Introduction to String Theory
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Summer 2006

Assignment # 5
(Due June 12, 2006)

1) Normal ordering and the quantum Virasoro algebra

In this excercise, we will show that, in the quantized bosonic string theory, the normal ordered Virasoro generators

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{m-n} \cdot \alpha_n :$$

satisfy the Virasoro algebra with a central charge [1] :

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12} m(m^2-1)\delta_{m+n}.$$

In order to become more familiar with the normal ordering prescription, we will do this by brute force methods, i.e., by simply using the definition of the normal ordered generators L_m and then calculating their commutators. We will proceed in several smaller steps.

a) Explain why the normal ordering in L_m only affects L_0 and why the Virasoro generators L_m can be written in the following form:

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{0} \alpha_n \cdot \alpha_{m-n} + \frac{1}{2} \sum_{n=1}^{\infty} \alpha_{m-n} \cdot \alpha_n \tag{1}$$

b) Using $[X, YZ] = [X, Y]Z + Y[X, Z]$ and $[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}$ prove that, for all $m, n \in \mathbf{Z}$,

$$[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}.$$

c) Decompose the sum

$$\sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{0} + \sum_{n=1}^{\infty}$$

[1] A central charge, T_0, of a Lie algebra is a generator that commutes with all generators of the Lie algebra, $[T_a, T_0] = 0$, but appears on the right hand side of some commutators, $[T_a, T_b] = cT_0 + \ldots$, for some T_a and T_b, with c being a constant. In the above Virasoro algebra, the rôle of T_0 is played by the term proportional to δ_{m+n}, which should be viewed as an additional generator in addition to the L_m.

as we did in (1) to "solve" the normal ordering condition. Use the result of part b) to show that

$$[L_m, L_n] = \frac{1}{2}\sum_{l=-\infty}^{0}\{(m-l)\alpha_l \cdot \alpha_{m+n-l} + l\alpha_{n+l}\cdot\alpha_{m-l}\} + \frac{1}{2}\sum_{l=1}^{\infty}\{(m-l)\alpha_{m+n-l}\cdot\alpha_l + l\alpha_{m-l}\cdot\alpha_{n+l}\}. \tag{2}$$

d) Make the substitution $p = n + l$ in the second and fourth term in (2) and verify

$$[L_m, L_n] = \frac{1}{2}\{\sum_{l=-\infty}^{0}(m-l)\alpha_l\cdot\alpha_{m+n-l} + \sum_{p=-\infty}^{n}(p-n)\alpha_p\cdot\alpha_{m+n-p} + \sum_{l=1}^{\infty}(m-l)\alpha_{m+n-l}\cdot\alpha_l + \sum_{p=n+1}^{\infty}(p-n)\alpha_{m+n-p}\cdot\alpha_p\}. \tag{3}$$

e) From now on, we will restrict ourselves to the case $n > 0$, as the other cases $n < 0$ and $n = 0$ are completely analogous. Show, therefore, that, for $n > 0$, the expression (3) in d) is equal to

$$[L_m, L_n] = \frac{1}{2}\{\sum_{p=-\infty}^{0}(m-n)\alpha_p\cdot\alpha_{m+n-p} + \sum_{p=1}^{n}(p-n)\alpha_p\cdot\alpha_{m+n-p} + \sum_{p=n+1}^{\infty}(m-n)\alpha_{m+n-p}\cdot\alpha_p + \sum_{p=1}^{n}(m-p)\alpha_{m+n-p}\cdot\alpha_p\} \tag{4}$$

Which of these four terms are already normal-ordered?

f) Prove

$$\sum_{p=1}^{n}(p-n)\alpha_p\cdot\alpha_{m+n-p} = \sum_{p=1}^{n}(p-n)\alpha_{m+n-p}\cdot\alpha_p + \sum_{p=1}^{n}(p-n)pD\delta_{m+n}$$

and insert this for the second term in the expression (4) of part e).

g) Show that your result from part e) is now equivalent to

$$[L_m, L_n] = \frac{1}{2}\sum_{l=-\infty}^{\infty}(m-n) :\alpha_l\cdot\alpha_{m+n-l}: + \frac{1}{2}D\sum_{l=1}^{n}(l^2 - nl)\delta_{m+n}.$$

h) Prove, e.g. by induction, the following identities:

$$\sum_{q=1}^{n} q^2 = \frac{1}{6}n(n+1)(2n+1)$$
$$\sum_{q=1}^{n} q = \frac{1}{2}n(n+1) \tag{5}$$

and use this to finally derive

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n}$$

from the expression in part g).

2) The quantum Virasoro algebra as a Lie algebra

In homework assignment #4, you showed that the classical Virasoro generators form a Lie algebra with respect to the Poisson bracket. In the quantized version, the Poisson brackets have now become true commutators between operators, and we also have a central charge:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n} \tag{6}$$

a) Show that these commutation relations still define a Lie algebra.

b) Do L_0, L_1 and L_{-1} still form a Lie subalgebra?

c) Do the L_m with $m > 0$ form a Lie subalgebra? And the L_m with $m < 0$?

d) We have calculated the quantum Virasoro algebra by using normal ordered generators L_m. This choice need not be the physically correct one. One might therefore now wonder, whether a redefinition

$$L_m \to \tilde{L}_m = L_m - \lambda\delta_{m,0}$$

with a constant λ that parameterizes a different ordering in L_0 could perhaps remove the troublesome central charge term in the Virasoro algebra. Show that this cannot happen, i.e., show that such a redefinition in (6) could at most change the linear term in m in the central charge, but not the cubic one.