Introduction to String Theory
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Assignment # 10

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1) Charges of Kaluza-Klein and winding modes

As was shown in lecture, the compactification of the closed string on a circle of radius R leads to two 25D vector fields $G_{\mu,25}$ and $B_{\mu,25}$ that are massless for arbitrary values of the radius R. These two vector fields correspond to the zero modes of the $(\mu, 25)$-components $(\mu, \nu, \ldots = 0, 1, \ldots, 24)$ of the 26D metric G_{MN} and the 26D two-form field B_{MN} $(M, N, \ldots = 0, 1, \ldots, 25)$, respectively. They give rise to a gauge group $U(1)_L \times U(1)_R$ that remains unbroken for all R.

In addition, the closed string spectrum also contains four more vector fields that can also become massless, but only at the self-dual radius $R = \sqrt{2} = \sqrt{\alpha'}$. They correspond to the excitations

$$|V^\mu_\pm\rangle = \alpha^\mu_{-1}|M = \pm 1, L = \pm 1\rangle \tag{1}$$
$$|V'^\mu_\pm\rangle = \bar{\alpha}^\mu_{-1}|M = \pm 1, L = \mp 1\rangle. \tag{2}$$

In the lecture, it was claimed that these additional vector fields combine with $G_{\mu,25}$ and $B_{\mu,25}$ to fill out the full adjoint representation of $SU(2)_L \times SU(2)_R$, which should thus be considered as the full gauge group, which is unbroken at the self-dual radius, but Higgsed to $U(1)_L \times U(1)_R$ at generic values of R. For this to be possible, $V^\mu_\pm$ and $V'^\mu_\pm$ have to be charged with respect to the $U(1)$'s gauged by $G_{\mu,25}$ and $B_{\mu,25}$ (just as the $W^\pm$-bosons have to be charged in the Standard Model). In this excercise, we will uncover the physical origin of this charge.

To understand this origin, we have to first understand how the closed string couples to the metric G_{MN} and the two-form B_{MN}. So far, we have only studied the propagation of strings in flat Minkowski spacetime (corresponding to $G_{MN} = \eta_{MN}$) and without any background two-form field B_{MN} turned on. The action in this simplified case is just the Polyakov action,

$$S_P = -\frac{T}{2}\int d^2\sigma \sqrt{h} h^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \eta_{MN} \tag{3}$$
$$\stackrel{h_{\alpha\beta}=\eta_{\alpha\beta}}{=} -\frac{T}{2}\int d^2\sigma\big(-\partial_\tau X^M\partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N\big)\eta_{MN}. \tag{4}$$

In a more general, curved, background with metric $G_{MN}(X)$, this is simply generalized by replacing the constant Minkowski metric η_{MN} by the curved metric $G_{MN}(X)$:

$$S_P = -\frac{T}{2}\int d^2\sigma\big(-\partial_\tau X^M\partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N\big)G_{MN}(X(\sigma,\tau)). \tag{5}$$

The coupling of a string to a non-vanishing background B_{MN}-field, on the other hand, is described by adding the action

$$S_B = \frac{T}{2}\int d^2\sigma \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N B_{MN}(X(\sigma,\tau)) \tag{6}$$

$$= T \int d\sigma d\tau \partial_\tau X^M \partial_\sigma X^N B_{MN}(X(\sigma,\tau)), \quad (7)$$

where $\epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}$, ($\epsilon^{\tau\sigma} = 1$) is the 2D epsilon tensor. This action is simply the integral of the pull-back of the two-form B_{MN} to the two-dimensional world sheet and can be viewed as a higher-dimensional analogue of the electromagnetic coupling of a point particle of charge q in d dimensions with worldline $x^\mu(\tau)$:

$$S_{e.m.} = \int d^d x j^\mu A_\mu = q \int d\tau (\partial_\tau x^\mu) A_\mu(x(\tau)), \quad (8)$$

with $j^\mu(x) = q\dot{x}^\mu \delta^{(d)}(x^\mu - x^\mu(\tau))$ being the current density. The integrand here is likewise nothing but the pull-back of the one-form $A_\mu(x)$ to the worldline of the particle.

a) Consider the mode expansions ($\alpha' = 2$)

$$\begin{aligned} X^\mu(\sigma,\tau) &= x^\mu + 2p^\mu\tau \quad (+\text{ oscillators}) & (9)\\ &\equiv x^\mu(\tau) \quad (+\text{ oscillators}) & (10)\\ X^{25}(\sigma,\tau) &= x^{25} + 2\frac{M}{R}\tau + LR\sigma \quad (+\text{ oscillators}). & (11) \end{aligned}$$

Setting all oscillator terms in (10) and (11) equal to zero and considering only constant [1] $G_{MN}(x^\mu, x^{25}) = G_{MN}$, calculate the term in (5) that is proportional to $G_{\mu,25}$, and compare this with $S_{e.m.}$ in (8) to infer that the charge of a string with respect to the 25D vector field $G_{\mu,25}$ is proportional to its Kaluza-Klein momentum number M. Does the winding number L also enter the charge, and if yes, what is the proportionality?

b) Make a similar analysis for the action S_B in equation (7) and show that the charge of a string without oscillators with respect to the (constant or "σ-averaged" part of the) 25D vector field $B_{\mu,25}$ is proportional to the winding number L. What is the dependence upon the Kaluza-Klein momentum number M in this case?

Conclusion: Due to the non-trivial Kaluza-Klein momentum numbers M and winding numbers L, the states $|V_\pm^\mu\rangle$ and $|V_\pm'^\mu\rangle$ are charged with respect to (linear combinations of) $B_{\mu,25}$ and $G_{\mu,25}$, as they should in order to fit in the adjoints of $SU(2)$ groups.

2) T-duality

In a circle compactification for the coordinate X^{25}, the T-duality transformation acts on the coordinate field $X^{25}(\sigma,\tau) = X_L^{25}(\tau+\sigma) + X_R^{25}(\tau-\sigma)$ as

$$X^{25}(\sigma,\tau) \to \tilde{X}^{25}(\sigma,\tau) := X_L^{25}(\tau+\sigma) - X_R^{25}(\tau-\sigma) \quad (12)$$

a) Using the expansions

$$\begin{aligned} X_L^{25}(\tau+\sigma) &= \frac{1}{2}x^{25} + \left(\frac{M}{R} + \frac{1}{2}LR\right)(\tau+\sigma) + \text{ oscillators} & (13)\\ X_R^{25}(\tau-\sigma) &= \frac{1}{2}x^{25} + \left(\frac{M}{R} - \frac{1}{2}LR\right)(\tau-\sigma) + \text{ oscillators}, & (14) \end{aligned}$$

[1] For a non-constant metric, the relevant vector field is the "σ-averaged" quantity $\bar{G}_{\mu,25}(\tau) := \int d\sigma G_{\mu,25}(X(\tau,\sigma))$.

show that $X^{25}(\sigma,\tau) \to \tilde{X}^{25}(\sigma,\tau)$ indeed swaps the rôles of the Kaluza-Klein momentum number M and the winding number L.

b) Show that the above T-duality leaves the energy momentum tensor $T_{\pm\pm} = \frac{1}{2}\partial_\pm X \cdot \partial_\pm X$ invariant.

c) An open string can have either Neumann (N) or Dirichlet (D) boundary conditions at each endpoint $\sigma = \sigma^* = 0, \pi$:

$$\begin{aligned} \partial_\sigma X^\mu|_{\sigma^*} &= 0 \qquad (N) && (15)\\ \partial_\tau X^\mu|_{\sigma^*} &= 0 \qquad (D). && (16) \end{aligned}$$

Show that the T-duality transformation $X \to \tilde{X} = X_L - X_R$ interchanges the two types of boundary conditions.

3) Torus compactifications

Just as a circle can equivalently be described as $\mathbf{R}/2\pi R\mathbf{Z}$, a d-dimensional Torus, T^d, can be described as $\mathbf{R}^d/2\pi\Lambda^d$, where Λ^d denotes a d-dimensional lattice generated by integral linear combinations of d basis vectors $(i,j = 1,\ldots,d)$,

$$\vec{V}_i = \frac{1}{\sqrt{2}} R_i \vec{e}_i \qquad \text{(no sum)}. \tag{17}$$

Here, the vectors $\vec{e}_i$ are normalized as

$$\vec{e}_i \cdot \vec{e}_i = 2, \tag{18}$$

so that $\vec{V}_i$ has length R_i. Denoting the Cartesian components of the vectors $\vec{e}_i$ by e_i^I $(I,J = 1,\ldots,d)$, the torus with radii R_i is then given by the identification

$$X^I \sim X^I + 2\pi \sum_{i=1}^{d} V_i^I n_i \equiv X^I + 2\pi L^I, \qquad (n_i \in \mathbf{Z}) \tag{19}$$

where

$$L^I := \sum_{i=1}^{d} V_i^I n_i = \frac{1}{\sqrt{2}} \sum_{i=1}^{d} n_i R_i e_i^I \tag{20}$$

are the Cartesian coordinates of the possible lattice vectors. A set of basis vectors $\vec{e}_i^*$ is said to be dual to the basis $\vec{e}_i$ if

$$\vec{e}_i \cdot \vec{e}_j^* \equiv \sum_{I=1}^{d} e_i^I e_j^{*I} = \delta_{ij}, \tag{21}$$

which also implies

$$\sum_{i=1}^{d} e_i^I e_i^{*J} = \delta^{IJ}. \tag{22}$$

The Euclidean metric δ_{IJ} on $\mathbf{R}^d$ can be expressed in terms of the bases $\vec{V}_i$ or $\vec{V}_i^* = \frac{\sqrt{2}}{R_i}\vec{e}_i^*$, in terms of which it reads

$$\begin{aligned} g_{ij} &= \vec{V}_i \cdot \vec{V}_j = \frac{1}{2} \sum_{I=1}^{d} R_i e_i^I R_j e_j^I && (23)\\ g_{ij}^* &= \vec{V}_i^* \cdot \vec{V}_j^* = 2 \sum_{I=1}^{d} \frac{1}{R_i} e_i^{*I} \frac{1}{R_j} e_j^{*I}. && (24) \end{aligned}$$

g_{ij} and g^*_{ij} play the rôle of the metric on, respectively, the lattice Λ^d and the dual lattice $(\Lambda^d)^*$ generated by $\vec{V}_i^*$.

a) Show that g^*_{ij} is actually the inverse of g_{ij}.

b) Choosing $\vec{e}_1 = (\sqrt{2}, 0)$ and $\vec{e}_2 = (1, 1)$ in the case of a 2-torus, find the dual basis $\vec{e}_i^*$ (either graphically or algebraically).

c) Show that the single-valuedness of a wave function of the form $e^{i\sum_{I=1}^d X^I p^I}$ requires

$$\sum_{I=1}^{d} L^I p^I \in \mathbf{Z}, \tag{25}$$

and hence,

$$p^I = \sum_{i=1}^{d} m_i V_i^{*I}, \qquad (m_i \in \mathbf{Z}). \tag{26}$$

The momentum vectors p^I are thus constrained to lie on the dual lattice $(\Lambda_d)^*$.

d) The mass formula in the absence of internal B_{MN} fields is given by

$$m^2 = N_L + N_R - 2 + \frac{1}{2}((\vec{p}_L)^2 + (\vec{p}_R)^2), \tag{27}$$

where

$$\vec{p}_{L,R} = \vec{p} \pm \frac{1}{2}\vec{L}. \tag{28}$$

Show that this is equal to

$$m^2 = N_L + N_R - 2 + \sum_{i,j=1}^{d} \Big(m_i g^*_{ij} m_j + \frac{1}{4} n_i g_{ij} n_j \Big). \tag{29}$$

e) Switching now on a non-trivial internal B_{MN}-field background, $B_{IJ} \neq 0$, and using a flat spacetime metric, $G_{MN} = \eta_{MN}$, the action of a string (cf. eqs. (5), (7)) becomes $(T = 1/4\pi)$

$$S = S_P + S_B = -\frac{1}{8\pi} \int d^2\sigma (-\partial_\tau X^M \partial_\tau X^N + \partial_\sigma X^M \partial_\sigma X^N)\eta_{MN} \tag{30}$$

$$+\frac{1}{4\pi} \int d^2\sigma \partial_\tau X^I \partial_\sigma X^J B_{IJ}. \tag{31}$$

Use

$$X^I(\sigma, \tau) = x^I + 2p^I \tau + L^I \sigma + \text{ oscillators}, \tag{32}$$

to show that the internal canonical momenta

$$\Pi^I = \frac{\delta S}{\delta(\partial_\tau X^I)} \tag{33}$$

are given by

$$\Pi^I = \frac{1}{2\pi}(p^I + \frac{1}{2} B_{IJ} L^J) + \text{ oscillators} \tag{34}$$

This implies that the internal canonical center of mass momenta π^I are now given by

$$\pi^I = p^I + \frac{1}{2} B_{IJ} L^J \tag{35}$$

instead of just p^I. Hence, we now have to require single-valuedness of $e^{i\sum_I \pi^I X^I}$ instead of $e^{i\sum_I p^I X^I}$, so that π^I and not p^I is now quantized:

$$\pi^I = \sum_{i=1}^{d} m_i V_i^{*I}. \tag{36}$$

f) While the canonical momentum has changed, it is still the mechanical momentum p^I that enters $p^I_{L,R}$, just as in (28), and the mass formula is still of the form (27). Reexpressing p^I in terms of π^I, show that

$$\begin{aligned} p^I_{L,R} &= \pi^I \pm \frac{1}{2}(\delta^{IJ} \mp B_{IJ})L^J & (37)\\ &= (m_i - \frac{1}{2}n_j B_{ij})V_i^{*I} \pm \frac{1}{2}n_i V_i^I, & (38) \end{aligned}$$

where sums over repeated indices are understood and $B_{ij} \equiv \sum_{I,J} V_i^I V_j^J B_{IJ}$.

Remark: Inserting (38) into (27), one finds that the mass again depends on g_{ij} (and its dual/inverse g^*_{ij}), but also on B_{ij}. Hence, there are $d(d+1)/2 + d(d-1)/2 = d^2$ continuous parameters g_{ij}, B_{ij} that label the different physically inequivalent configurations. In the low energy effective field theory, these parameters ("moduli") arise as the vev's of d^2 lower-dimensional scalar fields, which are simply the zero modes of the internal metric and 2-form field components. The scalar potential of these scalar fields is (classically) flat, and so their vev's are not dynamically fixed. Finding mechanisms that fix the moduli of string compactifications is an important problem in present day string theory research, and much progress has been achieved in recent years in this area.

4) D-brane mode expansion

Let x^μ ($\mu = 0, \ldots, p$) denote the spacetime coordinates tangential to a Dp-brane and use x^m ($m = p+1, \ldots, 25$) for the transverse directions.

a) What kind of boundary conditions (Neumann or Dirichlet) does one have to impose on $X^\mu(\sigma,\tau)$ and on $X^m(\sigma,\tau)$ if the string begins and ends on the Dp-brane?

b) What kind of boundary conditions ((N) or (D)) does one have to impose on $X^\mu(\sigma,\tau)$ and on $X^m(\sigma,\tau)$ if the string begins on the Dp-brane but has the other end moving freely in spacetime?

c) For a string that begins and ends on a Dp-brane at positions $\bar{x}^a$, we have, for the transverse coordinates $X^a(\sigma,\tau)$,

$$X^a(\tau, 0) = X^a(\tau, \pi) = \bar{x}^a. \tag{39}$$

Furthermore, the fact that X^a solves the 2D wave equation implies that

$$X^a(\sigma,\tau) = \frac{1}{2}(f^a(\tau+\sigma) + g^a(\tau-\sigma)) \tag{40}$$

for some as yet arbitrary functions f^a and g^a. Evaluate (40) at $\sigma = 0$ to show that

$$X^a(\sigma,\tau) = \bar{x}^a + \frac{1}{2}(f^a(\tau+\sigma) - f^a(\tau-\sigma)). \tag{41}$$

d) Use the boundary condition at $\sigma = \pi$ to derive

$$f^a(\tau + \pi) = f^a(\tau - \pi). \tag{42}$$

e) The result of part d) means that f^a is a periodic function of its argument with period 2π. Show that this forbids a linear term in τ in $X^a(\tau, \sigma)$. What is the physical significance of the absence of a linear term in τ in the mode expansion of $X^a(\tau, \sigma)$?

f) Does one have a linear term in τ in $X^\mu(\sigma, \tau)$? Putting everything together, what is the physical consequence of the observations in parts e) and f) for the open string states with both ends on a Dp-brane?