Introduction to String Theory
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Assignment # 8

Due: July 10, 2006

1) The OPE of the energy momentum tensor

In the complex plane, the Virasoro generators L_n are given by

$$L_n = \oint_{C_0} \frac{dz}{2\pi i} z^{n+1} T(z). \tag{1}$$

a) Show that

$$[L_n, L_m] = \oint_{C_0} \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{n+1} w^{m+1} T(z)T(w). \tag{2}$$

Here, C_0 denotes a contour about $w = 0$, C_w is a contour about $z = w$, and, as usual, the product $T(z)T(w)$ is meant to be the radially ordered product:

$$T(z)T(w) \equiv R(T(z) \cdot T(w)) = \left\{ \begin{array}{ll} T(z) \cdot T(w) & \text{for } |z| > |w| \\ T(w) \cdot T(z) & \text{for } |w| > |z| \end{array} \right\}., \tag{3}$$

where the dot stands for the ordinary operator product. (Hint: Write the commutator as a difference of two double contour integrals and use a contour deformation of the dz integration for fixed w, just as was done in lecture for $\delta_\xi \phi(w) = [T_\xi, \phi(w)]$.)

b) Use (2) and the (radially ordered) operator product

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \text{ (finite terms) },\tag{4}$$

as well as the Cauchy-Riemann formula,

$$\oint_{C_w} \frac{dz}{2\pi i} \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{(n-1)}(w), \tag{5}$$

to rederive the quantum Virasoro algebra:

$$[L_n, L_m] = \frac{c}{12}(n+1)n(n-1)\delta_{m+n} + (n-m)L_{n+m}. \tag{6}$$

c) Use (4) and the general formula for infinitesimal conformal transformations

$$\delta_\xi \phi(w) = [T_\xi, \phi(w)], \tag{7}$$

where

$$T_\xi = \oint_{C_0} \frac{dz}{2\pi i} \xi(z) T(z), \tag{8}$$

to show that

$$\delta_\xi T(z) = \frac{c}{12} \partial^3 \xi(z) + 2\partial\xi(z) T(z) + \xi(z) \partial T(z). \tag{9}$$

d) Using (9) and the infinitesimal transformation property of a (chiral) primary field [1] derived in Problem 5b) of Assignment # 7, explain why $T(z)$ is not primary.

e) Even though $T(z)$ does not transform as a tensor under arbitrary (infinitesimal) conformal transformations, it does so for those transformations for which $\partial^3\xi(z) = 0$. Hence, for these transformations, it makes sense to assign $T(z)$ a conformal weight h. Read off h from (9).

2) Fractional linear transformations

The group $SL(2, \mathbf{R})$ of (2×2)-matrices of unit determinant acts on the Riemann-sphere (i.e. on $\mathbf{C} \cup \{\infty\}$) by so-called fractional linear transformations:

$$z \to z' = \frac{az + b}{cz + d}, \tag{10}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R}). \tag{11}$$

a) Show that two succesive fractional linear transformations,

$$z \to z' = \frac{az + b}{cz + d}, \qquad z' \to z'' = \frac{ez' + f}{gz' + h}, \tag{12}$$

are equivalent to one fractional linear transformation

$$z \to z'' = \frac{jz + k}{lz + m}, \tag{13}$$

where the matrix

$$\begin{pmatrix} j & k \\ l & m \end{pmatrix} \in SL(2, \mathbf{R}). \tag{14}$$

is the product of the two $SL(2, \mathbf{R})$ matrices that correspond to the single transformations $z \to z'$ and $z' \to z''$.

b) Show that the fractional linear action of the inverse matrix of (11) on z' leads back to z, and hence corresponds to the inverse transformation $z' \to z$.

[1] A *chiral* field is a field that only depends on z and not on $\bar{z}$. A *chiral primary* field is a primary field that likewise only depends on z and that has conformal weight $\bar{h} = 0$ (but in general $h \neq 0$). $T(z)$ is obviously chiral, but, as you show above, not primary.