Introduction to String Theory
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Assignment # 2
(Due May 15, 2006)

Note:
Problems 1) through 4) of this assignment illustrate some aspects of Hamiltonian systems with constraints. As this is rarely covered in depth in many classical mechanics courses, a brief summary of this formalism together with a simple example is provided in an appendix. The background material reviewed in the appendix should be sufficient to solve the Problems, and it is highly recommended that you first carefully read the appendix before you start with Problems 1) - 4). Problem 5), finally, illustrates in what sense the Nambu-Goto action measures the area of the world sheet.

The Problems:

1) Consider the Lagrangian

$$L = \frac{1}{2}\dot{q}_3^2 - q_1 q_2$$

a) Calculate the canonical momenta p^1, p^2, p^3 and identify the two primary constraints of this system. At this stage, are they first class or second class?
b) Using the example worked out in the Appendix as a guideline, form the canonical and the total Hamiltonian and identify the two secondary constraints of this system. Taking now into account all four constraints, which of them are second class and which of them are first class?

2) Consider the Lagrangian of a relativistic point particle,

$$L_1 = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu}$$

a) Calculate the canonical momenta p_μ and show that

$$\phi = p^\mu p_\mu + m^2 = 0$$

arises as a primary constraint. Is it first or second class?
b) Compute the canonical and the total Hamiltonian. Show that the canonical Hamiltonian vanishes identically and, therefore, that the "time"-evolution w.r.t. τ is only generated by the constraint ϕ.
c) Are there secondary constraints?

3) Consider now the alternative Lagrangian

$$L_2 = \frac{1}{2}\left(e^{-1}\dot{x}^\mu \dot{x}_\mu - em^2\right)$$

a) Treating $e(\tau)$ as a dynamical variable just as $x^\mu(\tau)$, compute the corresponding canonical momenta $p_e \equiv \frac{\partial L_2}{\partial \dot{e}}$ and $p_\mu \equiv \frac{\partial L_2}{\partial \dot{X}^\mu}$. Show that there is one (very simple) primary constraint (it is *not* $(p^\mu p_\mu + m^2) = 0$!). Is it first class?
b) Compute the canonical and the total Hamiltonian and verify that the constancy of the primary constraint found in a) requires $(p^\mu p_\mu + m^2) = 0$ as a secondary constraint. Is this secondary constraint first or second class?
Conclusion:
Problems 2) and 3) show that the classification scheme primary vs. secondary constraints can give different results depending on which particular Lagrangian is used as the starting point. It is therefore not really an intrinsic classification scheme at the Hamiltonian level. A more intrinsic classification is according to the first and second class property, which turns out to be the same in problem 2) and 3) (as you should have found).

4) Consider the Nambu-Goto action of a relativistic string:

$$S_{NG} = -T\int d\tau d\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})}.$$

a) Calculate the canonical momenta

$$\Pi_\mu(\tau,\sigma) := \frac{\partial\mathcal{L}}{\partial\dot{X}^\mu}$$

b) Verify that $\phi_1 = \Pi^\mu X'_\mu = 0$ and $\phi_2 = \Pi^\mu\Pi_\mu + T^2 X^{\mu\prime}X'_\mu = 0$ arise as primary constraints.
c) Using the equal time Poisson brackets

$$\begin{aligned}
\{X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\} &= \eta^{\mu\nu}\delta(\sigma-\sigma') \\
\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} &= 0 \\
\{\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\} &= 0,
\end{aligned}$$

show that the constraints ϕ_1 and ϕ_2 commute at equal time (but with possibly different σ) with themselves and with each other, i.e. that they are first class. (Hint: One has to use the constraints themselves to argue that the result vanishes, and some care has to be given to the different arguments σ and σ' and the derivatives of the delta distribution.)
d) Verify that the canonical Hamiltonian, $H_{\text{can}} = \int d\sigma(\Pi^\mu\dot{X}_\mu - \mathcal{L})$, derived from the Nambu-Goto action vanishes identically. The "time"-evolution along τ is therefore entirely given by the two primary first class constraints.
Remark: A certain linear combination of these constraints also generates reparameterizations in the σ direction, i.e., the presence of the two first class constraints is ultimately a consequence of the reparameterization invariance w.r.t. τ and σ.

5) A two-sphere of fixed radius ρ in three-dimensional Euclidean space, $\mathbb{R}^3$, can be considered a Euclidean analogue of an (admittedly somewhat peculiar) string world sheet. Using $(\theta,\phi) \in [0,\pi]\times[0,2\pi]$ as the analogue of the world sheet coordinates (τ,σ), the standard spherical coordinates yield the embedding functions

$$\begin{aligned}
X^1(\theta,\phi) &= \rho\sin\theta\cos\phi \\
X^2(\theta,\phi) &= \rho\sin\theta\sin\phi \\
X^3(\theta,\phi) &= \rho\cos\theta,
\end{aligned}$$

which are the analogues of $X^\mu(\tau,\sigma)$ for the usual string.

a) Calculate the matrix

$$M = \begin{pmatrix} \frac{\partial \vec{X}}{\partial \theta} \cdot \frac{\partial \vec{X}}{\partial \theta} & \frac{\partial \vec{X}}{\partial \theta} \cdot \frac{\partial \vec{X}}{\partial \phi} \\ \frac{\partial \vec{X}}{\partial \phi} \cdot \frac{\partial \vec{X}}{\partial \theta} & \frac{\partial \vec{X}}{\partial \phi} \cdot \frac{\partial \vec{X}}{\partial \phi} \end{pmatrix} \tag{1}$$

b) Calculate the area of the two-sphere using the Euclidean analogue of the Nambu-Goto action:

$$A = \int_0^\pi \int_0^{2\pi} d\theta d\phi \sqrt{\det(M)}.$$

Appendix: Hamiltonian mechanics of constrained systems

In a theory with gauge (or "reparametrization") invariance, not all dynamical configurations are physically inequivalent. Rather, there may be gauge transformations that map some solutions to others without changing their physical content. The true phase space of physically distinct configurations is then only a subspace of the naive phase space spanned by the positions and momenta (q_i, p^i). This subspace can be described as the vanishing loci of certain phase space constraints $\phi_\alpha(q_i, p^i) = 0$ $(\alpha = 1, \ldots, n)$.

Constraints are conventionally classified according to the following two schemes, *which have nothing to do with each other*:

(I) Primary vs. secondary (or tertiary, etc.) constraints

This clasification scheme only has a meaning when the Hamiltonian system is obtained from a Lagrangian formulation via the standard Legendre transform: $L(q_i, \dot{q}_i) \longrightarrow H(q_i, p^i)$.

Primary Constraints:

Primary constraints are relations $\phi_A(q_i, p^i) = 0$ $(A = 1, \ldots, n_p)$ between the dynamical variables (usually the momenta) that follow from the mere *definition* of the momenta, $p^i \equiv \partial L / \partial \dot{q}_i$. Such relations occur when the matrix $\frac{\partial p^i}{\partial \dot{q}_j} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ is not invertible so that the map $\dot{q}_i \longrightarrow p^j$ cannot be inverted, and the p^i are not all independent. *No use of the equations of motion* is required to derive the primary constraints.

Secondary constraints:

Secondary constraints are additional constraints $\phi_M(q_i, p^i) = 0$ $(M =$

$1, \ldots, n_s$) that may follow from the requirement that the primary constraints be constant under time evolution:

$$0 \stackrel{!}{=} \dot{\phi}_A = \{\phi_A, H\} \Rightarrow \phi_M(q_i, p^i) = 0.$$

The analogous requirement of time-independence of the *secondary* constraints may itself impose *further* constraints, which, if they exist, are then called "tertiary", etc. The total number of constraints is then $n = n_p + n_s + n_t + \ldots$. Note that the derivation of secondary and higher constraints requires the *use of the equations of motion* via the Poisson bracket with the Hamiltonian.

It should be emphasized that the distinction between primary and secondary constraints is not always a very meaningful concept, as a different Lagrangian formulation for one and the same Hamiltonian system may exchange the rôle of primary and secondary constraints (see Problems 2) and 3)).

(II) First class vs. second class constraints

This classification scheme has nothing to do with the previous one and has a deeper dynamical significance, as it does not refer to a possible Lagrangian origin of the Hamiltonian system, but is more adapted to the concept of gauge invariance.

First class constraints:

Their defining property is that they commute with all other constraints with respect to the Poisson bracket:

$$\phi_\mu = \text{first class} \Leftrightarrow \{\phi_\mu, \phi_\beta\} = 0 \qquad \forall \beta = 1, \ldots, n$$

First class constraints typically generate gauge transformations via the Poisson bracket:

$$\delta F = \{F, \phi_\mu\},$$

and hence signal a residual gauge invariance on the constraint surface. As the time evolution is only unique modulo gauge transformations, the canonical Hamiltonian $H_{\text{can}} = p^i \dot{q}_i - L$ has to be supplemented by the first class constraints to give the "total" Hamiltonian

$$H = H_{\text{can}} + N^\mu \phi_\mu,$$

with some (possibly time-dependent) Lagrange multipliers N^μ. Note that even though ϕ_μ vanishes on the constraint surface, its Poisson bracket may still be non-zero as the Poisson bracket involves derivatives on the phase space.

Second class constraints:

Second class constraints do *not* Poisson-commute with at least one other constraint.

Fixing the residual gauge invariance associated with a first class constraint ϕ_μ, i.e., imposing a gauge-fixing condition (in other words, a new constraint) $\phi_{g.f.}(q_i, p^i) = 0$ with $\{\phi_{g.f.}, \phi_\mu\} \neq 0$, makes $(\phi_\mu, \phi_{g.f.})$ a pair of second class constraints. If all constraints are second class, one has no more gauge invariances left, and the constraint surface describes the pure, physical, degrees of freedom.

Example:

Consider the Lagrangian

$$L[q_1, q_2, \dot{q}_1, \dot{q}_2] = \frac{1}{2}\dot{q}_1^2$$

which depends trivially on the second coordinate. The canonical momenta are

$$p^1 = \dot{q}_1, \qquad p^2 = 0$$

Thus, we have a primary constraint

$$\phi_1(q_i, p^i) := p^2 = 0,$$

which is trivially first class (there are no other constraints it could not commute with). The total Hamiltonian is

$$H = H_{\text{can}} + N^1\phi_1 = p^1\dot{q}_1 - L + N^1\phi_1 = \frac{1}{2}(p^1)^2 + N^1 p^2.$$

Note that there is no term $p^2\dot{q}_2$ in H_{can}. This can be understood as follows: p^2, being identically zero, cannot be used to express $\dot{q}_2$ in terms of p^2. Thus, a term $p^2\dot{q}_2$ cannot be converted to a term $(p^2)^2$, but remains a term linear in p^2 with a time-dependent coefficient $\dot{q}_2(t)$. This however can always be absorbed into the constraint term $N^1(t)p^2$, where $N^1(t)$ denotes a (possibly time-dependent) Lagrange multiplier. The primary constraint ϕ_1 should be preserved in time. However, it is easy to see that this is automatically the case:

$$\dot{\phi}_1 = \dot{p}^2 = \{p^2, H\} = 0. \tag{2}$$

Thus, there are no secondary or higher order constraints, and the constraint ϕ_1 is really first class. From

$$\delta q_2 = \{q_2, N^1\phi_1\} = N^1$$

we see that q_2 is pure gauge. Making the gauge choice $\phi_{g.f.} := q_2 = 0$, we find

$$\{\phi_{g.f.}, \phi_1\} = 1,$$

i.e., ϕ_1 and $\phi_{g.f.}$ form a pair of second class constraints. Hence, the constraint surface $q_2 = p^2 = 0$ is free of any residual gauge invariances and describes the dynamics of only one particle parameterized by (q_1, p^1), as was to be expected from the form of the Lagrangian.