Introduction to String Theory
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Assignment # 9

Due: July 17, 2006

1) Asymptotic in and out states

A (chiral) primary field $\phi(z)$ with conformal weight h in the complex plane has the following mode expansion:

$$\phi(z) = \sum_{n \in \mathbf{Z}} z^{-n-h} \phi_n. \tag{1}$$

As $z \to 0$ corresponds to $\tau \to -\infty$ on the cylinder, a state of the form

$$|\phi_{in}\rangle := \lim_{z \to 0} \phi(z)|0\rangle \tag{2}$$

can be considered an asymptotic "in-state". For this state to be non-singular, we need to impose (cf. eq. (1))

$$\phi_n|0\rangle = 0 \qquad \forall n \geq 1 - h. \tag{3}$$

a) Verify that, if (3) is satisfied, we have

$$|\phi_{in}\rangle = \phi_{-h}|0\rangle, \tag{4}$$

with

$$\phi_{-h} = \oint_{C_0} \frac{dz}{2\pi i} \frac{\phi(z)}{z} \tag{5}$$

being the coefficient ϕ_n for $n = -h$.

b) Using

$$[L_n, \phi_m] = [n(h-1) - m]\phi_{n+m} \tag{6}$$

and

$$L_n|0\rangle = 0 \qquad \forall n \geq -1, \tag{7}$$

show

$$\begin{aligned} L_0|\phi_{in}\rangle &= h|\phi_{in}\rangle && (8)\\ L_n|\phi_{in}\rangle &= 0, \qquad n > 0, && (9) \end{aligned}$$

i.e., that $|\phi_{in}\rangle$ is a highest weight state of a representation of the Virasoro algebra (alias "Verma module").

Remark: The primary operators of a CFT are obviously in one-to-one correspondence with highest weight states of Verma modules. As was explained in lecture, a full Verma module is generated by acting with the L_{-n} $(n > 0)$ on a lowest weight state. The states so-generated are called descendant states. They can be generated directly from the vacuum by acting with descendant fields, which are operators that occur in (possibly multiple) operator products of the primary field with the energy momentum tensor.

c) For a non-chiral primary field $\phi(z,\bar{z})$ with conformal weights $(h,\bar{h})$, one defines in a similar way:

$$|\phi_{in}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle = \phi(0,0)|0\rangle, \tag{10}$$

which results in

$$\overset{(-)}{L}_0|\phi_{in}\rangle = \overset{(-)}{h}\,|\phi\rangle \tag{11}$$

$$\overset{(-)}{L}_n|\phi_{in}\rangle = 0 \qquad \forall n>0. \tag{12}$$

From this, read off what extra condition has to be imposed on the primary operator $\phi(z,\bar{z})$ for the state $|\phi_{in}\rangle$ to be *physical* in the sense of the old covariant quantization of the closed string. (I.e., we assume that $\phi(z,\bar{z})$ is a primary operator in the conformal field theory that describes the quantized closed bosonic string).

Remark: Primary operators that satisfy the additional constraint to be found in part c) are called vertex operators. They can be used to create physical in and out states from the vacuum. In string theory, scattering amplitudes of asymptotic in and out states are thus determined by vacuum expectation values of products of the vertex operators that correspond to those in and out states. The vertex operator for the closed string tachyon, for example, is given by $:e^{ik_\rho X^\rho(z,\bar{z})}:$ with $k^2=2$, whereas the massless graviton, Kalb-Ramond field and dilaton correspond to particular linear combinations of the vertex operators $:\partial X^\mu(z)\bar{\partial}\bar{X}^\nu(\bar{z})e^{ik_\rho X^\rho(z,\bar{z})}:$ with $k^2=0$. To verify that these are primary operators and that they satisfy the additional requirement of part c), one has to calculate the operator product with the (normal-ordered) energy momentum tensor using Wick's theorem for products of normal-ordered operators (cf. the lecture).

2) The closed string on S^1: Mass shell and level matching condition

Compactifying the closed bosonic string on a circle, leads to a quantization of the momentum in the compactified dimension, $p^{25}=\frac{M}{R}$ $(M\in\mathbf{Z})$, and introduces the string winding number $L\in\mathbf{Z}$ as a new (topological) quantum number.

Using

$$\alpha_0^{25} = \frac{M}{R} - \frac{RL}{2} \tag{13}$$

$$\bar{\alpha}_0^{25} = \frac{M}{R} + \frac{RL}{2} \tag{14}$$

as well as

$$L_0 = \frac{1}{2}\alpha_0^M\alpha_{0M} + N \tag{15}$$

$$\bar{L}_0 = \frac{1}{2}\bar{\alpha}_0^M\bar{\alpha}_{0M} + \bar{N} \tag{16}$$

where $M,N,\ldots = 0,1,2,\ldots,25$, derive the mass shell condition

$$m^2_{(25D)} = \frac{M^2}{R^2} + \frac{1}{4}L^2R^2 + N + \bar{N} - 2 \tag{17}$$

for the 25D mass squared, as well as the level matching condition

$$N - \bar{N} = ML \tag{18}$$

from the physical state conditions

$$(L_0 + \bar{L}_0 - 2)|\text{phys}\rangle = 0, \qquad (L_0 - \bar{L}_0)|\text{phys}\rangle = 0. \tag{19}$$

3) The bosonic string on $\mathbf{S}^1$: The spectrum

As was shown in lecture, the sector with vanishing Kaluza-Klein momentum and zero winding number (i.e., the sector with $M = L = 0$), simply corresponds to the ordinary field theoretic zero-modes of the 26D fields when they are dimensionally reduced on the circle. In particular, the sector with $N = \bar{N} = 0$ describes the 25D remnant of the 26D tachyon, whereas the sector with $N = \bar{N} = 1$ describes the 25D massless fields that descend from the massles 26D fields G_{MN} (the metric), B_{MN} (the Kalb-Ramond field) and Φ (the dilaton). Under this dimensional reduction, the 26D metric G_{MN} decomposes into the 25D metric $G_{\mu\nu}$ ($\mu, \nu, \ldots = 0, 1, 2, \ldots, 24$), a 25D vector field $G_{\mu,25}$ and a 25D scalar $G_{25,25}$, whereas the two-form B_{MN} gives rise to a 25D two-form $B_{\mu\nu}$ and a 25D vector $B_{\mu,25}$, and the dilaton Φ simply leads to a 25D scalar. The vacuum expectation value $\langle G_{25,25}\rangle \sim R$ of the scalar $G_{25,25}$ describes the (dynamically undetermined) size of the circle, and the two vector fields gauge two $U(1)$'s.

A prime example for truely stringy states without a point particle analogue, on the other hand, is given by the states with $(M, L) = (\pm 1, \pm 1)$ and $(M, L) = (\pm 1, \mp 1)$. For these states, the level matching condition (18) implies $N - \bar{N} = 1$ and $N - \bar{N} = -1$, respectively. The lowest lying modes in this sector correspond to $(N, \bar{N}) = (1, 0)$ and $(N, \bar{N}) = (0, 1)$, respectively. As was shown in lecture, each of these two cases leads to two 25D vector fields and two 25D scalars with a radius dependent mass

$$m^2_{(25D)} = \frac{1}{R^2} + \frac{R^2}{4} - 1 \geq 0. \tag{20}$$

For $R = \sqrt{2} = \sqrt{\alpha'}$, these states become massless, and the four vector fields combine with $G_{\mu,25}$ and $B_{\mu,25}$ to fill out the adjoint representation of $SU(2)_L \times SU(2)_R$, which is Higgsed to $U(1)_L \times U(1)_R$ by the scalar field $G_{25,25}$ at generic values $R \neq \sqrt{\alpha'}$.

In this excercise, we take a closer look at some of the other states in the spectrum that were not yet discussed in the lecture.

a) Show that in the sectors with $(M, L) = (\pm 1, \pm 1)$ and $(M, L) = (\pm 1, \mp 1)$ and $(N \cdot \bar{N}) > 0$, there can be no massless or tachyonic states, no matter how the radius R is chosen.

b) Consider now the sector with $|ML| > 1$. Can there be states in this sector that can become massless or tachyonic at some particular values of R? If yes, give the corresponding values of R.

c) Consider now the sector $M \neq 0$ and $L = 0$. What is the constraint on the occupation numbers N and $\bar{N}$ for these states?

d) Show that for any given $M \neq 0$ with $L = 0$ and $N = \bar{N} = 0$, all three cases (tachyonic, massless, massive) can be realized by dialing R appropriately. What is the mass squared value for the special radius $R = \sqrt{2} = \sqrt{\alpha'}$?

e) Show that, for $M \neq 0$, $L = 0$ and $N = \bar{N} \geq 1$ ($R < \infty$), there can only be massive states.

f) Repeat parts c) through e) for the states of the form $M = 0$, $L \neq 0$.