Introduction to String Theory
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Summer 2006

Assignment # 3
(Due May 22, 2006)

1) Differential Geometry for General Relativity

Consider the metric of a 2-sphere of radius a:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = a^2[d\theta^2 + \sin^2\theta d\phi^2].$$

The metric encodes all information on the geometry of a manifold. In Assignment # 2, you already calculated the surface area of the two-sphere. In the present problem, we will determine all those geometric quantities that are relevant for general relativity:

a) *The metric*
Choosing $x^1 = \theta$ and $x^2 = \phi$, read off the matrix $g_{\mu\nu}$.

b) *The Christoffel symbols*
The Christoffel symbols are defined as

$$\Gamma^\kappa_{\lambda\mu} = \frac{1}{2} g^{\kappa\nu} \left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\nu\lambda}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu} \right).$$

They enter covariant derivatives such as $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda$, where the correction term with the Christoffel symbols ensures that the covariant derivative indeed transforms "covariantly" under arbitrary coordinate transformations $x^\mu \to x^{\mu\prime}(x^\nu)$, i.e.,

$$\nabla_\mu V^\nu \to (\nabla_\mu V^\nu)' = \frac{\partial x^\lambda}{\partial x^{\mu\prime}} \frac{\partial x^{\nu\prime}}{\partial x^\rho} \nabla_\lambda V^\rho,$$

without second derivatives in the coordinates.

Compute the non-vanishing Christoffel symbols for the two-sphere. (Hint: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, so only a few components have to be computed explicitly.)

c) *The Riemann tensor*
The Riemann curvature tensor has the form

$$R^\kappa_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa_{\nu\lambda} - \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\eta_{\nu\lambda} \Gamma^\kappa_{\mu\eta} - \Gamma^\eta_{\mu\lambda} \Gamma^\kappa_{\nu\eta}.$$

Calculate the non-vanishing components of $R^\kappa_{\lambda\mu\nu}$ for the two-sphere (Hint: Use the antisymmetry in μ and ν to avoid redundant computations).

Remark: The Riemann tensor measures the curvature of a space, for instance by quantifying the non-commutativity of the covariant derivatives:

$$[\nabla_\mu, \nabla_\nu] V^\lambda = R^\lambda_{\kappa\mu\nu} V^\kappa.$$

A space with vanishing $R^\kappa_{\lambda\mu\nu}$ is flat, i.e., the metric can be brought to the standard Minkowskian (or Euclidean) form by means of a coordinate transformation.

d) *The Ricci tensor*
The Ricci tensor is defined as

$$\mathrm{Ric}_{\mu\nu} = R^\lambda_{\mu\lambda\nu}.$$

Calculate $\mathrm{Ric}_{\mu\nu}$ for S^2.

e) *The scalar curvature*
The scalar curvature is given as

$$\mathcal{R} = g^{\mu\nu}\mathrm{Ric}_{\mu\nu}.$$

Calculate $\mathcal{R}$ for S^2. How does the scalar curvature behave in the limit $a \to \infty$? Interpret this behaviour.

f) *The Einstein tensor*
The Einstein equation is the field equation of general relativity, and it relates the curvature of spacetime to the matter distribution:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where G denotes Newton's constant, $T_{\mu\nu}$ is the energy momentum tensor, and $G_{\mu\nu}$ denotes the Einstein tensor:

$$G_{\mu\nu} = \mathrm{Ric}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\mathcal{R}.$$

Calculate $G_{\mu\nu}$ for S^2.

2) The Polyakov action: I) The field equations
Consider the Polyakov action,

$$S_P = -\frac{T}{2}\int d^2\sigma \sqrt{h} h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}.$$

a) Remembering $\det(\exp A) = \exp(\mathrm{Tr} A)$, show that

$$\delta h = -h_{\alpha\beta}(\delta h^{\alpha\beta}) h,$$

where $h = -\det(h_{\alpha\beta})$

b) The energy momentum tensor $T_{\alpha\beta}$ describes the response of the action to changes in the metric:

$$\delta S = -T \int d^2\sigma \sqrt{h} T_{\alpha\beta} \delta h^{\alpha\beta} \quad \Longleftrightarrow \quad T_{\alpha\beta} = -\frac{1}{T\sqrt{h}} \frac{\delta S}{\delta h^{\alpha\beta}}.$$

Compute $T_{\alpha\beta}$ for the Polyakov action.

c) Find the equations of motion for $h^{\alpha\beta}$ and show that, after some manipulation and re-insertion into S_P, one re-obtains the Nambu-Goto action.

d) Show that adding a "cosmological constant term",

$$S_1 = \lambda \int d^2\sigma \sqrt{h}$$

to the Polyakov action leads to inconsistent field equations for $h_{\alpha\beta}$ in the combined system $S_P + S_1$ when $\lambda \neq 0$.

3) The Polyakov action: II) The symmetries

a) Show in one line that the Weyl invariance $S_P[e^{2\Lambda} h_{\alpha\beta}, X^\mu] = S_P[h_{\alpha\beta}, X^\mu]$ automatically implies $h^{\alpha\beta} T_{\alpha\beta} = 0$ without the use of the equations of motion.

b) Verify the tracelessness of $T_{\alpha\beta}$ directly by using your result for $T_{\alpha\beta}$ from Problem 2) b).

c) How does $h_{\alpha\beta}$ have to transform under arbitrary reparameterizations $(\tau, \sigma) \to (\tilde{\tau}(\tau, \sigma), \tilde{\sigma}(\tau, \sigma))$ for S_P to be invariant?

4) Lightcone coordinates

In conformal gauge and after a Weyl rescaling, the metric $h_{\alpha\beta}$ can be brought to the standard Minkowskian form:

$$ds^2 = \eta_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = -d\tau^2 + d\sigma^2.$$

a) Rewrite the metric in terms of light cone coordinates

$$\sigma^\pm := \tau \pm \sigma,$$

and read off the components η_{++}, η_{+-} and η_{--}.

b) Determine the components η^{++}, η^{--} and η^{+-} of the inverse metric and use it to derive the relation between the components (V^+, V^-) and the components (V_+, V_-) of a 2D vector by raising the indices.

c) Determine the derivatives $\partial_\pm$ in terms of ∂_τ and ∂_σ. (Use $\partial_\pm \sigma^\pm = 1$ and $\partial_\mp \sigma^\pm = 0$).