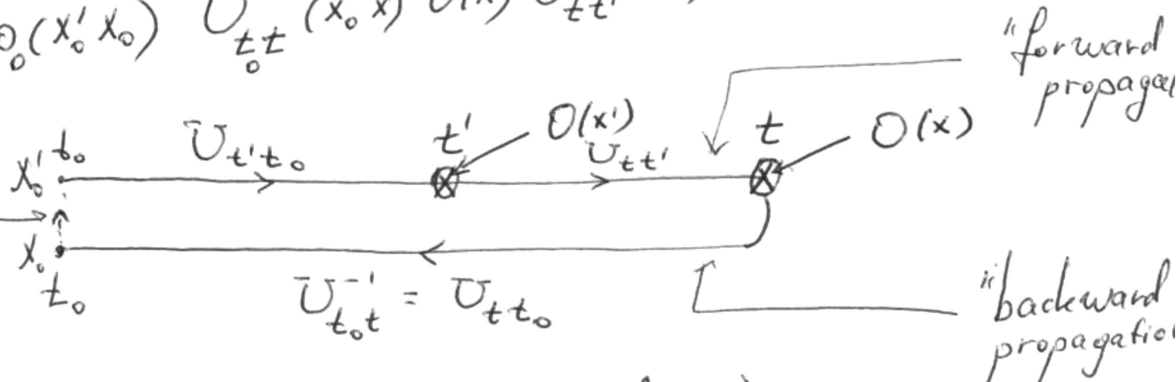
$$\hat{O}(t) = e^{i\hat{\mathcal{H}}t}\,\hat{O}\,e^{-i\hat{\mathcal{H}}t}, \quad \hat{O} = \hat{O}(\hat{x},\hat{p})$$

Observables (example) $\langle \psi_0 | \hat{O}(\hat{x}t)\,\hat{O}(\hat{x}'t') | \psi_0 \rangle$

(or multi-point corr.)

Explicitely

$$\int d^4x\; \rho_0(x_0' x_0)\; U^{-1}_{t_0 t}(x_0 x)\, O(x)\, U_{tt'}(x x')\, O(x')\, U_{t' t_0}(x' x_0')$$



$$\rho_0(x_0', x_0) = \langle x_0' | \psi_0 \rangle \langle \psi_0 | x_0 \rangle = \langle x_0' | \hat{\rho} | x_0 \rangle$$

For statistical description density matrix

$$\hat{\rho}_0 = \sum_n |n\rangle \rho_n \langle n|, \quad |n\rangle \text{ - all eigenstates}$$

At zero temperature $T=0$ we should keep only ground state $|\psi_0\rangle = |n=0\rangle$ and "backward" branch is redundant

$$\langle \psi_0 | e^{i\hat{\mathcal{H}}(t-t_0)} \ldots | \psi_0 \rangle = e^{iE_0(t-t_0)} \langle \psi_0 | \ldots | \psi_0 \rangle$$

Keldysh contour II

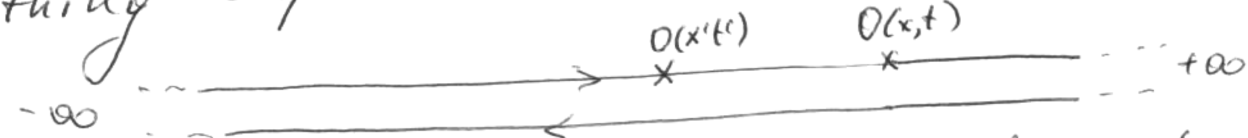
$$S = S_{forward}(x_0', t_0 \to x', t')$$
$$+ S_{forward}(x', t' \to x, t)$$
$$+ S_{backward}(x, t \to x_0, t_0)$$

$$S(x_i, t_i \to x_j, t_j) = \int_{t_i}^{t_j} d\tau\, L \quad \text{on trajectories} \quad \begin{matrix} x(\tau = t_i) = x_i \\ x(\tau = t_j) = x_j \end{matrix}$$

Right-hand-side edge of contour can be always moved to $+\infty$:

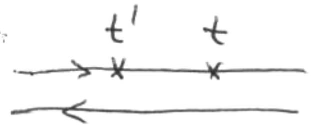
$$\mathrm{Tr}\, \hat{\rho}\, \hat{O}(x t)\, \underbrace{e^{i\hat{H}(t-T)} \cdot e^{-i\hat{H}(t-T)}}_{=1,\ \text{take } T \to +\infty}\, \hat{O}(x' t')$$

Left-hand-side edge at t_0 can be moved to $-\infty$ IF there is a reason that nothing depends on initial state $\hat{\rho}(t_0)$

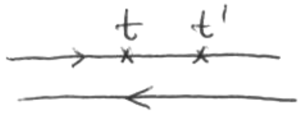


Observables are always located at upper (forward propagating) branch of contour.

Keldysh III



$\Rightarrow \operatorname{Tr} \hat{\rho}\, \hat{O}(t)\, \hat{O}(t')$



$\Rightarrow \operatorname{Tr} \hat{\rho}\, \hat{O}(t')\, \hat{O}(t)$

For quantum mechanisc $\hat{\rho} = |\psi_0\rangle\langle\psi_0|$

using $\phi^+ \hat{A} \psi = \sum A_{ij} \psi_j \bar{\phi}_i = tr\, \hat{\rho} \hat{A}$, $\rho_{ij} = \psi_j \phi_i$

$$\rightarrow \int \mathcal{D}x\, \mathcal{D}p\; O(xt)\, O(x't')\, e^{iS} = \langle \hat{T}_K\, \hat{O}(xt)\, \hat{O}(x't') \rangle$$

Path integral gives time-ordered correlator.

$$\hat{T}\, \hat{a}(t)\, \hat{b}(t') = \begin{cases} \hat{a}(t)\, \hat{b}(t'), & t > t' \\ \hat{b}(t')\, \hat{a}(t), & t < t' \end{cases}$$

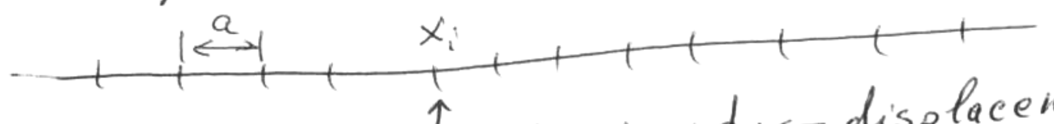
$\hat{T}_K$ - orders times on contour (Keldys

Actually any closed loop in complex time plane would do

Many-particle problem in 1st quantization

QFT – quantum mechanics with ∞ number of degrees of freedom.

Example: elastic string



$x_i = x_{i0} + \phi_i \leftarrow$ displacement

$$V_{int}(\hat{x}_{i+1} - \hat{x}_i) = \frac{m\omega^2}{2}(\hat{x}_{i+1} - \hat{x}_i - a)^2 = \frac{m\omega^2}{2}(\hat{\phi}_{i+1} - \hat{\phi}_i)^2$$

$$\begin{cases} \hat{H} = \sum \frac{\hat{p}_i^2}{2m} + \frac{m\omega^2}{2}(\hat{\phi}_{i+1} - \hat{\phi}_i)^2 \\ [\hat{p}_i, \hat{\phi}_j] = -i\delta_{ij} \end{cases}$$

Low-energy Hamiltonian $\phi_{i+1} \approx \phi_i$

Continuous limit $\hat{H} = \int \frac{dx}{a}\left[\frac{\hat{p}^2(x)}{2m} + \frac{m\omega^2 a^2}{2}(\partial_x \hat{\phi})^2\right]$

Density $\rho = \frac{m}{a}$ $\hat{H} = \frac{1}{2}\int dx \left[\frac{1}{a^2}\hat{p}^2(x) + \rho\omega^2 a^2 (\partial_x \hat{\phi})^2\right]$

Quantization $\left[\frac{1}{a}\hat{p}_i, \hat{\phi}_j\right] = -i\frac{1}{a}\delta_{ij}$

$\hat{\Pi}(x) = \frac{1}{a}\hat{p}(x)$ $\left[\hat{\Pi}(x), \hat{\phi}(x')\right] = -i\delta(x-x')$

$\omega^2 a^2 = c^2$

Elastic string (phonons)

$$\hat{\mathcal{H}} = \frac{1}{2}\int dx \left[\frac{1}{\rho}\hat{\Pi}^2 + \rho c^2 (\partial_x \hat{\Phi})^2\right]$$

$$[\hat{\Pi}(x), \hat{\Phi}(x')] = -i\hbar\,\delta(x-x')$$

$$\hat{\Pi}(x) = \int\frac{dq}{2\pi}\,\hat{\Pi}_q e^{iqx}, \quad \hat{\Phi}(x) = \int\frac{dq}{2\pi}\,\hat{\Phi}_q e^{iqx}$$

$$\mathcal{H} = \frac{1}{2}\int\frac{dq}{2\pi}\left[\frac{1}{\rho}\hat{\Pi}_{-q}\hat{\Pi}_q + \rho\omega_q^2\hat{\Phi}_{-q}\hat{\Phi}_q\right], \quad \omega_q^2 = c^2q^2$$

$$[\hat{\Pi}_q, \hat{\Phi}_{q'}] = -2\pi i\hbar\,\delta(q+q')$$

Real filds $\begin{cases}\hat{\Phi}^+(x) = \hat{\Phi}(x)\\ \hat{\Pi}^+(x) = \hat{\Pi}(x)\end{cases} \Rightarrow \begin{cases}\hat{\Phi}_q^+ = \hat{\Phi}_{-q}\\ \hat{\Pi}_q^+ = \hat{\Pi}_{-q}\end{cases}$

Transformation

$$\begin{cases}\hat{\Phi}_q = \alpha_q[\hat{b}_q^+ + \hat{b}_{-q}^+] \\ \hat{\Pi}_q = -i\beta_q[\hat{b}_q - \hat{b}_{-q}^+]\end{cases} \qquad \alpha_q = \sqrt{\frac{\hbar}{2\rho\omega_q}} \qquad \beta_q = \sqrt{\frac{\hbar\rho\omega_q}{2}}$$

brings Hamiltonian to the form

$$\mathcal{H} = \int\frac{dq}{2\pi}\,\frac{\hbar\omega_q}{2}\left(\hat{b}_q^+\hat{b}_q + \hat{b}_q\hat{b}_q^+\right) = \int\frac{dq}{2\pi}\,\hbar\omega_q\,\hat{b}_q^+\hat{b}_q + \text{zero point energy}$$

where we used $[\hat{b}_q, \hat{b}_{q'}^+] = 2\pi\delta(q-q')$

Bosons = phonons

Elastic string II

$$\hat{\mathcal{H}} = \frac{1}{2}\int dx \left[\frac{1}{\rho}\hat{\Pi}^2(x) + \rho c^2 (\partial_x \hat{\phi})^2\right]$$

$$[\hat{\Pi}(x), \hat{\phi}(x')] = -i\,\delta(x-x')$$

Classically Hamilton eq-ns

$$\dot{\phi}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \frac{p_i}{m} \Rightarrow \partial_t \phi = \frac{1}{\rho}\Pi(x) \left[= \frac{\delta \mathcal{H}}{\delta \Pi(x)}\right]$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial \phi_i} = m\omega^2(\phi_{i+1}+\phi_{i-1}-2\phi_i) \Rightarrow \partial_t \Pi = \rho c^2 \partial_x^2 \phi \left[= -\frac{\delta \mathcal{H}}{\delta \phi}\right]$$

$$(\partial_t^2 - c^2\partial_x^2)\phi = 0$$

Lagrangian

$$L = \Pi \partial_t \phi - \mathcal{H}(\Pi, \phi)$$
$$= \frac{\rho}{2}\left[(\partial_t \phi)^2 - c^2 (\partial_x \phi)^2\right]$$

Lagrange eq-ns

$$\partial_\mu \frac{\delta L}{\delta \partial_\mu \phi} = \frac{\delta L}{\delta \phi}$$

$$\partial_t \underbrace{\frac{\delta L}{\delta \partial_t \phi}}_{=\rho \partial_t \phi} + \partial_x \underbrace{\frac{\delta L}{\delta \partial_x \phi}}_{=-\rho c^2 \partial_x \phi} = 0$$

$$(\partial_t^2 - c^2 \partial_x^2)\,\phi(x,t) = 0$$

1st quantization many-particle basis

$$\begin{cases} \hat{q}\,|q\rangle = q\,|q\rangle \\ \hat{p}\,|p\rangle = p\,|p\rangle \end{cases} \Rightarrow \begin{cases} \hat{q}_i\,|q_1 \ldots q_i \ldots q_N\rangle = q_i\,|q_1 \ldots q_i \ldots q_N\rangle \\ \hat{p}_i\,|p_1 \ldots p_i \ldots p_N\rangle = p_i\,|p_1 \ldots p_i \ldots p_N\rangle \end{cases}$$

Generalization for ∞ number of degrees of freedom

$$\begin{cases} \hat{\phi}(x)\,|\phi\rangle = \phi(x)\,|\phi\rangle \\ \hat{\Pi}(x)\,|\Pi\rangle = \Pi(x)\,|\Pi\rangle \end{cases}$$

Generalizations for continuous limit

$$|\phi\rangle = |\phi_1 \dots \phi_N\rangle = |\phi(x_1) \dots \phi(x_N)\rangle$$

Scalar product $\langle \phi | \phi' \rangle = \prod_{i=1}^{N} \delta(\phi_i - \phi_i') \xrightarrow[N\to\infty]{} \delta[\phi - \phi']$

δ- functional

$$\langle \phi | \Pi \rangle = e^{i\int dx\, \Pi(x)\phi(x)} \equiv e^{i\Pi\phi}$$

Closure $\int \mathcal{D}\phi\, |\phi\rangle\langle\phi| = \int \mathcal{D}\Pi\, |\Pi\rangle\langle\Pi| = 1$

"Matrix" element of evolution operator

$$\langle \phi | e^{-i\hat{\mathcal{H}}(t-t')} | \phi' \rangle = \langle \phi | \hat{U}_{tt'} | \phi' \rangle$$

The same time-slicing as above for QM

$$\langle \phi | \hat{U}_{tt'} | \phi' \rangle = \int_{\phi_0 = \phi'}^{\phi_N = \phi} \mathcal{D}\phi\, \mathcal{D}\Pi \cdot \prod \langle \phi_{n+1} | \Pi_{n+1} \rangle \langle \Pi_{n+1} | e^{-i\hat{\mathcal{H}}\Delta t} | \phi_n \rangle$$

$$= \int \mathcal{D}\phi\, \mathcal{D}\Pi\; e^{iS} \quad ; \quad \begin{array}{l} \phi(t') = \phi' \\ \phi(t) = \phi \end{array}$$

$$S = \sum \Pi_{n+1}(\phi_{n+1} - \phi_n) - \mathcal{H}(\Pi_{n+1}, \phi_n)\Delta t$$

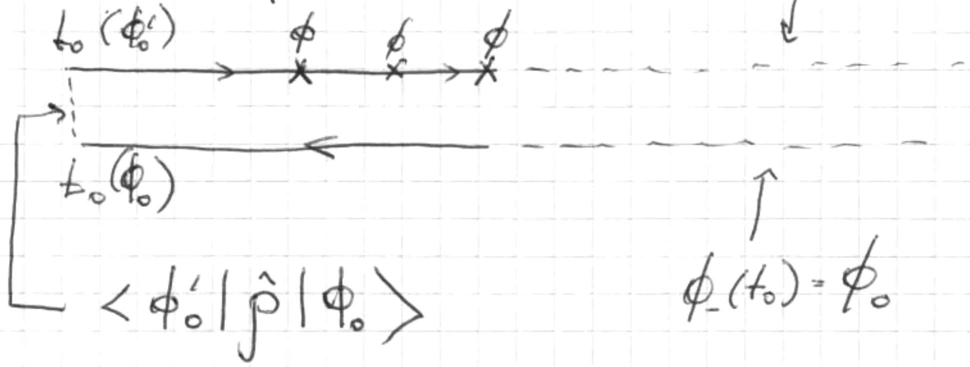
$$\to \int dx\, dt \left[\Pi(xt)\partial_t \phi(xt) - \mathcal{H}(\Pi(xt), \phi(xt)) \right]$$

String $\mathcal{H} \to \frac{1}{2}\left[\Pi^2(xt) + (\partial_x \phi(xt))^2\right]$, $\rho = 1$

Like it was for QM (quantum mechanics or 1-particle case)

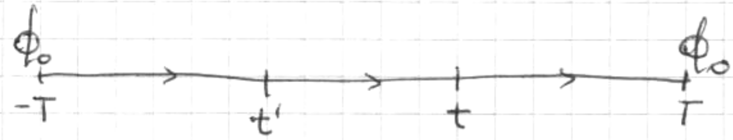
$$\langle T_K \prod_{i=1}^{n} \hat{\phi}(x_i, t_i) \rangle = \int \mathcal{D}\phi \prod_{i=1}^{n} \phi(x_i, t_i)\, e^{iS} \rho(\phi_0', \phi_0)$$

$$S = \int_K dt\, L(t) \qquad\qquad \phi_+(t_0) = \phi_0'$$



If we calculate average between ground state (which corresponds to $T=0$ zero-temp limit) we can be satisfied with a single branch

$$\langle \phi_0 | \underbrace{e^{i\hat{\mathcal{H}}t}}_{\Downarrow\; e^{-i\hat{\mathcal{H}}(T-t)}} \hat{\phi}(x)\, e^{-i\hat{\mathcal{H}}(t-t')}\, \hat{\phi}(x') \underbrace{e^{-i\hat{\mathcal{H}}t'}}_{\Downarrow\; e^{-i\hat{\mathcal{H}}(t'-T)}} | \phi_0 \rangle \quad (t>t', \text{example})$$



since $\hat{\mathcal{H}} | \phi_0 \rangle = E_0 | \phi_0 \rangle$ and $T \to \infty$

$$\langle T\, \hat{\phi}(xt)\, \hat{\phi}(x't') \rangle = \int_{\phi(t=-\infty)=\phi_0}^{\phi(t=+\infty)=\phi_0} \mathcal{D}\phi\; \phi(xt)\, \phi(x't')\, e^{i\int_{-\infty}^{\infty} dt\, L(t)}$$

Assuming that corr. function (Green function) does not depend on "initial" moment $t = \pm\infty$

$$i\, G^c(xt, x't') = \int\limits_{\phi(t=-\infty)=0}^{\phi(t=+\infty)=0} \mathcal{D}\phi\; \phi(xt)\,\phi(x't')\, e^{\,i\int_{-\infty}^{\infty} dt\, L(t)}$$

G^c = casual Green function (time-ordered along single branch)

$$S = \frac{1}{2}\int dx\,dt\left[(\partial_t\phi)^2 - (\partial_x\phi)^2\right] = -\frac{1}{2}\int dx\,dt\;\phi\left(\partial_t^2 - \partial_x^2\right)\phi$$

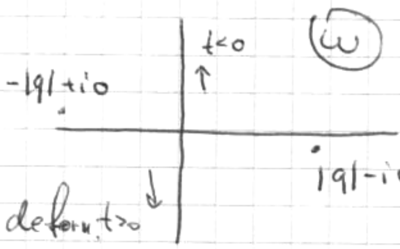
$$i\,G^c = \int \mathcal{D}\phi\;\phi(xt)\,\phi(x't')\, e^{+\frac{i}{2}\int \phi\, \hat{G}^{-1}\phi} = \int \mathcal{D}\phi\;\phi(xt)\,\phi(x't')\, e^{-\frac{i}{2}\phi\hat{\Box}\phi}$$

$$\hat{G}^{-1} = -\left(\partial_t^2 - \partial_x^2\right) = -\hat{\Box}$$

$$-\hat{\Box}\,\hat{G} = \hat{1} \;\Rightarrow\; \left(\partial_t^2 - \partial_x^2\right) G^c(x-x', t-t') = -\delta(x-x')\,\delta(t-t')$$

$$G^c(x,t) = \int \frac{d\omega\, dq}{(2\pi)^2}\, \frac{e^{iqx - i\omega t}}{\omega^2 - q^2 + i0}$$

$G^c(x, t\to\infty) \to 0$



We did not mention statistics. Particles were distinguishable: $\hat{x}_i = x_{i0} + \hat{\phi}_i$

$x_{i0} = i \cdot a$

Statistics (if any) is in $|\phi_0\rangle$ or $\hat{\rho}_0$.
But excitations above ground state turned out to be bosons (phonons)

$T \neq 0$ / Arbitrary initial state

$$G^c_q(t) = -i\langle\phi|\,\hat{T}\,\hat{\phi}_{-q}(t)\,\hat{\phi}_q\,|\phi\rangle = \theta(t)\,G^>_q(t) + \theta(-t)\,G^<_q(t)$$

$$\hat{\phi}_q(t) = \alpha_q\left[\hat{b}_q(t) + \hat{b}^+_{-q}(t)\right] = \alpha_q\left[\hat{b}_q e^{-i\omega_q t} + \hat{b}^+_{-q} e^{i\omega_q t}\right]$$

$$\omega_q = |q|$$

$$G^c(q,\omega) = \frac{N_q+1}{\omega^2-\omega_q^2+i0} - \frac{N_q}{\omega^2-\omega_q^2-i0} = \frac{1}{\omega^2-\omega_q^2+i0} - 2\pi i\, N_q\, \delta(\omega^2-\omega_q^2)$$

$$N_q \equiv \langle\phi|\hat{b}^+_q \hat{b}_q|\phi\rangle$$

Inversion of $\omega^2 - \omega_q^2$ or $\partial_t^2 - \partial_x^2$ should be done with care of behaviour of functions at $t = \pm\infty$ at both branches.

Operator is well defined only when defined its action on a function, plus the class of functions allowed.