Introduction to String Theory
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Assignment # 4
(Due May 29, 2006)

1) Lie algebras

a) Consider a three-dimensional vector space V with the basis elements T_1, T_2, T_3. Define an antisymmetric product $[\ ,\ ]$ as follows:

$$[T_1, T_2] = -[T_2, T_1] = iT_2 \tag{1}$$
$$[T_2, T_3] = -[T_3, T_2] = iT_3 \tag{2}$$
$$[T_1, T_3] = -[T_3, T_1] = iT_1 \tag{3}$$

Verify whether this defines a Lie algebra (**Note:** There is an appendix at the end of this Problem Assignment which recalls some background material on Lie algebras and symmetries in classical Hamiltonian systems).

b) What relations do, respectively, the antisymmetry and the Jacobi identity impose on the structure constants f_{ab}^c of a Lie algebra?

2) The residual conformal transformations

a) Using light cone coordinates $\sigma^\pm$, the world sheet metric in conformal gauge reads

$$ds^2 = -\Omega^2 d\sigma^+ d\sigma^-,$$

where the conformal factor $\Omega(\sigma^+, \sigma^-)$ can be absorbed by a Weyl transformation to make the metric flat. Show that transformations of the type

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+), \qquad \sigma^- \to \tilde{\sigma}^-(\sigma^-)$$

do not lead one out of the conformal gauge. These transformations are called *conformal transformations* and correspond to a residual freedom in choosing the worldsheet coordinates even after one has gone to conformal gauge.

b) Using $T_{\pm\pm} = \frac{1}{2}\partial_\pm X \cdot \partial_\pm X$ and the Poisson brackets in conformal gauge,

$$\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\} = \{\dot{X}^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = 0 \tag{4}$$
$$\{X^\mu(\sigma,\tau), \dot{X}^\nu(\sigma',\tau)\} = \frac{1}{T}\eta^{\mu\nu}\delta(\sigma - \sigma'), \tag{5}$$

calculate the Poisson brackets

$$\{T_{\pm\pm}(\sigma,\tau), X^\mu(\sigma',\tau)\}.$$

c) Use the definition

$$L_\xi := 2T \int_0^{\bar{\sigma}} d\sigma \xi(\sigma^+) T_{++}(\sigma^+)$$

and the result of part b) to calculate the Poisson bracket

$$\{L_\xi, X^\mu(\sigma,\tau)\}$$

and show that the L_ξ generate infinitesimal conformal transformations via the Poisson bracket.

d) For the closed string, one can also define the analogous quantities for T_{--} and decompose the functions $\xi(\sigma^\pm)$ into Fourier components $e^{im\sigma^\pm}$. The resulting generators L_m and $\bar{L}_m$ then form two copies of the classical Virasoro algebra with respect to the Poisson bracket, i.e.,

$$\{L_m, L_n\} = -i(m-n)L_{m+n}$$

and similarly for the $\bar{L}_m$. Verify explicitly that the above commutation relations satisfy the Jacobi identity, i.e., form a Lie algebra.

e) Show that the generators L_0, L_1 and L_{-1} form a Lie subalgebra.

f) Show that the combination $(\bar{L}_0 - L_0) = T\int_0^{2\pi} d\sigma \dot{X}\cdot X'$ generates rigid σ-translations along the closed string.

3) Oscillator expansion

Consider the mode expansion of the closed string:

$$\begin{aligned} X_R^\mu(\tau-\sigma) &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau-\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in(\tau-\sigma)} && (6)\\ X_L^\mu(\tau+\sigma) &= \frac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu(\tau+\sigma) + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu e^{-in(\tau+\sigma)}. && (7)\end{aligned}$$

Use the Poisson brackets

$$\begin{aligned} \{\alpha_m^\mu, \alpha_n^\nu\} &= \{\bar{\alpha}_m^\mu, \bar{\alpha}_n^\nu\} = -im\delta_{m+n}\eta^{\mu\nu} && (8)\\ \{x^\mu, p^\nu\} &= \eta^{\mu\nu} && (9)\end{aligned}$$

to reproduce the Poisson brackets for X^μ and $\dot{X}^\mu$ given in Problem 2 b).

Appendix

In this appendix, we briefly recall some elementary terminology concerning the implementation of continuous symmetries in classical Hamiltonian mechanics.

Continuous symmetries, Lie groups and Lie algebras

In general, a symmetry of a physical system denotes a transformation of the system's dynamical variables that leaves the kinematical and dynamical structure of the system invariant. The combination of two symmetry transformations is again a symmetry transformation, and the set of all symmetry transformations of a system naturally carries the structure of a group, the "symmetry group".

In physical systems, one encounters discrete as well as continuous symmetry groups. Discrete symmetry groups such as, e.g., parity or charge conjugation have only finitely many group elements. A continuous symmetry group, on the other hand, has infinitely many elements that can be parameterized by one or more real parameters λ^a

$(a = 1, \ldots, n)$. Viewing the parameters λ^a as local coordinates, such a group can be considered a manifold. Groups that are differentiable manifolds such that the group operations are differentiable are called *Lie groups*. Important examples of Lie groups are provided by the classical matrix groups such as $SO(n)$, $SU(n)$ etc.

The group $SU(2)$, for example, is defined to be the group of unitary (2×2)-matrices of unit determinant. Any such group element, U, can be written as

$$U = e^{i\lambda^a T_a}$$

where $T_a = \sigma_a$ $(a = 1, 2, 3)$ are the three Pauli matrices. Thus, $SU(2)$ is parameterized by three real parameters $\lambda^1, \lambda^2, \lambda^3$ and is thus a differentiable manifold of real dimension three (which turns out to be the three-sphere S^3).

The set of all possible linear combinations of the Pauli matrices, $v = v^a \sigma_a$ $(v^a \in \mathbf{R}^3)$, forms a three-dimensional vector space V. As the Pauli matrices close into themselves under commutation:

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c,$$

one can use the commutator of the σ_a to define a product on the whole of V:

$$[v, w] = [v^a \sigma_a, w^b \sigma_b] := v^a w^b [\sigma_a, \sigma_b] = 2i v^a w^b \epsilon_{abc} \sigma_c.$$

This product satisfies:

(i) Antisymmetry:

$$[v, w] = -[w, v]$$

(ii) Jacobi identity:

$$[v, [w, u]] + [u, [v, w]] + [w, [u, v]] = 0.$$

In general, a vector space V with a product $[\ ,\ ] : V \times V \to V$ that satisfies the conditions (i) and (ii) above is called a *Lie algebra*. Denoting a set of basis elements of V by T_a $(a = 1, \ldots, \dim(V))$, the product $[\ ,\ ]$ is completely specified by the commutation relations

$$[T_a, T_b] = i f_{ab}^c T_c$$

of the basis elements. Here, f_{ab}^c are called the *structure constants* of the Lie algebra, which for the Lie algebra of $SU(2)$ are simply given by the epsilon tensor $f_{ab}^c = 2\epsilon_{abc}$.

For small parameters λ^a, an $SU(2)$ group element $U = e^{i\lambda^a \sigma_a}$ is well approximated by the two lowest order terms:

$$U \cong \mathbf{1}_2 + i\lambda^a \sigma_a$$

In this sense, the Lie algebra of $SU(2)$ describes the infinitesimal neighbourhood of the group $SU(2)$ at the unit element $e = \mathbf{1}_2$. This is a general phenomenon for any Lie group: The tangent space of a d-dimensional Lie group G at the unit element, e, is a d-dimensional vector space, $\mathbf{g}$, which naturally inherits a product, $[\ ,\ ]$, from the group multiplication in G. This product satisfies (i) and (ii) and hence makes the tangent space of G at e a Lie algebra.

As vector spaces are easier to handle than curved manifolds, physicists usually use the Lie algebra of a continuous symmetry group and the language of infinitesimal symmetries to describe the symmetries of a physical system. For classical Hamiltonian systems, this is done as follows:

Symmetries in classical Hamiltonian mechanics

In classical Hamiltonian mechanics, a physical system is described by a phase space, $\mathbf{P}$, which is endowed with a Poisson bracket $\{\ ,\ \}$ and a Hamiltonian $H(q,p)$ that generates the system's time evolution via

$$\dot{F} = \{F, H\},$$

where $F(q,p)$ denotes an arbitrary function on the phase space [1]. A symmetry of such a classical Hamiltonian system is then a map $\Phi : \mathbf{P} \to \mathbf{P}$ that leaves the kinematical structure (i.e, the Poisson brackets) and dynamics (i.e., the time evolution generated by H) invariant. More precisely, a symmetry transformation Φ has to satisfy

$$\{F, G\} \circ \Phi = \{F \circ \Phi, G \circ \Phi\} \tag{10}$$

$$H \circ \Phi = H \tag{11}$$

where $F(q,p)$ and $G(q,p)$ denote arbitrary functions on the phase space. Transformations that satisfy the first property (10) are generally called *canonical transformations*, i.e., symmetries are canonical transformations that also leave the Hamiltonian invariant.

If the symmetries form a d-dimensional Lie group, G, parameterized by some parameters λ^a ($a = 1, \dots, d$), the corresponding canonical transformations Φ likewise form a d-parameter family $\Phi[\lambda^a]$. For infinitesimal λ^a, one obtains an infinitesimal symmetry transformation on the phase space. According to a well-known theorem in Hamiltonian mechanics, for any infinitesimal canonical transformation $\delta\Phi$, there exists locally a generating function $Q(q,p)$ on the phase space that implements the infinitesimal canonical transformation via the Poisson bracket:

$$\delta F = \{F, Q\},$$

where δF denotes the change of an arbitrary function $F(q,p)$ on the phase space under the infinitesimal canonical transformation $\delta\Phi$.

If the infinitesimal canonical transformation $\delta\Phi$ is a symmetry, it also has to leave the Hamiltonian invariant (cf. the second property (11)). In terms of Q this translates to

$$\delta H = \{H, Q\} = 0.$$

This, however, also means that

$$\dot{Q} = \{Q, H\} = 0,$$

i.e., Q is a conserved quantity.

At the infinitesimal level, the symmetry group G is described by its Lie algebra $\mathbf{g}$, which has $d = \dim(G)$ linearly independent generators T_a. According to the above, there will therefore be d linearly independent conserved quantities Q_{T_a} that generate the infinitesimal transformations corresponding to T_a on the phase space. They implement the symmetry algebra $\mathbf{g}$ on the phase space via the Poisson bracket:

$$\{Q_{T_a}, Q_{T_b}\} = Q_{[T_b, T_a]}$$

[1] The choices $F = q_i$ and $F = p^i$ give the standard form of the Hamilton equations, $\dot{q}_i = \partial H/\partial p^i$, $\dot{p}^i = -\partial H/\partial q_i$.