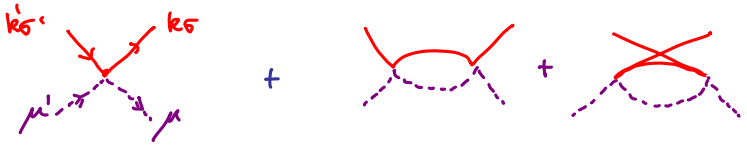
## Kondo effect in quantum dots: Anderson model

AM1

Main results of lecture 1:

Kondo Model: $H = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + J \sum_{kk'\sigma\sigma'} (c^\dagger_{k\sigma} \tfrac{1}{2}\vec{\sigma}_{\sigma\sigma'} c_{k'\sigma'}) \cdot \vec{S}$ (1)

Spin-flip scattering:




enhanced at low temp: $T \lesssim T_K = D \exp[-\frac{1}{\nu J}]$ $\Rightarrow$ ground state = spin singlet $\downarrow\uparrow$ (2)

Scattering phase shifts at T = 0: $\delta_\uparrow(0) = -\delta_\downarrow(0) = \pi/2$ (3)

How do magnetic moments form in metals?
Answer provided by "Anderson impurity model" (AM) [1961], relevant also to describe transport through quantum dots, which also show Kondo effect [1998].

## Single-impurity Anderson model

AM2

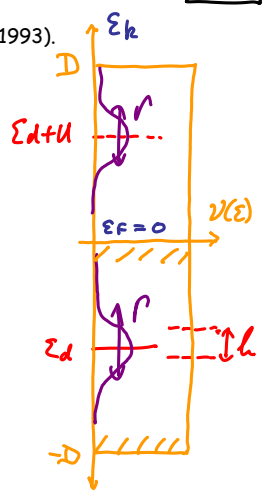
Anderson, Phys. Rev. 124, 41 (1961); Hewson, "The Kondo Problem to Heavy Fermions", Cambridge (1993).

Conduction band: (flat DOS, "wide-band limit": $D \gg$ all other scales )
$H_{band} = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma}$ (1)

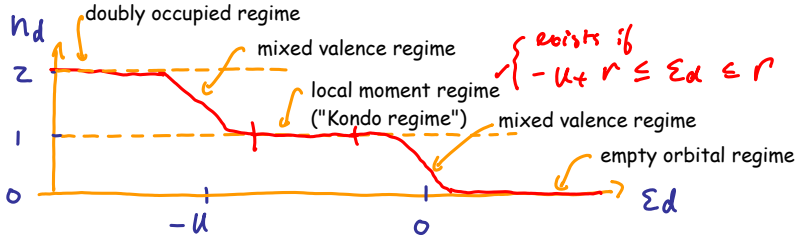
Localized impurity level:
$H_{loc} = \sum_\sigma (\varepsilon_d + \sigma \underbrace{h}_{=0}) \underbrace{d^\dagger_\sigma d_\sigma}_{\hat{n}_{d\sigma}} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow}$ (2)

Hybridization: [Usual convention: $v_k = v$ = real]
$H_{hyb} = \sum_{k\sigma} v_k [c^\dagger_{k\sigma} d_\sigma + h.c.]$ (3)

Level width: (from golden rule)
$\Gamma = \pi \nu v^2$ (4)

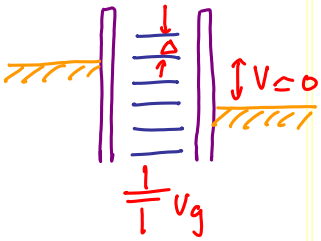


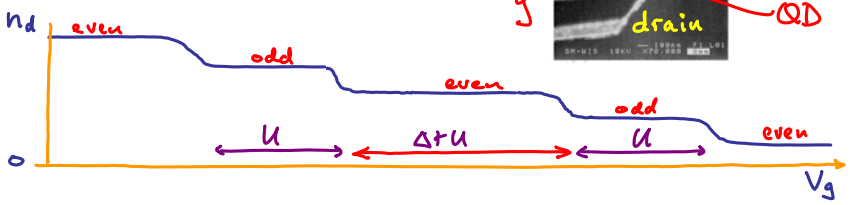

Level occupancy:
$n_d = \langle n_{d\uparrow} + n_{d\downarrow} \rangle$

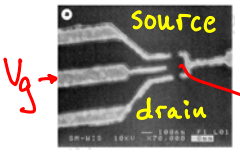


## Conductance anomalies for quantum dot in "Kondo regime"

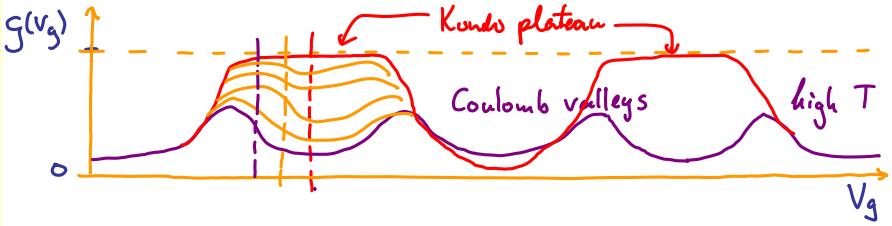
AM3



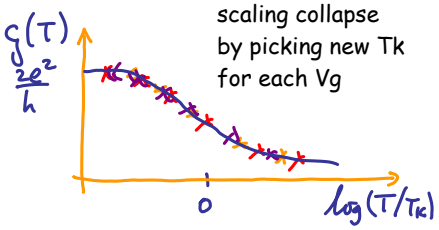



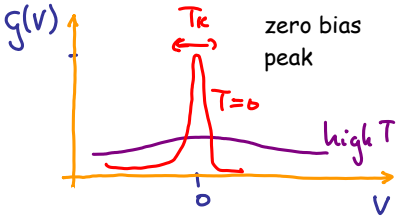




Linear conductance:
Odd Coulomb valleys become Kondo plateaus




Fixed gate voltage:
Anomalous T- and V-dependence

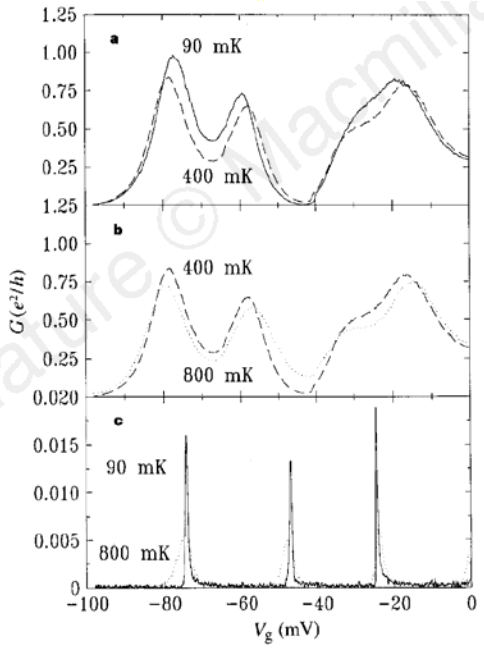



## Conductance anomalies: real data

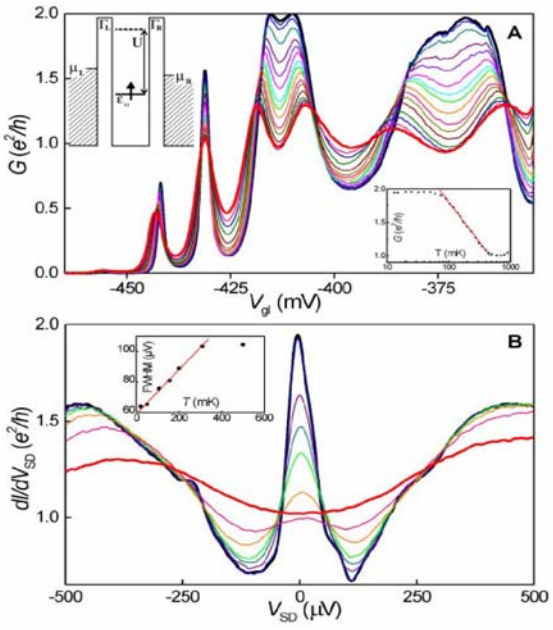
AM4

Weak Kondo effect




Goldhaber-Gordon et al., Nature **391**, 156

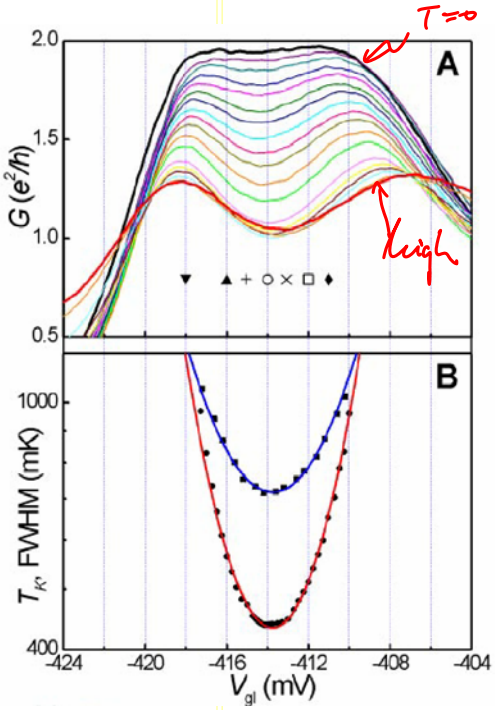
Strong Kondo effect

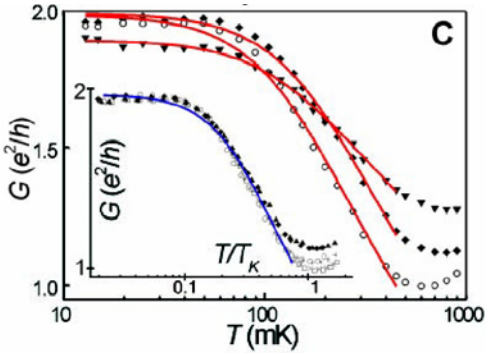



van der Wiel et al., Sience 289, 2105 (2000)

## Ideal Kondo effect

van der Wiel et al., Sience 289, 2105 (2000)




- Beautiful Kondo plateau observed
- T-dependence follows universal form when scaled by Tk
- This allows determination of Tk
- Observed dependence of Tk on εd agrees with theoretical prediction:

$$\log T_K = c_1\, \varepsilon_d(\varepsilon_d+U) + c_2$$

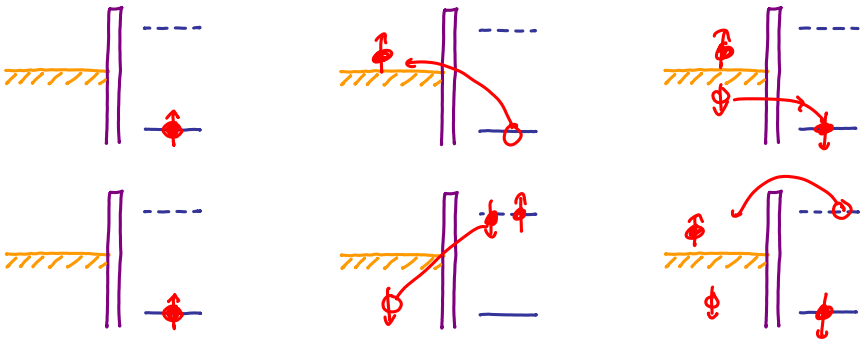


## When does "Kondo plateau" arise?

Occurs for $T \to 0 \quad (T \ll T_K)$

and when $n_d \simeq 1$ (odd) $\Rightarrow$ localized spin

Loc. spin + cond. band : Kondo model $\Rightarrow$ Spin-flip scatt. !!



$\Gamma = \pi v v^2$

Spin-flip processes occur via virtual intermediate states.

Effective spin-flip rate: $\nu \dfrac{v^2}{\varepsilon_d+U} - \dfrac{v^2}{\varepsilon_d} = -\dfrac{U\Gamma/\pi}{\varepsilon_d(\varepsilon_d+U)} =: J \quad (>0)$

(1)

## Schrieffer-Wolff transformation

Phys Rev 149, 491 (1966)

AM 7

$H_{AM} = \underbrace{H_{band} + H_{loc}}_{H_0} + \underbrace{H_{yb}}_{H_1 = O(v)}$

Idea: seek effective $\tilde{H}$ in subspace of $n_d = 1$

Try unitary transf.: $\tilde{H} = e^{A} H e^{-A}$, with $A^\dagger = -A$ (1)

A has pert. exp. in $v$ : $A = 0 + O(v) + O(v^2) + \dots$ (2)

Expand $\tilde{H}$ : $\tilde{H} \overset{(2)}{=} (H_0 + H_1) + \underbrace{[A, H_0]}_{:=0} + [A, H_1] + \frac{1}{2}[A, \underbrace{[A, H_0]}_{(4)\ -H_1}] + O(v^3)$ (3)

Demand: $\tilde{H}$ contains no $O(v)$ : $H_1 \overset{(3)}{=:} -[A, H_0]$, (4) $\Rightarrow \tilde{H} \overset{(3)}{=} H_0 + \frac{1}{2}[A, H_1] + O(v^3)$ (5)

check algebra yourself!

(5) is satisfied by: $A = \sum_{k\sigma} v \left[ \frac{1}{\varepsilon_k - \varepsilon_d} c^\dagger_{k\sigma} d_\sigma + \frac{U}{(\varepsilon_d - \varepsilon_k)(\varepsilon_d + U - \varepsilon_k)} d^\dagger_{-\sigma} d_{-\sigma} c^\dagger_{k\sigma} d_\sigma \right] - h.c.$ (6)

## Effective Hamiltonian for $n_d = 1$ yields Kondo model

AM 8

(7.5) yields: $\tilde{H}\big|_{n_d=1} = \sum_{k\sigma} \varepsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{kk'} \tilde{v}^{(2)}_{kk'} \vec{s}_{kk'} \cdot \vec{S} + (c^\dagger_k c_{k'}\text{-term})$ (1)

potential scattering

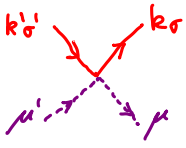
local spin operators: $S^z = \frac{1}{2}\left(d^\dagger_\uparrow d_\uparrow - d^\dagger_\downarrow d_\downarrow\right) := \frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $S^+ = d^\dagger_\uparrow d_\downarrow := \begin{pmatrix}0&1\\0&0\end{pmatrix}$, $S^- = d^\dagger_\downarrow d_\uparrow := \begin{pmatrix}0&0\\1&0\end{pmatrix}$ (2)

conduction band spin operators: $\vec{s}_{kk'} = \frac{1}{2} \sum_{\sigma\sigma'} c^\dagger_{k\sigma} \vec{\sigma}_{\sigma\sigma'} c_{k'\sigma'}$ (3)

coupling: $\tilde{v}^{(2)}_{kk'} = \frac{-\frac{1}{2} v_k v_{k'} U}{(\varepsilon_d - \varepsilon_k)(\varepsilon_d + U - \varepsilon_k)} + k \to k' \xrightarrow{|\varepsilon_k|, |\varepsilon_{k'}| \ll |\varepsilon_d|, |\varepsilon_d + U|} \simeq \frac{v_{k\sigma}^2 U}{|\varepsilon_d|\,|\varepsilon_d + U|} =: J$ (4)

Low-en. properties of AM for nd = 1 described by KM: $H_{Kondo} = H_{band} + J \sum_{kk',\mu\mu'} \vec{s}_{kk'} \cdot \vec{S}_{\mu\mu'}$ (6)

SU(2) symmetric!



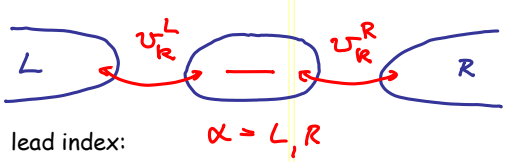

Eff. Kondo temp: (observed: see AM 5!) $T_K \overset{(K6.5)}{=} D e^{-\frac{1}{\nu J}} \overset{(4)}{=} D \exp\left[ -\frac{\pi |\varepsilon_d|\,|\varepsilon_d + U|}{\Gamma U} \right]$ (7)

agrees with Bethe Ansatz, except for prefactor

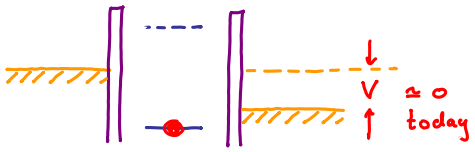
## Single-level quantum dot with two leads

Pustilnik, Glazman, PRL 87, 216601 (2001)

Recent review: "Nanophysics: Coherence and Transport," eds. H. Bouchiat et al., pp. 427-478 (Elsevier, 2005).




lead index: $\alpha = L, R$




Two-lead Hamiltonian:

$$H = \sum_{k\alpha\sigma} \varepsilon_k c^\dagger_{k\alpha\sigma} c_{k\alpha\sigma} + \sum_{k\alpha\sigma} v_k^\alpha c^\dagger_{k\alpha\sigma} d_\sigma + \text{h.c.} + H_{loc} \quad (1)$$

($H_{leads}$)

Schrieffer-Wolff as before, with

$$\sum_k v_k \longrightarrow \sum_{k\alpha} v_k^\alpha ( \dots )^\alpha \quad (2)$$

Effective Hamiltonian for nd = 1:

$$H_{\substack{\text{Kondo}\\ \text{scat.}}} = \sum_{\alpha\alpha'} \underbrace{\left( - \frac{v_{kF}^\alpha v_{kF}^{\alpha'} U}{\varepsilon_d(\varepsilon_d+U)} \right)}_{:= J_{\alpha\alpha'}} \underbrace{\left( \sum_{kk'\sigma\sigma'} c^\dagger_{k\alpha\sigma} \vec{\sigma}_{\sigma\sigma'} c_{k\alpha'\sigma'} \right)}_{\vec{\sigma}_{\alpha\alpha'}} \cdot \vec{S} = \underbrace{\sum_{\alpha\alpha'} J_{\alpha\alpha'} \vec{\sigma}_{\alpha\alpha'}}_{\text{Tr}(J\vec{\sigma})} \cdot \vec{S} \quad (3)$$

Coupling matrix:

$$J_{\alpha\alpha'} = \tilde{c} \begin{bmatrix} v_L^2 & v_L v_R \\ v_R v_L & v_R^2 \end{bmatrix}, \quad \text{with} \quad v_\alpha = v_{kF}^\alpha, \quad \tilde{c} = \frac{-U}{\varepsilon_d(\varepsilon_d+U)} \quad (4)$$

Determinant:

$$\det J_{\alpha\alpha'} = \tilde{c}^2 \left[ v_L^2 v_R^2 - (v_L v_R)^2 \right] = 0 \quad (5)$$

one eigenvalue = 0



## Diagonalization of coupling matrix J

J is diagonalized by:

$$W = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad \tan\theta = -v_R/v_L \quad (1)$$

diagonal form:

$$\tilde{J} = W J W^\dagger = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} = \begin{bmatrix} \tilde{c}(v_L^2 + v_R^2) & 0 \\ 0 & 0 \end{bmatrix} \quad (2)$$

Rotate basis:

$$\psi_{kf\sigma} = \sum_\alpha c_{k\alpha\sigma} W_{\alpha f} \quad (3)$$

$$\sum_{kk'\sigma\sigma'} (\psi^\dagger_{k1\sigma} \vec{\sigma}_{\sigma\sigma'} \psi_{k'1\sigma'})$$

$$\tilde{H}_{\substack{\text{Kondo}\\ \text{scat}}} \overset{(9.2)}{=} \text{Tr}\left[ \underbrace{W^\dagger J W}_{\tilde{J}} \underbrace{W^\dagger \vec{\sigma} W}_{\vec{\sigma}_1} \right] \cdot \vec{S} = J_1 \vec{\sigma}_1 \cdot \vec{S} \quad (4)$$

J-diagonal Kondo Hamiltonian:

$$H = \sum_{kf\sigma} \varepsilon_k \psi^\dagger_{kf\sigma} \psi_{kf\sigma} + J_1 \vec{\sigma}_1 \cdot \vec{S} \quad (5)$$

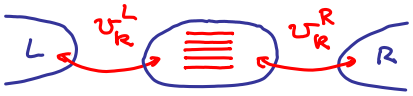
Important conclusion: One mode yields Kondo-Hamiltonian, other mode decouples completely!

[ Comment: for multilevel AM, coupling matrix is more complicated: $v_k^{\alpha j} c^\dagger_{k\alpha\sigma} d_{j\sigma}$

Then J2 can be nonzero, because $\det J_{\alpha\alpha'} \propto \sum_{jj'} \left( v_j^L v_{j'}^R - v_j^R v_{j'}^L \right)^2 \neq 0$ (6) ]

## Conductance through (many-level) QD with 2 leads

AM11




lead index: $\alpha = L, R$

Consider $T=0$, $\beta \neq 0$ but $\to \infty$
which ensures non-degenerate ground state

Then incident electrons experience only potential scattering, described by 2x2 S-matrix:

$$S_{\sigma,\sigma'}(0) = W^\dagger D_\sigma W, \qquad D_\sigma = \begin{bmatrix} e^{2i\delta_{1\sigma}} & 0 \\ 0 & e^{2i\delta_{2\sigma}} \end{bmatrix} \quad (1)$$

($W$: same as in (10.1))

Phase shifts: $\delta_{\gamma\sigma}$, with $\gamma = 1,2$, $\sigma = \uparrow, \downarrow$

Landauer formula for conductance:

$$g(T=0) = \frac{e^2}{h} \sum_\sigma |S^\gamma_{\sigma,RL}(0)|^2 \overset{(10.1)}{=} g_0 \tfrac{1}{2} \sum_\sigma \sin^2(\delta_{1\sigma} - \delta_{2\sigma}) \quad (2)$$

Prefactor:

$$g_0 = \frac{2e^2}{h} \sin^2 2\theta = \frac{2e^2}{h} \frac{4(v_L v_R)^2}{(v_L^2 + v_R^2)^2} = \frac{2e^2}{h} \text{ if } v_L = v_R \quad (3)$$

Important conclusion: T = 0 conductance is determined purely by phase shifts!

## Conductance through 1-level QD with 2 leads

AM12

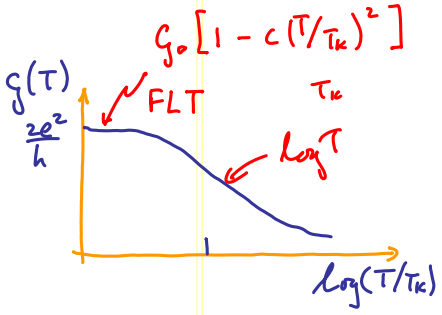
For 1-level AM: $J_z \overset{(10.2)}{=} 0$, $\to \delta_{2\sigma} = 0$ (1)

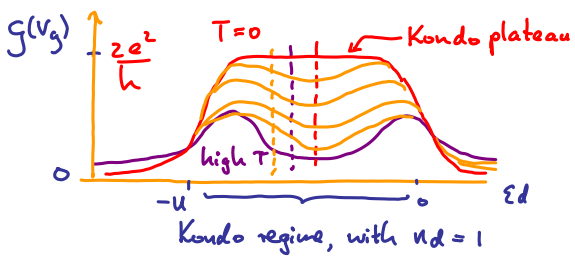
From lecture 1, (K9.4): at $T=0$, $\delta_{1\uparrow} = -\delta_{1\downarrow} = \pi/2$ (2)

Conductance at T = 0:

$$g(T=0) \overset{(11.2)}{=} g_0 \tfrac{1}{2} \sum_\sigma \sin^2(\delta_{1\sigma} - \delta_{2\sigma}) \overset{\pi/2 \;\; 0}{=} g_0 \quad (3)$$

for symmetric couplings ($v_L = v_R$) $= \frac{2e^2}{h}$ = "unitarity limit", maximal possible value, as though channel were completely open! (4)



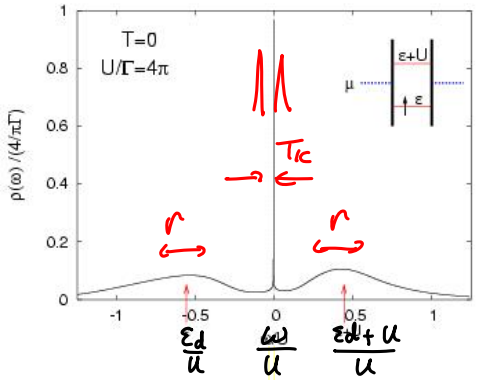
## Kondo-Abrikosov-Suhl resonance in local spectral function

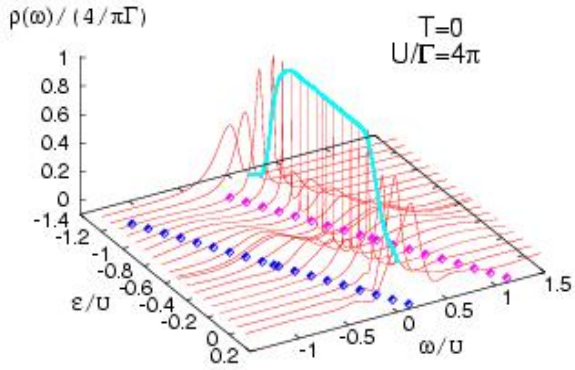
Local Green's function:

$$G^R_{d\sigma}(\omega) = \int_0^\infty dt\, e^{-i\omega t}(-i)\langle\{d_\sigma(t), d_\sigma(0)\}\rangle$$

Spectral function = local density of states (LDOS):

$$A_{d\sigma}(\omega) = -\frac{1}{\pi}\,\mathrm{Im}\,G^R_{d\sigma}(\omega) = \nu_{loc}(\omega) = \rho(\omega)$$




For T < Tk, LDOS develops Kondo resonance...

which is observed directly in V-dep. of G (see AM4)

Numerical Renormalization Group calculations by Michael Sindel, 2004