Introduction to String Theory
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Assignment # 7

Due: July 3, 2006

NOTE:
Assignments #6 and #7 have been posted at the same time, so please check the due dates and make sure that you don't forget Assignment # 6, which is due a week earlier.

1) Path integral and Faddeev-Popov determinant

The Polyakov action $S_P[X^\mu, h_{\alpha\beta}]$ treats the embedding coordinates X^μ and the world sheet metric $h_{\alpha\beta}$ as dynamical variables. S_P is invariant under the (infinite-dimensional) group $diff \times Weyl$ of world sheet diffeomorphisms (corresponding to arbitrary reparameterizations $\sigma^\alpha \rightarrow \tilde{\sigma}^\alpha(\sigma^\beta)$) and local Weyl rescalings of the metric $h_{\alpha\beta} \rightarrow e^{2\Lambda(\tau,\sigma)} h_{\alpha\beta}$. These gauge symmetries allow one to classically eliminate $h_{\alpha\beta}$ as a dynamical variable by going to the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$.

In the *quantum* theory, this gauge fixing requires some more care and might even be impossible, as is best seen in the path integral formulation (For a more detailed account on path integrals and the Faddeev-Popov procedure see eg. Peskin Schroeder, Ch. 9 and 16). The naive vacuum to vacuum amplitude, or partition function,

$$Z = \int \frac{\mathcal{D}[h]\mathcal{D}[X]}{V_{diff \times Weyl}} e^{iS_P} \tag{1}$$

sums over all possible field configurations $[X^\mu(\tau,\sigma), h_{\alpha\beta}(\tau,\sigma)]$ between some fixed initial and final values and weighs them with the exponential of the classical action. This path integral contains a huge overcounting, as all gauge equivalent field configurations are independently integrated over. Formally, one should therefore normalize this expression by dividing by the "volume" $V_{diff \times Weyl}$ of the local symmetry group, which, however, is itself infinite. In order to make the naive expression (1) more meaningful, one should therefore use a change of integration variables

$$\mathcal{D}[h]\mathcal{D}[X] \rightarrow \mathcal{D}[\text{gauge equivalent}]\mathcal{D}[\text{gauge inequivalent}] \tag{2}$$

so that the redundant integration over gauge equivalent configurations can be factored out and formally be "cancelled" by the volume factor, leaving an integration over the physically independent configurations only. Just as for finite-dimensional integrals, this change of variables comes with a Jacobian, the so-called Faddeev-Popov determinant Δ_{FP}, which has to be included in the remaining integration over the gauge inequivalent configurations,

$$Z = \int \mathcal{D}[\text{gauge inequivalent}] \Delta_{FP} e^{iS_P}. \tag{3}$$

For the factorization into integrations over gauge equivalent and gauge inequivalent configurations to be possible, the integration measure of the original path integral has to be gauge invariant. If this is not the case, the gauge degrees of freedom cannot be consistently decoupled in the quantum theory, and the theory becomes anomalous.

In our case, the gauge symmetries act non-trivially on $h_{\alpha\beta}$, their infinitesimal action being given by (c.f. the lecture)

$$\begin{aligned} \delta h_{\alpha\beta} &= \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha + 2\Lambda h_{\alpha\beta} && (4) \\ &= (P\xi)_{\alpha\beta} + 2\tilde{\Lambda} h_{\alpha\beta}, && (5) \end{aligned}$$

with $(P\xi)_{\alpha\beta} \equiv \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha - (\nabla_\gamma \xi^\gamma) h_{\alpha\beta}$ and $2\tilde{\Lambda} = 2\Lambda + (\nabla_\gamma \xi^\gamma)$. As $h_{\alpha\beta}$ is completely gauge, one can write

$$\mathcal{D}[h] = \mathcal{D}[P\xi]\mathcal{D}[\tilde{\Lambda}] = \mathcal{D}[\xi]\mathcal{D}[\Lambda] \left| \frac{\delta(P\xi, \tilde{\Lambda})}{\delta(\xi, \Lambda)} \right|. \tag{6}$$

a) Using $\frac{\delta(P\xi)}{\delta\xi} = P$, show that, formally, the matrix

$$\begin{pmatrix} \frac{\delta(P\xi)}{\delta\xi} & \frac{\delta(P\xi)}{\delta\Lambda} \\ \frac{\delta\tilde{\Lambda}}{\delta\xi} & \frac{\delta\tilde{\Lambda}}{\delta\Lambda} \end{pmatrix} \tag{7}$$

has lower triangular form.

b) Use this to infer that, formally,

$$\left| \frac{\delta(P\xi, \tilde{\Lambda})}{\delta(\xi, \Lambda)} \right| = \det P. \tag{8}$$

Hence, $\det P$ plays the rôle of the Faddeev-Popov determinant, and one has

$$Z = \int \mathcal{D}[X](\det P) e^{iS_P[X, h_{\alpha\beta} = \eta_{\alpha\beta}]}. \tag{9}$$

2) Faddeev-Popov ghosts and Grassmann numbers

Determinants of operators such as the above Faddeev-Popov determinant $\Delta_{FP} = \det P$ can formally be written as a separate path integral over a new set of auxiliary variables. In order for this to be possible, these auxiliary variables have to be anti-commuting rather than ordinary commuting numbers. Two anti-commuting numbers (or Grassmann numbers) θ and η satisfy

$$\theta\eta = -\eta\theta \tag{10}$$

and hence $\theta^2 = 0$. Because of this, the most general function of one Grassmann variable θ is

$$f(\theta) = A + B\theta \tag{11}$$

with $A, B \in \mathbf{C}$.

Integrals over Grassmann variables ("Berezin integrals") are defined by

$$\int d\theta [A + B\theta] := B. \tag{12}$$

a) Defining the derivative

$$\frac{d}{d\theta}\theta = 1, \qquad \frac{d}{d\theta}A = 0 \qquad (A \in \mathbf{C}), \tag{13}$$

show that the Berezin integral of a total derivative is zero and that the Berezin integral is translation invariant, i.e.,

$$\int d\theta \frac{d}{d\theta} f(\theta) \;=\; 0 \tag{14}$$

$$\int d\theta f(\theta + a) \;=\; \int d\theta f(\theta) \qquad \text{for } a \in \mathbf{C}. \tag{15}$$

These properties mimic similar properties of ordinary integrals of the type $\int_\infty^\infty dx f(x)$, which is the motivation for the unusual definition (12). Note that, for Grassmann variables, integration and differentiation are equivalent operations.

b) If one has several linearly independent Grassmann variables θ_i $(i = 1, \ldots, n)$, where

$$\theta_i \theta_j = -\theta_j \theta_i \qquad \forall i, j, \tag{16}$$

one defines

$$\int d\theta_1 \ldots d\theta_n f(\theta_i) = c, \tag{17}$$

where c is the coefficient in front of the $\theta_n \theta_{n-1} \ldots \theta_1$-term in $f(\theta^i)$ (note the order):

$$f = \ldots + c\theta_n \theta_{n-1} \ldots \theta_1. \tag{18}$$

Let n be even and split the θ_i into two sets ψ_m, χ_m $(m = 1, \ldots, \frac{n}{2})$:

$$(\theta_1, \ldots, \theta_n) = (\psi_1, \chi_1, \psi_2, \chi_2, \ldots, \psi_{\frac{n}{2}}, \chi_{\frac{n}{2}}). \tag{19}$$

Show that

$$\Big(\Pi_{m=1}^{\frac{n}{2}} \int d\psi_m d\chi_m\Big) e^{\sum_{m=1}^{\frac{n}{2}} \chi_m \lambda_m \psi_m} = \Pi_{m=1}^{\frac{n}{2}} \lambda_m, \tag{20}$$

where $\lambda_m \in \mathbf{C}$ are ordinary c-numbers and the exponetntial is defined via its power series expansion.

c) If the λ_m are the eigenvalues of an operator Λ, one thus obtains

$$\Big(\Pi_{m=1}^{\frac{n}{2}} \int d\psi_m d\chi_m\Big) e^{\sum_{m,l=1}^{\frac{n}{2}} \chi_m \Lambda_{ml} \psi_l} = \det \Lambda, \tag{21}$$

or, in a path integral context with Grassman-valued fields $\psi(x)$, $\chi(x)$ and a differential operator Δ,

$$\int \mathcal{D}[\psi]\mathcal{D}[\chi] e^{\int d^d x \chi \Delta \psi} = \det \Delta. \tag{22}$$

Using similar arguments (see, e.g., Polchinski, Chapter 3.3 for a detailed account), one obtains

$$\det P = \int \mathcal{D}[c_\alpha]\mathcal{D}[b^{\beta\gamma}] \exp\Big[-\frac{i}{4\pi}\int d^2\sigma \sqrt{h} b^{\alpha\beta} (Pc)_{\alpha\beta}\Big], \tag{23}$$

where $b^{\alpha\beta}(\sigma) = b^{\beta\alpha}(\sigma)$ is a symmetric traceless anti-commuting field, and $c_\alpha(\sigma)$ is an anti-commuting world sheet vector field. Show that, due to the symmetry and tracelessness of $b^{\alpha\beta}$, one can write

$$\det P = \int \mathcal{D}[c_\alpha]\mathcal{D}[b^{\beta\gamma}] \exp\Big[-\frac{i}{2\pi}\int d^2\sigma \sqrt{h} b^{\alpha\beta} \nabla_\alpha c_\beta\Big]. \tag{24}$$

d) It is more convenient to use $b_{\alpha\beta}$ ("anti-ghost") and c^α ("ghost") as the independent fields, as they turn out to be neutral under Weyl transformations, whereas $b^{\alpha\beta}$ and c_α are not due to additional powers of the (inverse) metric. Use

$$S_{ghost} = -\frac{i}{2\pi}\int d^2\sigma\sqrt{h}b_{\alpha\beta}\nabla^\alpha c^\beta \tag{25}$$

to derive the ghost action in flat world sheet light cone coordinates:

$$S_{ghost} = \frac{i}{\pi}\int d^2\sigma(c^+\partial_- b_{++} + c^-\partial_+ b_{--}). \tag{26}$$

e) Derive the equations of motion for $c^\pm$ and $b_{\pm\pm}$ from (26).

f) The total gauge fixed path integral is now

$$Z = \int \mathcal{D}[X]\mathcal{D}[c]\mathcal{D}[b]e^{i[S_P+S_{ghost}]_{h_{\alpha\beta}=\eta_{\alpha\beta}}}, \tag{27}$$

and one clearly sees that it would have been inconsistent to simply set $h_{\alpha\beta} = \eta_{\alpha\beta}$ and drop the $\mathcal{D}[h]$ integration, as that would have missed the ghost contribution. To appreciate the ghost contribution, one notes that the total energy momentum tensor $T_{\alpha\beta}$ now also gets a contribution from the ghost action,

$$T_{\alpha\beta} = T_{\alpha\beta}[X] + T_{\alpha\beta}[b,c] \tag{28}$$

which modifies the central charge term in the Virasoro algebra to

$$A(m) = \frac{D}{12}m(m^2-1) + \frac{1}{6}(m - 13m^3) + 2am. \tag{29}$$

A non-vanishing total A(m) translates to an anomaly of the local Weyl transformations. Verify that this anomaly is absent if and only if $D = 26$ and $a = 1$.

3) Conformal transformations as area preserving maps

In general, a conformal transformation $x \to \tilde{x}(x)$ is defined to be a transformation that preserves the metric up to a local scale factor:

$$\tilde{g}_{pq}(\tilde{x}(x)) = \Omega^2(x)\frac{\partial x^m}{\partial \tilde{x}^p}\frac{\partial x^n}{\partial \tilde{x}^q}g_{mn}(x). \tag{30}$$

Specializing from now on to positive definite curved metrics (i.e., Euclidean signature) the angle α between two vector fields $v^m(x)$ and $w^m(x)$ at a point x_0 is defined by

$$\cos\alpha(v,w)(x_0) := \frac{v^m w^n g_{mn}}{||v||\,||w||}\Big|_{x=x_0}. \tag{31}$$

Here, $||v|| := (v^m v^n g_{mn})^{1/2}$ is the length, or norm, of a vector v^m.

a) Show that a conformal transformation is angle-preserving, i.e., that

$$\cos\alpha(\tilde{v},\tilde{w})(\tilde{x}(x_0)) = \cos\alpha(v,w)(x_0), \tag{32}$$

where

$$\tilde{v}^m(\tilde{x}(x)) = \frac{\partial\tilde{x}^m}{\partial x^n}v^n(x) \tag{33}$$

is the transformed vector field.

b) In conformal gauge, the 2D Lorentzian world sheet metric is

$$\begin{aligned} ds^2 &= \Omega^2(\sigma,\tau)(-d\tau^2 + d\sigma^2) && (34)\\ &= -\Omega^2(\sigma^+,\sigma^-)d\sigma^+ d\sigma^-. && (35)\end{aligned}$$

Performing the Wick rotation

$$\sigma^{\pm} = (\tau \pm \sigma) \rightarrow -i(\tau \pm i\sigma), \tag{36}$$

write down the resulting Euclidean metric both in terms of the (Wick-rotated) (τ, σ) and the complex coordinates

$$z' = \tau - i\sigma, \qquad \bar{z}' = \tau + i\sigma. \tag{37}$$

c) Show that all holomorphic coordinate transformations

$$z' \rightarrow \tilde{z}(z'), \qquad \bar{z}' \rightarrow \bar{\tilde{z}}(\bar{z}') \tag{38}$$

change the metric only by a local rescaling $\Omega^2(z', \bar{z}') \rightarrow f(z', \bar{z}')\Omega^2(z', \bar{z}')$, i.e., that they are conformal.

4) The complex plane and the cylinder

A special case of such a conformal/holomorphic transformation is the map

$$z' \rightarrow z = e^{z'}, \qquad \bar{z}' \rightarrow \bar{z} = e^{\bar{z}'}, \tag{39}$$

which maps the cylinder (i.e., the Wick-rotated world sheet of a non-interacting closed string) to the complex plane.

a) Calculate the rescaling function $f(z, \bar{z})$ introduced in Problem 3) c) for this conformal transformation.

b) What is the image of a curve of constant τ under this transformation?

c) Determine how σ translations $\sigma \rightarrow \sigma + \theta$ and time translations $\tau \rightarrow \tau + a$ operate on the new coordinates z and $\bar{z}$ on the complex plane and interprete the result geometrically.

5) Primary fields

A primary field $\phi(z, \bar{z})$ is a tensor field under conformal transformations $z \rightarrow z'$, $\bar{z} \rightarrow \bar{z}'$ in the sense that

$$\phi(z, \bar{z}) \rightarrow \phi'(z, \bar{z}) = \left(\frac{\partial z'}{\partial z}\right)^h \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{\bar{h}} \phi(z'(z), \bar{z}'(\bar{z})), \tag{40}$$

where $z', \bar{z}'$ denote arbitrary holomorhic functions, i.e., not necessarily the inverse maps of the cylinder to the complex plane used in the previous problem.

a) How does a primary field with conformal weights $(h, \bar{h})$ transform under dilatations $z \rightarrow e^{\lambda} z$ and rotations $z \rightarrow e^{i\theta} z$ with $\lambda, \theta \in \mathbf{R}$?

b) Consider the infinitesimal conformal transformation

$$z' = z + \xi(z), \qquad \bar{z}' = \bar{z} + \bar{\xi}(\bar{z}). \tag{41}$$

Show that the infinitesimal transformation of a primary field is given by

$$\delta_{\xi,\bar{\xi}}\phi(z, \bar{z}) = (h(\partial_z \xi) + \bar{h}(\partial_{\bar{z}}\bar{\xi}) + \xi\partial_z + \bar{\xi}\partial_{\bar{z}})\phi(z, \bar{z}). \tag{42}$$