Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM

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New formulations for scattering amplitudes
Ludwig Maximilians Universität, München
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work with J. Plefka, D. Müller, C. Vergu (1509.05403);
and with A. Garus, M. Rosso (in progress)

Integrability $\Longleftrightarrow$ Yangian Symmetry

Spectrum of Local Operators:
symmetry broken by boundaries (annulus)

Scattering Amplitudes /
Null Polygonal Wilson Loops:
symmetry broken by IR/UV divergences

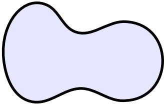
Smooth Maldacena–Wilson loops:
finite observable with disc topology!

I. Yangian Symmetry and Wilson Loops

Maldacena–Wilson Loops

Can define finite Wilson loops in $\mathcal{N}=4$ SYM: Couple scalars [Maldacena hep-th/9803002]

$$W = \mathrm{P}\exp\int_0^1 \left(A_\mu dx^\mu + \Phi_m q^m d\tau\right).$$



Maldacena–Wilson loops where $|dx| = |q|d\tau$:

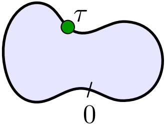
- path is non-null in 4D;
- path is null in 10D (4 spacetime + 6 internal);
- locally supersymmetric object;
- no perimeter divergence (perimeter is null).

Yangian symmetry: [Müller, Münkler Plefka, Pollok, Zarembo]

- finite observable: could make meaningful statements;
- requires superconformal transformations; best done in superspace;
- Yangian symmetry demonstrated at leading order in θ's;
- symmetry up to subtleties regarding boundary terms.

Conformal and Yangian Symmetry

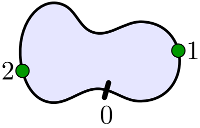
Conformal action (level-zero Yangian) by path deformation.
Action equivalent to Wilson line with single insertion



$$\mathrm{J}^k W = \int d\tau \, W[1,\tau] \, \mathrm{J}^k A(\tau) \, W[\tau,0].$$

Level-one Yangian action: bi-local insertion follows coproduct

$$\widehat{\mathrm{J}}^k W = f^k_{mn} \iint_{\tau_2>\tau_1} d\tau_1 \, d\tau_2 \, W[1,\tau_2] \, \mathrm{J}^m A(\tau_2) \, W[\tau_2,\tau_1] \, \mathrm{J}^n A(\tau_1) \, W[\tau_1,0].$$



Yangian is symmetry if (higher levels follow)

$$\left\langle \mathrm{J}^k \operatorname{Tr} W \right\rangle = 0, \qquad \left\langle \widehat{\mathrm{J}}^k \operatorname{Tr} W \right\rangle = 0.$$

Important issue: Yangian normally does not respect cyclicity.

Open Questions

How about full superspace (all orders in θ)?

How about boundary terms?

Difficulties:

[NB, Müller Plefka, Vergu]

How to deal with boundary terms?

How about regularisation and local terms?

Is the action consistent with the constraints?

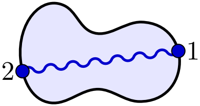
Does the Yangian algebra close (and how)?

II. Yangian Invariance

Conformal Symmetry of Wilson Loops

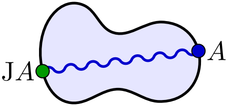
Wilson loop expectation value at order $\mathcal{O}(g^2)$:

$$\langle \operatorname{Tr} W \rangle \sim \int\int \langle A_1\, A_2 \rangle.$$



Conformal action on propagator is non-trivial

$$\begin{aligned} \mathrm{J}^k \langle A_1\, A_2 \rangle &= \langle \mathrm{J}^k A_1\, A_2 \rangle + \langle A_1\, \mathrm{J}^k A_2 \rangle \\ &= d_1 H^k_{12} + d_2 H^k_{21}. \end{aligned}$$



Total derivatives: cancel on closed Wilson loop; finite in IR limit.

Conformal action: Two choices (note $A = dX^A A_A(X)$)

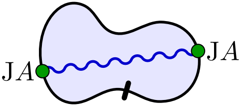
$$\mathrm{J}A := d(\mathrm{J}X^A)A_A + dX^A\, \mathrm{J}X^B\, \partial_B A_A \qquad \to \text{simpler results,}$$
$$\mathrm{J}A := dX^A\, \mathrm{J}X^B\, F_{BA} \qquad \to \text{gauge covariant;}$$

differ by total derivative $D(\mathrm{J}x^A A_A)$; equivalent for closed Wilson loop.

Yangian Symmetry of Wilson Loops

Yangian level-one action on propagator:

$$\widehat{\mathrm{J}}^k \big\langle A_1\, A_2 \big\rangle = f^k_{mn} \big\langle \mathrm{J}^m A_1\, \mathrm{J}^n A_2 \big\rangle.$$



Two alternative actions $\mathrm{J}A$ yield substantially different results:

first, simple definition

$$\widehat{\mathrm{J}}^k \big\langle A_1\, A_2 \big\rangle = d_1 \hat{H}^k_{12} - d_2 \hat{H}^k_{21}.$$

- not sufficient for cancellation on Wilson loops, bulk-boundary terms remain,
- definition not cyclic,
- definition not gauge invariant.

second, covariant definition

$$\widehat{\mathrm{J}}^k \big\langle A_1\, A_2 \big\rangle = d_1 d_2 R^k_{12}.$$

- sufficient for cancellation on Wilson loops, bulk and boundary terms remain,
- definition gauge invariant,
- definition cyclic.

Yangian Invariance of Propagator

Level-one generators almost annihilate gauge propagator $\langle A_1 A_2\rangle$.
Proof: consider instead $\langle dA_1 A_2\rangle$; "integration by parts" on $\mathrm{J}F_1$

$$\begin{aligned}
\widehat{\mathrm{J}}^k \big\langle dA_1 A_2 \big\rangle &= f^k_{mn} \big\langle \mathrm{J}^m F_1\, \mathrm{J}^n A_2 \big\rangle \\
&= f^k_{mn} \mathrm{J}^m \big\langle F_1\, \mathrm{J}^n A_2 \big\rangle - f^k_{mn} \big\langle F_1\, \mathrm{J}^m \mathrm{J}^n A_2 \big\rangle \\
&= f^k_{mn} \mathrm{J}^m \big\langle F_1\, (\mathrm{J}^n X \cdot F)_2 \big\rangle - \tfrac{1}{2} f^k_{mn} \big\langle F_1\, [\mathrm{J}^m, \mathrm{J}^n] A_2 \big\rangle = 0.
\end{aligned}$$

- first term zero due to conformal symmetry,
- second due to $f^k_{mn} f^{mn}_l = f^k_{mn}\, \mathrm{J}^m X^A\, \mathrm{J}^n X^B\, F_{AB} = 0$ ($\mathcal{N} = 4!$).

Therefore action on gauge propagator yields double total derivative:

$$\widehat{\mathrm{J}}^k \big\langle A_1 A_2 \big\rangle = d_1 d_2 R^k_{12}.$$

Scalar fields as components of superspace field strength $\Phi \in F$:

$$\widehat{\mathrm{J}}^k \big\langle \Phi_1 A_2 \big\rangle = \widehat{\mathrm{J}}^k \big\langle \Phi_1 \Phi_2 \big\rangle = 0.$$

Yangian Invariance of Wilson Loop

Action on Wilson line leaves a local contribution:

$$\int\int_{1<2} d_1 d_2 R^k_{12} = \int_2 (d_2 R^k_{12})|_{1=2} - \int_2 d_2 R^k_{02} = \int_2 (d_2 R^k_{12})|_{1=2}.$$

Need to adjust local action of Yangian

$$\widehat{\mathrm{J}}^k W = \widehat{\mathrm{J}}^k_{\text{bi-local}} W + \int d\tau\, W[1,\tau]\, \widehat{\mathrm{J}}^k A(\tau)\, W[\tau,0],$$

$$\widehat{\mathrm{J}}^k A_1 = -(d_1 R^k_{21})|_{2=1}.$$

Wilson loop expectation value at $\mathcal{O}(g^2)$ is **Yangian invariant!**

However, not paid attention to divergences yet...

Level-One Momentum Generator

Compute R for some explicit level-one generator.
Level-one momentum (dual conformal) $\widehat{\mathrm{P}}$ easiest:

$$\widehat{\mathrm{P}} \simeq \mathrm{P} \wedge \mathrm{D} + \mathrm{P} \wedge \mathrm{L} + \mathrm{Q} \wedge \bar{\mathrm{Q}}.$$

For level-one momentum $\widehat{\mathrm{P}}$ find explicitly (mixed chiral propagator):

$$\widehat{\mathrm{P}}_\mu \big\langle A_1^+ \, A_2^- \big\rangle \sim d_1 d_2 \, \frac{1}{(x_{12} - i\theta_{12}\bar{\theta}_{12})^\mu} \, .$$

Regularisation and renormalisation:

- Local term divergent $(d_2 R_{21}^{\mathrm{P}})|_{2=1} \sim \epsilon^{-2}$ for cut-off ϵ.
- Need to renormalise local and boundary part of Yangian action.
- Proper cancellation after renormalisation.

What now?

Still invariant at higher orders?

Are there other invariant objects?

Can we prove the symmetry in general?

III. Yangian Symmetry?

Would like to show:

$$\widehat{\mathrm{J}}\,\mathcal{S} = 0$$

Invariance of the Action

Aim: Show Yangian invariance of the (planar) action. [NB, Garus, Rosso (in progress)]

How to apply $\widehat{\mathrm{J}}$ to the action $\mathcal{S}$?

- distinction of planar and non-planar parts not evident
- which representation: free, non-linear, quantum?

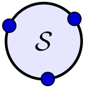
Hints:

- Propagator is Yangian invariant (up to gauge artefacts)
- OPE of (invariant) Wilson loops contains Lagrangian $\mathcal{L}$
- Lagrangian is a sequence of fields

Use above (classical, non-linear) representation!

Essential features of the action:

- action is single-trace (disc topology)
- action is conformal (required for cyclicity)
- action is not renormalised (no anomalies)



Equations of Motion

Application on the action needs extra care.
Consider the equations of motion instead:

$$\widehat{\mathrm{J}}(\text{e.o.m.}) \overset{?}{\sim} \text{e.o.m.}$$

Need this for consistency! Quantum formalism usually on-shell ...

Dirac equation is easiest:

$$D{\cdot}\Psi + [\Phi, \bar{\Psi}] = \partial{\cdot}\Psi + i[A, \Psi] + [\Phi, \bar{\Psi}] = 0.$$

Bi-local action of $\widehat{\mathrm{J}}^k$ on the Dirac equation:

$$i f^k_{mn} \{\mathrm{J}^m A, \mathrm{J}^n \Psi\} + f^k_{mn} \{\mathrm{J}^m \Phi, \mathrm{J}^n \bar{\Psi}\} \overset{?}{=} 0.$$

Many terms cancel, however, some have no counterparts.

Local Terms in Yangian Action

How to make equations of motion invariant?
Not yet specified action on single fields. Level-one momentum $\widehat{\mathrm{P}}$:

$$
\begin{aligned}
&\widehat{\mathrm{P}}_{\dot{\alpha}\beta} A_{\dot{\gamma}\delta} \sim \varepsilon_{\dot{\alpha}\dot{\gamma}} \varepsilon_{\beta\delta} \{\Phi^{ef}, \bar{\Phi}_{ef}\},\\
&\widehat{\mathrm{P}}_{\dot{\alpha}\beta} \Psi_{\gamma}{}^{d} \sim \varepsilon_{\beta\gamma} \{\Phi^{de}, \bar{\Psi}_{e\dot{\alpha}}\},\\
&\widehat{\mathrm{P}}_{\dot{\alpha}\beta} \Phi^{cd} = 0.
\end{aligned}
$$

All terms cancel properly. Dirac equation Yangian-invariant!

$$
D\cdot\widehat{\mathrm{P}}\Psi + i[\widehat{\mathrm{P}}A, \Psi] + [\Phi, \widehat{\mathrm{P}}\bar{\Psi}] + f^{\mathrm{P}}_{mn}\big[i\{\mathrm{J}^m A, \mathrm{J}^n \Psi\} + \{\mathrm{J}^m \Phi, \mathrm{J}^n \bar{\Psi}\}\big] \stackrel{!}{=} 0.
$$

- shown invariance for all equations of motion: Φ, Ψ and A;
- other Yangian generators $\widehat{\mathrm{J}}^k$ follow from algebra.

$$\boxed{\textbf{Equations of Motion Yangian-invariant!}}$$

Issues of Lagrangian

Invariance of e.o.m. not sufficient for correlation functions (sources).

Would like to show invariance of action $\mathcal{S} = \int dx^4 \, \mathcal{L}$

$$\widehat{\mathrm{J}} \, \mathcal{L} \overset{?}{=} \partial_\mu \widehat{\mathcal{K}}^\mu .$$

Difficulties:

- cyclicity: where to cut open trace?
  conformal symmetry should help...
- non-linearity: how to deal with terms of different length (2,3,4)?
  terms of given length not gauge invariant (∂ vs. A).
- complexity: many terms, spinor algebra, signs, traces, ...?
- should we use equations of motion? rather not ...
- attempts to write proper statement failed.

Revisit this issue later.

Invariance of the Action Equivalent

How to express Yangian symmetry of theory?
Reconsider conformal symmetry of action, write as variations:

$$0 = \mathrm{J}\mathcal{S} = \int dx\,\mathrm{J}\phi_a(x)\,\frac{\delta\mathcal{S}}{\delta\phi_a(x)}\,.$$

Vary statement again $\delta\mathrm{J}\mathcal{S} = 0$:

$$0 = \int dx\,\delta\phi_c\,\frac{\delta(\mathrm{J}\mathcal{S})}{\delta\phi_c} = \int dx\,\delta\phi_c\left[\mathrm{J}\phi_a\,\frac{\delta^2\mathcal{S}}{\delta\phi_c\,\delta\phi_a} + \frac{\delta(\mathrm{J}\phi_a)}{\delta\phi_c}\,\frac{\delta\mathcal{S}}{\delta\phi_a}\right].$$

Invariance of e.o.m. follows:

$$\mathrm{J}\,\frac{\delta\mathcal{S}}{\delta\phi_c} + \frac{\delta(\mathrm{J}\phi_a)}{\delta\phi_c}\,\frac{\delta\mathcal{S}}{\delta\phi_a} = 0.$$

- Stronger statement than invariance of e.o.m.: no use of e.o.m.!
- $\delta\mathrm{J}\mathcal{S} = 0$ practically equivalent to $\mathrm{J}\mathcal{S} = 0$. (constant term in $\mathrm{J}\mathcal{S}$?!)
- Extra term related to adding sources $\check{\phi}^a$: $\mathcal{S}_{\mathrm{src}} = \int dx\,\phi_a\check{\phi}^a$.

Exact Invariance of the E.o.M.

How to interpret above statement?

- Invariance of action is generating functional for exact invariance of the equations of motion.
- exact invariance sufficient to derive invariance of quantum correlators (up to anomalies).
- no traces, no cyclicity issues!
- Can apply to Yangian.

Take inspiration from formal variation. **Magic Identity:**

$$\widehat{\mathrm{J}}^k \frac{\delta \mathcal{S}}{\delta \phi_c} + \frac{\delta \mathcal{S}}{\delta \phi_a} \frac{\delta(\widehat{\mathrm{J}}^k \phi_a)}{\delta \phi_c} \pm f^k_{mn} \left((\mathrm{J}^m \phi_a) \frac{\delta}{\delta \phi_a} \wedge \frac{\delta \mathcal{S}}{\delta \phi_b} \frac{\delta}{\delta \phi_c} \right) (\mathrm{J}^n \phi_b) = 0.$$

Reduces to previous statement $\widehat{\mathrm{J}}^k(\mathrm{e.o.m.}) \sim \mathrm{e.o.m.}$.

$\boxed{\text{Proper definition of integrability!}}$

IV. Implications

Higher Loops

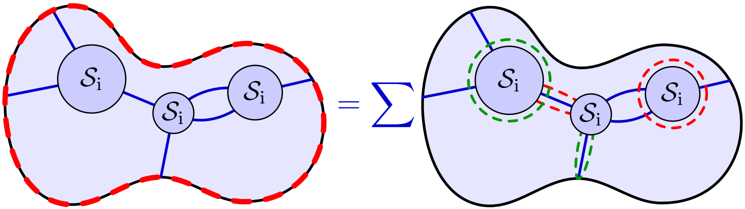
Above proof of Yangian symmetry is:

- classical, but
- non-linear in the fields.

How about quantum effects? We know:

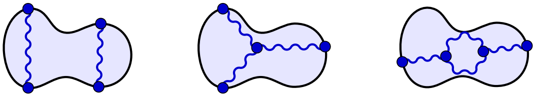
- variation of action is *exactly* invariant: $\delta\mathcal{S}$;
- propagators are *linearly* invariant: $1/\mathcal{L}_0$ (up to gauge);
- interaction vertices are *not* invariant: $\mathcal{S}_\mathrm{i} = \mathcal{S} - \mathcal{S}_0$.

Non-linear composition in Feynman diagrams is invariant!



Non-Linear Cancellations

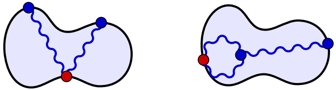
Example: conformal symmetry at $\mathcal{O}(g^4)$. Three diagrams (II, Y, Φ):



Conformal invariance requires all of them:

- conformal symmetry acts on perimeter (linearly and non-linearly);
- vertices and propagators are conformal;
- linear equation of motion contracts a propagator to a point;
- non-linear action cancels other end of propagator;
- residual gauge terms shift along perimeter; cancel by themselves.

Cancellation at interface terms: V (II$\leftrightarrow$Y), collapsed Y (Y$\leftrightarrow\Phi$)

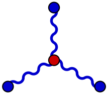


Exact invariance of e.o.m. sufficient, but all e.o.m. terms needed!

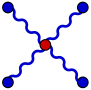
Correlators of Fields

Try to derive some implication of Yangian symmetry concretely.
Consider correlators of fields:

$$\langle A_1\, A_2\, A_3 \rangle =$$



$$\langle A_1\, A_2\, A_3\, A_4 \rangle =$$



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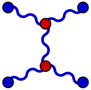

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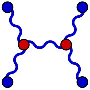


Yangian symmetry should imply some relationships.

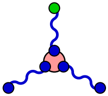
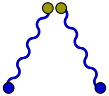
Notes:

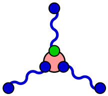
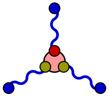
- restrict to planar / colour-ordered contributions;
- similar to scattering amplitudes, but in position space and off-shell;
- avoids complications/singularities due to mass shell condition;
- ignore gauge artefacts (total derivative terms to correlators).

Conformal Symmetry of 3-Field Correlator

Start simple: non-linear conformal invariance of correlator

$$\begin{aligned}
\mathrm{J}\big\langle A_1\, A_2\, A_3 \big\rangle &= \big\langle \mathrm{J} A_1\, A_2\, A_3 \big\rangle + \big\langle A_1\, \mathrm{J} A_2\, A_3 \big\rangle + \big\langle A_1\, A_2\, \mathrm{J} A_3 \big\rangle \\
&= \big\langle \mathrm{J}_{\mathrm{lin}} A_1\, A_2\, A_3 \big\rangle + \big\langle \mathrm{J}_{\mathrm{nlin}} A_1\, A_2\, A_3 \big\rangle + \ldots
\end{aligned}$$

$=$  $+$  $+ \ldots$

$=$  $+$  $+ \ldots = 0.$

Used $\mathrm{J}_{\mathrm{lin}} S_2 = 0$ and $\delta^4 = (\delta S_2/\delta\phi)G$ to invent a propagator.

Resulting statement equivalent to

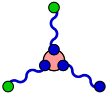
$$\mathrm{J}_{\mathrm{lin}} S_3 + \mathrm{J}_{\mathrm{nlin}} S_2 = 0.$$

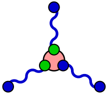
Invariance of action implies invariance of correlator.

Yangian Symmetry of 3-Field Correlator

Non-linear Yangian action on correlator of 3 fields

$$\widehat{\mathrm{J}}\big\langle A_1\,A_2\,A_3\big\rangle = \big\langle \mathrm{J}A_1\,\mathrm{J}A_2\,A_3\big\rangle + \ldots + \big\langle \widehat{\mathrm{J}}A_1\,A_2\,A_3\big\rangle + \ldots$$

$= \ldots +$  $+ \ldots$

$= \ldots +$  $+ \ldots = 0.$

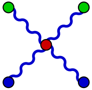
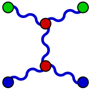
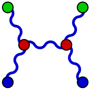
Invariance based on:

- conformal invariance of propagator and 3-vertex,
- Yangian invariance of propagator,
- magic identity for equations of motion.

Can be formulated as identity of 3-vertices: action Yangian-invariant?!

Yangian Symmetry of 4-Field Correlator

Non-linear Yangian action on correlator of 4 fields

$$\widehat{\mathrm{J}}\langle A_1\, A_2\, A_3\, A_4\rangle = \ldots +$$

$$+$$

$$+$$

$$+ \ldots$$
$$= \ldots = 0.$$

Invariance based on:

- conformal invariance of propagator, 3-vertex and 4-vertex,
- Yangian invariance of propagator and 3-vertex,
- magic identity for equations of motion.

New identity for 4-vertices:

- action almost certainly Yangian-invariant.
- how to formulate invariance of action consistently?

Anomalies?

Classical symmetries may suffer from quantum anomalies:

- Not clear how to deal with anomalies for non-local action (in colour-space not necessarily in spacetime).
- Violation of (non-local) current? Cohomological origin?

However:

- Not an issue for Wilson loop expectation value at one loop.
- Anomalies from interplay of symmetry deformation by regularisation and quantum divergences; $\mathcal{N}=4$ is finite, no anomalies expected.

Gauge Fixing

Yangian action is gauge invariant:
by construction all fields mapped to gauge covariant fields.

What impact does gauge fixing have on Yangian symmetry?

- gauge equations of motion change; introduction of ghosts.
- should consider BRST symmetry:
- how to represent symmetry on unphysical fields and ghosts?

Consider **conformal symmetry** first

- assume all conformal transformations act trivially on ghosts;
- action conformal modulo BRST exact terms;
- equations of motion conformal modulo particular terms; follows from variation of action.

Can apparently proceed like this for **Yangian action**.
Gauge fixing does not seem to disturb Yangian symmetry.

Algebra

Does the algebra close as a Yangian?

Algebra comprises Yangian and gauge transformations, e.g.

$$[\mathrm{J}^k, \mathrm{J}^l] \sim f^{kl}_m \mathrm{J}^m + \mathrm{G}^{kl}.$$

Several terms left over at level one:

$$[\mathrm{J}^k, \widehat{\mathrm{J}}^l] \sim f^{kl}_m \widehat{\mathrm{J}}^m + \widehat{\mathrm{G}}^{kl}$$

with "bi-local gauge transformations" $\widehat{\mathrm{G}}$:

- evidently symmetries of $\mathcal{N} = 4$ SYM,
- also symmetries of all planar conformal theories;
- form an ideal of Yangian algebra, can be divided out.

Serre relations non-linear and very complicated. Not settled…

V. Conclusions

Conclusions

Yangian Invariance of Wilson Loops:

- How to act with Yangian on Wilson loops.
- Yangian symmetry of Wilson loop expectation value at $\mathcal{O}(g^2)$.

Yangian Symmetry of Planar $\mathcal{N} = 4$ SYM:

- Equations of motion & action variation invariant.
- Planar $\mathcal{N} = 4$ SYM integrable.
- Identities for vertices due to Yangian symmetry.
- No anomalies to be expected?!
- Symmetry compatible with BRST gauge fixing.