1) Differential Geometry for General Relativity

Consider the metric of a 2-sphere of radius $a$:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = a^2[d\theta^2 + \sin^2\theta d\phi^2].$$

The metric encodes all information on the geometry of a manifold. In Assignment #2, you already calculated the surface area of the two-sphere. In the present problem, we will determine all those geometric quantities that are relevant for general relativity:

a) The metric
Choosing $x^1 = \theta$ and $x^2 = \phi$, read off the matrix $g_{\mu\nu}$.

b) The Christoffel symbols
The Christoffel symbols are defined as

$$\Gamma^\kappa_{\lambda\mu} = \frac{1}{2}g^{\kappa\nu}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right).$$

They enter covariant derivatives such as $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda}V^\lambda$, where the correction term with the Christoffel symbols ensures that the covariant derivative indeed transforms “covariantly” under arbitrary coordinate transformations $x^\mu \to x'^\mu (x^\nu)$, i.e.,

$$\nabla_\mu V^\nu \to (\nabla_\mu V^\nu)' = \frac{\partial x'^\lambda}{\partial x^\mu} \frac{\partial x'^\rho}{\partial x^\nu} \nabla_\lambda V^\rho,$$

without second derivatives in the coordinates.

Compute the non-vanishing Christoffel symbols for the two-sphere. (Hint: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, so only a few components have to be computed explicitly.)

c) The Riemann tensor
The Riemann curvature tensor has the form

$$R^\kappa_{\lambda\mu\nu} = \partial_{[\mu} \Gamma^\kappa_{\nu\lambda]} - \partial_{[\nu} \Gamma^\kappa_{\mu\lambda]} + \Gamma^\kappa_{\rho\lambda} \Gamma^\rho_{\mu\nu} - \Gamma^\kappa_{\rho\mu} \Gamma^\rho_{\nu\lambda}.$$

Calculate the non-vanishing components of $R^\kappa_{\lambda\mu\nu}$ for the two-sphere (Hint: Use the antisymmetry in $\mu$ and $\nu$ to avoid redundant computations).
**Remark:** The Riemann tensor measures the curvature of a space, for instance by quantifying the non-commutativity of the covariant derivatives:

\[ [\nabla_\mu, \nabla_\nu] V^\lambda = R^\lambda_{\mu\rho\nu} V^\rho. \]

A space with vanishing \( R^\lambda_{\mu\rho\nu} \) is flat, i.e., the metric can be brought to the standard Minkowskian (or Euclidean) form by means of a coordinate transformation.

d) *The Ricci tensor*

The Ricci tensor is defined as

\[ \text{Ric}_{\mu\nu} = R^\lambda_{\mu\rho\nu}. \]

Calculate \( \text{Ric}_{\mu\nu} \) for \( S^2 \).

e) *The scalar curvature*

The scalar curvature is given as

\[ \mathcal{R} = g^{\mu\nu} \text{Ric}_{\mu\nu}. \]

Calculate \( \mathcal{R} \) for \( S^2 \). How does the scalar curvature behave in the limit \( a \to \infty \)? Interpret this behaviour.

f) *The Einstein tensor*

The Einstein equation is the field equation of general relativity, and it relates the curvature of spacetime to the matter distribution:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}, \]

where \( G \) denotes Newton’s constant, \( T_{\mu\nu} \) is the energy momentum tensor, and \( G_{\mu\nu} \) denotes the Einstein tensor:

\[ G_{\mu\nu} = \text{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R}. \]

Calculate \( G_{\mu\nu} \) for \( S^2 \).

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2) *The Polyakov action: I) The field equations*

Consider the Polyakov action,

\[ S_P = -\frac{T}{2} \int d^2 \sigma \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \]

a) Remembering \( \det(\exp A) = \exp(\text{Tr} A) \), show that

\[ \delta h = -h_{\alpha\beta} (\delta h^{\alpha\beta}) h, \]

where \( h = -\det(h_{\alpha\beta}) \)
b) The energy momentum tensor $T_{\alpha\beta}$ describes the response of the action to changes in the metric:

$$\delta S = -T \int d^2\sigma \sqrt{h} T_{\alpha\beta} \delta h^{\alpha\beta} \quad \iff \quad T_{\alpha\beta} = -\frac{1}{T \sqrt{h}} \frac{\delta S}{\delta h^{\alpha\beta}}.$$  

Compute $T_{\alpha\beta}$ for the Polyakov action.

c) Find the equations of motion for $h^{\alpha\beta}$ and show that, after some manipulation and re-insertion into $S_P$, one re-obeys the Nambu-Goto action.

d) Show that adding a “cosmological constant term”,

$$S_1 = \lambda \int d^2\sigma \sqrt{h}$$

to the Polyakov action leads to inconsistent field equations for $h_{\alpha\beta}$ in the combined system $S_P + S_1$ when $\lambda \neq 0$.

3) The Polyakov action: II) The symmetries

a) Show in one line that the Weyl invariance $S_P[e^{2\Delta} h_{\alpha\beta}, X^\mu] = S_P[h_{\alpha\beta}, X^\mu]$ automatically implies $h^{\alpha\beta} T_{\alpha\beta} = 0$ without the use of the equations of motion.

b) Verify the tracelessness of $T_{\alpha\beta}$ directly by using your result for $T_{\alpha\beta}$ from Problem 2) b).

c) How does $h_{\alpha\beta}$ have to transform under arbitrary reparameterizations $(\tau, \sigma) \to (\tilde{\tau}(\tau, \sigma), \tilde{\sigma}(\tau, \sigma))$ for $S_P$ to be invariant?

4) Lightcone coordinates

In conformal gauge and after a Weyl rescaling, the metric $h_{\alpha\beta}$ can be brought to the standard Minkowskian form:

$$ds^2 = \eta_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = -d\tau^2 + d\sigma^2.$$  

a) Rewrite the metric in terms of light cone coordinates

$$\sigma^\pm := \tau \pm \sigma,$$

and read off the components $\eta_{++}, \eta_{+-}$ and $\eta_{--}.$

b) Determine the components $\eta^{++}, \eta^{--}$ and $\eta^{+-}$ of the inverse metric and use it to derive the relation between the components $(V^+, V^-)$ and the components $(V_+, V_-)$ of a 2D vector by raising the indices.

c) Determine the derivatives $\partial_\pm$ in terms of $\partial_\tau$ and $\partial_\sigma.$ (Use $\partial_\pm \sigma^\pm = 1$ and $\partial_\mp \sigma^\pm = 0.$)