Introduction to String Theory
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Assignment # 6

Due: June 26, 2006

1) Old covariant quantization

In the old covariant quantization procedure, all fields $X^\mu(\tau,\sigma)$ $(\mu = 0,\ldots,(D-1))$ are kept as dynamical variables. The corresponding naive Fock space $\mathcal{H}_{\text{Fock}}$, which is generated by acting with all possible combinations of raising operators α^μ_{-m} (and $\bar{\alpha}^\mu_{-m}$ for the closed string) $(m > 0)$ on the Fock vacuum $|0,p\rangle$, always contains negative norm states. These negative norm states, however, are completely harmless provided they are all projected out by the physical state conditions

$$L_m|\text{phys}\rangle = 0 \qquad (m > 0) \tag{1}$$

$$(L_0 - 1)|\text{phys}\rangle = 0 \tag{2}$$

and similarly for the $\bar{L}_m$ in the case of the closed string. These physical state conditions are the quantum implementation of the classical Virasoro constraints.

a) Use the second of these constraints, eq. (2), as well as the relation $\sqrt{2\alpha'}p^\mu = \alpha^\mu_0$ to derive the mass shell condition for the open string:

$$\alpha' m^2 = (N-1)\,, \tag{3}$$

where

$$N = \sum_{m>0} \alpha_{-m}\cdot\alpha_m \tag{4}$$

denotes the number operator. States that satisfy the relation (3) automatically solve the constraint (2).

b) Consider the following open string state:

$$|\Phi\rangle = \frac{1}{2}\Big[\alpha_{-1}\cdot\alpha_{-1} + \frac{(D-1)}{5}p\cdot\alpha_{-2} + \frac{(D+4)}{10}(p\cdot\alpha_{-1})^2\Big]|0,p\rangle. \tag{5}$$

In the rest of this problem, we set

$$\alpha' = \frac{1}{2}. \tag{6}$$

We now want to verify whether this state is physical, i.e., whether this state satisfies (1) and (2). Use first (3) from part a) to derive the constraint imposed on $p^\mu p_\mu = -m^2$ by (2).

c) Use $[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}$ and (remembering $\alpha' = \frac{1}{2}$) $p^\mu = \alpha^\mu_0$ to show that $|\Phi\rangle$ is annihilated by L_1, L_2 and $L_{m\geq 3}$. This implies that $|\Phi\rangle$ is physical for all spacetime dimensions D.

d) Calculate the norm of $|\Phi\rangle$ and show that there are negative norm states in the physical spectrum for $D > 26$.

2) Light cone quantization

In light cone quantization, the residual coordinate freedom in the conformal gauge (cf. Assignment # 4, Problem 2) is fixed to eliminate the variable $X^+(\tau,\sigma)$ from the dynamics. The use of spacetime light cone coordinates then allows one to explicitly solve the Virasoro constraints $T_{\alpha\beta} = 0$ by expressing $X^-(\tau,\sigma)$ in terms of $X^i(\tau,\sigma)$ $(i = 1,\ldots,(D-2))$. The transverse oscillations $X^i(\tau,\sigma)$ thus remain as the only dynamical degrees of freedom. The mass operator of the open string now becomes

$$m^2 = \frac{1}{\alpha'}[N^\perp - 1], \tag{7}$$

where

$$N^\perp = \sum_{n>0} \alpha^i_{-n}\alpha^i_n \tag{8}$$

is the transverse number operator.

a) Rank the following open string states according to their mass squared (lowest first):

$$\begin{aligned} |A\rangle = \alpha^i_{-3}|0,p\rangle, \qquad |B\rangle = \alpha^i_{-2}\alpha^j_{-1}|0,p\rangle, \qquad |C\rangle = |0,p\rangle \\ |D\rangle = \alpha^i_{-2}\alpha^j_0|0,p\rangle, \qquad |E\rangle = (\alpha^i_{-1}\alpha^j_{-1} + \alpha^j_{-1}\alpha^i_{-1})|0,p\rangle \end{aligned} \tag{9}$$

b) With the p^μ satisfying the mass constraint (7), which of the states in part a) still have to be projected out by the Virasoro constraints for $n > 0$?

3) The graviton as a closed string excitation

In the light cone quantization of the closed string, the first excited states are massless states of the form

$$\alpha^i_{-1}\bar{\alpha}^j_{-1}|0,p\rangle, \tag{10}$$

where $i,j = 1,\ldots,(D-2)$ denote the $(D-2)$ transverse directions. The above states decompose into three irreducible representations of $SO(24)$: a symmetric traceless tensor, the trace and an antisymmetric tensor. The latter two describe one-particle excitations of a massless scalar ϕ and a 2-form field $B_{\mu\nu} = -B_{\nu\mu}$, respectively. In this excercise, we want to understand in what sense the symmetric traceless part describes the graviton. To this end, we consider weak gravitational fields, i.e., metrics that are close to the flat Minkowski metric:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \tag{11}$$

with $h_{\mu\nu} \ll 1$. The matter-free Einstein equation $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ then yields the following linearized field equation for $h_{\mu\nu}$:

$$\partial^2 h^{\mu\nu} - \partial_\rho(\partial^\mu h^{\nu\rho} + \partial^\nu h^{\mu\rho}) + \partial^\mu\partial^\nu h = 0, \tag{12}$$

where all indices are raised with the flat Minkowski metric $\eta^{\mu\nu}$, and

$$h \equiv \eta^{\mu\nu} h_{\mu\nu} \tag{13}$$

denotes the trace of $h_{\mu\nu}$.

a) Check that (12) is invariant under the following gauge transformations

$$\delta h^{\mu\nu}(x) = \partial^\mu \epsilon^\nu(x) + \partial^\nu \epsilon^\mu(x). \tag{14}$$

Remark: These gauge transformations are simply inherited from the invariance under infinitesimal coordinate transformations,

$$x^{\mu\prime} = x^\mu + \epsilon^\mu(x). \tag{15}$$

b) Making a Fourier transform $h_{\mu\nu}(x) \to h_{\mu\nu}(p)$, the above field equation and the gauge transformations become:

$$\begin{aligned} 0 &= p^2 h^{\mu\nu} - p_\rho(p^\mu h^{\nu\rho} + p^\nu h^{\mu\rho}) + p^\mu p^\nu h & (16) \\ \delta h^{\mu\nu}(p) &= ip^\mu \epsilon^\nu(p) + ip^\nu \epsilon^\mu(p). & (17) \end{aligned}$$

Read off $\delta h^{++}(p)$, $\delta h^{+-}(p)$ and $\delta h^{+i}(p)$, and show that for $p^+ \neq 0$ (which will be assumed throughout the rest of this excercise), these gauge transformations are just enough to gauge away all $h^{\mu\nu}$ with at least one $+$ index:

$$h^{++} = h^{+-} = h^{+i} = 0. \tag{18}$$

c) The remaining degrees of freedom are thus carried by (h^{--}, h^{-i}, h^{ij}). Use (16) for $\mu = \nu = +$ and the gauge choice (18) to infer that $h = 0$, and verify that this implies

$$\delta^{ij} h_{ij} = 0, \tag{19}$$

i.e., the tracelessness of the transverse components.

d) Show that eq. (16) for $\mu = +$, together with $h = 0$ and (18), imply

$$p_\rho h^{\nu\rho} = 0. \tag{20}$$

e) Using the gauge choice (18) and the result of part d) for $\nu = i$ and $\nu = -$, show that h^{i-} and h^{--} can be expressed in terms of h^{ij}.

This implies that all physical degrees of freedom of a weak gravitational field can be described by a purely transverse, symmetric traceless tensor h^{ij}. Inserting (20) into (16), one easily sees that $p^2 h^{ij} = 0$, i.e., the corresponding particle states ("gravitons") are indeed massless, just as the corresponding closed string states.