2nd Quantization

Hilbert space for distinguishable particles has basis $|\lambda_1 \dots \lambda_N\rangle = \underset{\text{1st particle}}{|\lambda_1\rangle} \otimes |\lambda_2\rangle \dots \otimes \underset{\text{N-th particle}}{|\lambda_N\rangle}$

Indistinguishability (basis)

Bosons $|\lambda_1 \dots \lambda_N\rangle = \sqrt{\frac{\prod n_i!}{N!}} \sum_P |\lambda_{P_1}\rangle \otimes \dots \otimes |\lambda_{P_n}\rangle$

Fermions $|\lambda_1 \dots \lambda_N\rangle = \frac{1}{\sqrt{N!}} \sum_P \mathrm{sgn} P \cdot |\lambda_{P_1}\rangle \otimes \dots \otimes |\lambda_{P_N}\rangle$

$P \equiv$ permutations of $\{\lambda_1 \dots \lambda_N\}$

1-particle operator $\hat{H}^{(1)} = \sum_{i=1}^{N} \hat{h}_i$. Its matrix elements

Bosons $\langle \dots \lambda_\mu \dots | \hat{H}^{(1)} | \dots \lambda_\nu \dots \rangle = h_{\mu\nu} \sqrt{(n_\mu + 1) n_\nu}$, $h_{\mu\nu} = \langle \mu | \hat{h} | \nu \rangle$

occupation number representation

Removing redundancy $\langle \dots n_\mu + 1 \dots n_\nu - 1 \dots | \hat{H}^{(1)} | \dots n_\mu \dots n_\nu \dots \rangle = h_{\mu\nu} \sqrt{(n_\mu + 1) n_\nu}$

The same can be achieved by

$$\hat{H}^{(1)} = \sum h_{\mu\nu} \hat{b}^+_\mu \hat{b}_\nu$$

with $\langle n_\mu + 1, n_\nu - 1 | \hat{b}^+_\mu \hat{b}_\nu | n_\mu, n_\nu \rangle = \sqrt{(n_\mu + 1) n_\nu}$

or
$$\begin{cases} \langle n_\mu + 1, n_\nu - 1 | \hat{b}^+_\mu | n_\mu, n_\nu - 1 \rangle = \sqrt{n_\mu + 1} \\ \langle n_\mu, n_\nu - 1 | \hat{b}_\nu | n_\mu, n_\nu \rangle = \sqrt{n_\nu} \end{cases}$$

which means $[\hat{b}_\mu, \hat{b}_\nu] = [\hat{b}^+_\mu, \hat{b}^+_\nu] = 0$, $[\hat{b}_\mu, \hat{b}^+_\nu] = \delta_{\mu\nu}$

2nd Quantization

Calculating matrix element of $\hat{H}^{(1)}$ between two anti-symmetric wave-functions (determinants)

$$\langle \ldots n_\mu+1 \ldots n_\nu-1 \ldots | \hat{H}^{(1)} | \ldots n_\mu \ldots n_\nu \ldots \rangle = h_{\mu\nu} \sqrt{(n_\mu+1)n_\nu} \cdot (-1)^{\Sigma_{\mu\nu}}$$

where $\Sigma_{\mu\nu} = \sum_{\mu<\lambda<\nu} n_\lambda$

Can again be achieved with substitution

$$\hat{H}^{(1)} = \sum h_{\mu\nu} \hat{f}_\mu^+ \hat{f}_\nu$$

but $\{f_\mu, f_\nu\} = \{f_\mu^+, f_\nu^+\} = 0, \quad \{f_\mu, f_\nu^+\} = \delta_{\mu\nu}$

Field operators:

<u>Bosons</u> $\hat{\phi}(x) = \sum_\mu \phi_\mu(x) \hat{b}_\mu$

$$[\hat{\phi}(x), \hat{\phi}(x')] = [\hat{\phi}^+(x), \hat{\phi}^+(x')] = 0$$

$$[\hat{\phi}(x), \hat{\phi}(x')] = \delta(x-x')$$

<u>Fermions</u> $\hat{\psi}(x) = \sum_\mu \psi_\mu(x) \cdot \hat{f}_\mu$

$$\{\hat{\psi}(x), \hat{\psi}(x')\} = \{\hat{\psi}^+(x), \hat{\psi}^+(x')\} = 0$$

$$\{\hat{\psi}(x), \hat{\psi}^+(x')\} = \delta(x-x')$$

$$\hat{H}_B^{(1)} = \int dx \; \hat{\phi}^+(x) \hat{h}(x) \hat{\phi}(x) \qquad \hat{h}(x) = -\frac{\hbar^2}{2m} \partial_x^2 + V(x)$$

$$\hat{H}_F^{(1)} = \int dx \; \hat{\psi}^+(x) \hat{h}(x) \hat{\psi}(x)$$

2nd Quantization

Interaction in 1st Quantization $\hat{H}^{(2)} = \frac{1}{2} \sum_{ij} V(\hat{x}_i - \hat{x}_j)$

In 2nd Quantization:

$$\hat{H}^{(2)} = \hat{\mathcal{H}}_{int} = \frac{1}{2} \int dx\, dx'\, \hat{\psi}^+(x) \hat{\psi}^+(x') V(x-x') \hat{\psi}(x') \hat{\psi}(x)$$

for bosons $\hat{\psi}(x) \to \hat{\phi}(x)$, $[\hat{\phi}(x), \hat{\phi}^+(x')] = \delta(x-x')$

fermions $\{\hat{\psi}(x), \hat{\psi}^+(x')\} = \delta(x-x')$

$$\hat{\mathcal{H}} = \mathcal{H}(\hat{\psi}^+, \hat{\psi}) \text{ or } \mathcal{H}(\hat{\phi}^+, \hat{\phi}) \text{ in 2nd Quant.}$$

Example: single state bosons $\Rightarrow \hat{\mathcal{H}} = \mathcal{H}(\hat{b}^+, \hat{b})$

with $[\hat{b}, \hat{b}^+] = 1$

Like in 1st quantized version with $\hat{\mathcal{H}} = \mathcal{H}(\hat{p}, \hat{x})$, where we used basis $|p\rangle$ and $|x\rangle$, we need bases "$|b\rangle$"

$$\hat{b}|b\rangle = b|b\rangle \text{ - coherent state}$$

Occupation number representation

states $= |n\rangle$ (n = # of bosons) $\qquad |n\rangle = \frac{1}{\sqrt{n!}} (\hat{b}^+)^n |0\rangle$

$$\langle n | m \rangle = \delta_{nm}$$

Expanding coherent state over "n-states"

$$|b\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

Coherent states (bosons)

$$\hat{b}|b\rangle = b|b\rangle \qquad \begin{cases} \hat{b}|n\rangle = \sqrt{n}\,|n-1\rangle \\ \hat{b}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle \end{cases}$$

$$\hat{b}\sum_{n=0}^{\infty} C_n |n\rangle = b\sum_{n=0}^{\infty} C_n |n\rangle$$

$$\sum_{n=1}^{\infty} C_n \sqrt{n}\,|n-1\rangle = b\sum_{n=0}^{\infty} C_n |n\rangle$$

$$C_{n+1}\sqrt{n+1} = b C_n \quad \Rightarrow \quad C_n = \frac{b^n}{\sqrt{n!}}$$

Coherent state $\quad |b\rangle = \sum_{n=0}^{\infty} \frac{b^n \hat{b}^{+n}}{n!}|0\rangle = e^{b\hat{b}^{+}}|0\rangle$

$$|b\rangle = \sum_{n=0}^{\infty} \frac{b^n}{\sqrt{n!}}|n\rangle$$

Scalar product of two different coherent states:

$$\langle b|b'\rangle = \sum_{n=0}^{\infty} \frac{(\bar{b}b')^n}{n!}\langle n|n\rangle = e^{\bar{b}b'}$$

Closere $\quad \int \frac{d\bar{b}\,db}{\pi} e^{-|b|^2} |b\rangle\langle b| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{1}$

integral over complex plane

sum of countable set

Coherent states - "over-complete" bas

we used $\quad \int \frac{d\bar{b}\,db}{\pi} e^{-|b|^2} |b|^{2n} = n!$

Coherent state path integral
(functional integral)

Slicing again $\quad t' \; t_1 \; \ldots \; t_{N-1} \; t$

$$\langle b | e^{-i\hat{\mathcal{H}}(t-t')} | b' \rangle = \lim_{N\to\infty} \int \frac{d^N \bar{b}\, d^N b}{\pi^{N-1}} \prod_{i=0}^{N-1} \langle b_{i+1} | e^{-i\hat{\mathcal{H}}\Delta} | b_i \rangle \cdot E$$

$$E = e^{-\sum_{i=1}^{N-1} |b_i|^2}$$

$$\langle b_{i+1} | e^{-i\mathcal{H}(\hat{b}^+,\hat{b})\Delta} | b_i \rangle \approx e^{-i\mathcal{H}(\bar{b}_{i+1}, b_i)\Delta} \langle b_{i+1} | b_i \rangle$$

if Hamiltonian is normally-ordered
(i.e. all $\hat{b}^+$'s are at the left wrt all $\hat{b}$'s)

Matrix element of evolution operator:

$$\langle b | e^{-i\hat{\mathcal{H}}(t-t')} | b' \rangle = \int_{b(t')=b'}^{b(t)=b} \mathcal{D}\bar{b}\, \mathcal{D}b \; e^{iS}$$

$$S = i \sum_{i=0}^{N-1} \bar{b}_{i+1}(b_{i+1} - b_i) - \mathcal{H}(\bar{b}_{i+1}, b_i) \; ; \qquad b_0 = b'$$
$$b_N = b$$

$$S \to \int_{t'}^{t'} [\bar{b}\, i\partial_t b - \mathcal{H}(\bar{b}, b)]$$

$$b(t') = b', \quad b(t) = b$$

Lagrangian $\quad L = \bar{b}\, i\partial_t b - \mathcal{H}(\bar{b}, b)$

"momentum" $\quad$ "coordinate"

b and $\bar{b}$ - conjugate variables

Least action principle $\Rightarrow \quad i\partial_t b = \dfrac{\delta \mathcal{H}}{\delta \bar{b}(t)}$ - classical eq-n

Coherent states path integral

$$\mathrm{Tr}\,\hat{A} = \sum_{n=0}^{\infty} \langle n|\hat{A}|n\rangle = \int \frac{d\bar{b}\,db}{\pi} e^{-|b|^2} \langle b|\hat{A}|b\rangle$$

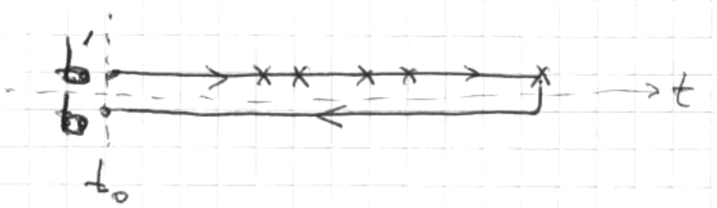
$$\langle \phi_0|\hat{A}|\phi_0\rangle = \mathrm{Tr}\,\hat{\rho}\hat{A} \quad , \quad \hat{\rho} = |\phi_0\rangle\langle\phi_0| \to \sum_{\mu} |\phi_\mu\rangle \rho_\mu \langle\phi_\mu|$$

$$\mathrm{Tr}\,\hat{\rho}\hat{A} = \int \frac{d\bar{b}\,db}{\pi} \frac{d\bar{b}'\,db'}{\pi} e^{-|b|^2-|b'|^2} \langle b'|\hat{\rho}|b\rangle \langle b|\hat{A}|b'\rangle$$

For $\hat{A} = T \prod_i \hat{b}(t_i)\hat{b}^+(t_i')$, $\quad \hat{b}(t) = e^{i\hat{\mathcal{H}}(t-t_0)}\hat{b}\,e^{-i\hat{\mathcal{H}}(t-t_0)}$

$$\langle b|\hat{A}|b'\rangle = \int \mathcal{D}\bar{b}\,\mathcal{D}b \;\; A[b(t_i), \bar{b}(t_i')]\, e^{iS}$$

$$S = \int_K dt \left[\bar{b}\, i\partial_t b - \mathcal{H}(\bar{b}, b)\right] = S_{forward} + S_{backward}$$

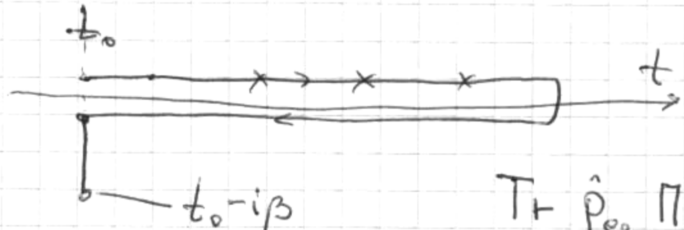


$$\begin{cases} b(t_0+i0) = b' \\ b(t_0-i0) = b \end{cases}$$

or

$$\begin{cases} \text{upper contour: } b_+(t_0) = b' \\ \text{lower contour: } b_-(t_0) = b \end{cases}$$

Equilibrium (Gibbs) : $\hat{\rho} = e^{-\beta\mathcal{H}}$ (Wick rotation $\tau = it$)



$$\mathrm{Tr}\,\hat{\rho}_{eq} \prod \hat{b}(t_i)\hat{b}^+(t_i') = \oint \mathcal{D}\bar{b}\,\mathcal{D}b \prod_i b\bar{b}'\, e^{iS}$$

$$S = S_f + S_b + S_\beta$$

$$b(t+i0) = b(t_0 - i\beta)$$

Coherent states (fermions)

Again example: single state fermion

$$\hat{\mathcal{H}} = \mathcal{H}(f^+, f) \qquad \{\hat{f}, \hat{f}^+\} = 1$$

Coherent state is defined as

$$\hat{f}\,|f\rangle = f\,|f\rangle \qquad |f\rangle = |0\rangle + \eta\,|1\rangle$$

Grassmann variables

$$|f\rangle = |0\rangle - f|1\rangle = (1 - f\cdot\hat{f}^+)|0\rangle = e^{-f\hat{f}^+}|0\rangle$$

Scalar product $\langle f|f'\rangle = e^{\bar{f}f'}$ (like bosons)

Closure $\int d\bar{f}\,df\, e^{-\bar{f}f}|f\rangle\langle f| = 1$ (like bosons)

<u>if</u> $\int df = \int d\bar{f} = 0 \qquad \int df\cdot f = \int d\bar{f}\cdot\bar{f} = 1$

Matrix element of evolution operator

$$\langle f|e^{-i\hat{\mathcal{H}}(t-t')}|f'\rangle = \int_{f_0=f'}^{f_N=f} d^N(\bar{f},f)\prod_{i=0}^{N-1}\langle f_{i+1}|f_i\rangle e^{-i\mathcal{H}(\bar{f}_{i+1},f_i)-\bar{f}_if_i}$$

$$\to \int_{f(t')=f'}^{f(t)=f}\mathcal{D}\bar{f}\,\mathcal{D}f\; e^{iS[f]}$$

$\mathrm{Tr}\,\hat{A} = \int d\bar{f}\,df\, e^{-\bar{f}f}\langle -f|\hat{A}|f\rangle \Rightarrow$ All rules are the same except one $\boxed{f_{in} = -f_{final}}$

Equlibrium: $\mathrm{Tr}\,\hat{\rho}_{eq}\hat{A} = \int_{f(t+i0)=f_0}^{f(t-i\beta)=f}\mathcal{D}\bar{f}\,\mathcal{D}f\, e^{iS}$

$\hat{A} = T\Pi\, f(t_i)f^+(t_i')$

Many-particle functional integrals

$$\hat{\Phi}(x) = \sum \phi_n(x)\, \hat{b}_n\,, \qquad [\hat{b}_n, \hat{b}_m^+] = \delta_{nm}$$

$$[\hat{\Phi}(x), \hat{\Phi}^+(x')] = \delta(x-x')$$

Coherent state $\quad \hat{\Phi}(x)|\phi\rangle = \phi(x)|\phi\rangle$

$$|\phi\rangle = e^{\phi\hat{\Phi}^+}|0\rangle \equiv e^{\int dx\, \phi(x)\hat{\Phi}^+(x)}|0\rangle$$

Repeating everything (just extra index "x" for creation/annihilation operators.

Equilibrium $\quad \langle \ldots \rangle = \mathrm{Tr}\, \hat{\rho}_{eq} \ldots$

$$\langle T_K\, \hat{\Phi}(xt)\hat{\Phi}^+(x't')\rangle = \int_{\phi_{in} = \phi_{final}} \mathcal{D}\phi\; \phi(xt)\,\phi(x't')\, e^{iS[\phi]}$$

$$S = \int dt\, [\bar{\Phi}\, i\partial_t \phi - \mathcal{H}(\bar{\phi},\phi)]$$

$$\mathcal{H} = \int dx\, \bar{\Phi}\, \hat{h}^{(1)}\phi + \frac{1}{2}\int dx\, dx'\, \bar{\Phi}\bar{\Phi} V \phi\phi$$

Fermions :

$$\langle T_K\, \hat{\Psi}(xt)\hat{\Psi}^+(x't')\rangle = \int_{\psi_{in} = -\psi_{final}} \mathcal{D}\psi\; \psi(xt)\,\psi(x't')\, e^{iS[\psi]}$$

ψ's – Grassmann fields; $\quad S = \int dt\, [\bar{\psi}\, i\partial_t \psi - \mathcal{H}(\bar{\psi},\psi)]$

$$\mathcal{H} = \int dx\, \bar{\psi}\,\hat{h}^{(1)}\psi + \frac{1}{2}\int dx\, dx'\, \bar{\psi}\bar{\psi} V \psi\psi$$

Coherent states : $\quad |\psi\rangle = e^{-\psi\hat{\Psi}^+}|0\rangle \equiv e^{-\int dx\, \psi(x)\hat{\Psi}^+(x)}|0\rangle$