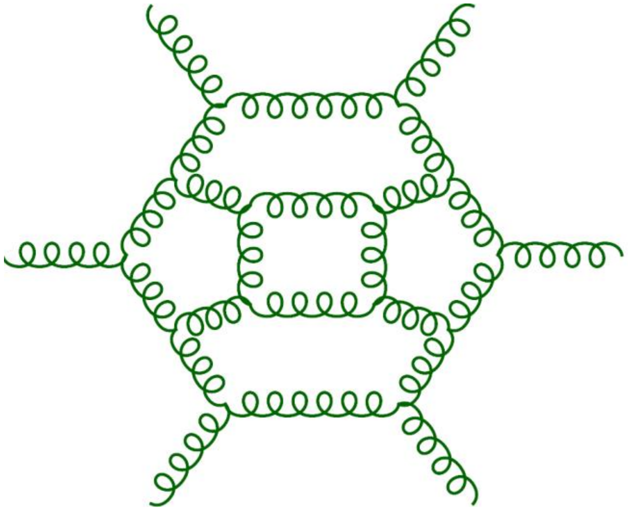
The Steinmann-Assisted Bootstrap at Five Loops



Lance Dixon (SLAC)

S. Caron-Huot, LD, M. von Hippel, A. McLeod,

1609.00669

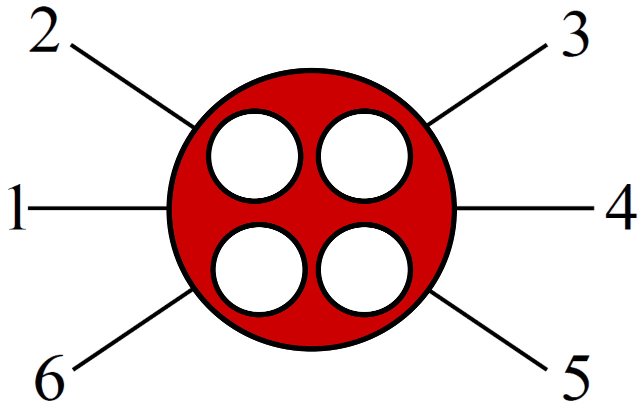
"New Formulations for Scattering Amplitudes"

LMU, Munich 5 Sept., 2016

Hexagon function bootstrap

LD, Drummond, Henn, 1108.4461, 1111.1704;
Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington, 1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
Drummond, Papathanasiou, Spradlin, 1412.3763

Use analytical properties of perturbative amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops



First step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

Outline of program

1. Ansatz for IR finite versions of 6 gluon scattering amplitudes as linear combination of "hexagon functions"
2. NEW: Steinmann relations dramatically reduce size of ansatz at high loop orders!
3. Use precise "boundary value data" to fix constants in ansatz.
4. Cross check.

- Works fantastically well for 6-gluon amplitude, first "nontrivial" amplitude in planar N=4 SYM

  → 5 loops for both MHV = (--++++) and NMHV = (---+++)
- Steinmann constraints can be used for (n>6) point amplitudes too.

Other inputs

Global: Dual superconformal "Descent Equation" or $\bar{Q}$-equation Bullimore, Skinner; Caron-Huot, He (2011)

Boundary data:

- OPE limit Basso, Sever, Vieira (2013,…)
- Multi-Regge-limit Bartels, Lipatov, Sabio-Vera, Schnitzer (2008,…); Basso, Caron-Huot, Sever (2014)
- NMHV multi-particle- factorization limit Bern, Chalmers (1995); LD, von Hippel, 1408.1505; BSV,…
- Self-crossing limit Georgiou, 0904.4675; LD, Esterlis, 1602.02107

BDS Ansatz

Bern, LD, Smirnov, hep-th/0505205

- Captures all IR divergences of amplitude
- Accounts for anomaly in dual conformal invariance due to IR divergences
- **Fails for** *n = 6,7,...*
- Failure (remainder function) is dual conformally invariant

$$\mathcal{A}_n^{\mathrm{BDS}} = \mathcal{A}_n^{\mathrm{tree}} \times \exp\Big[\sum_{l=1}^{\infty} [\frac{\lambda}{8\pi^2}]^l \Big(f^{(l)}(\epsilon) M_n^{(1)}(l\epsilon; s_{ij}) + C^{(l)} + \mathcal{O}(\epsilon)\Big)\Big]$$

constants, indep.of kinematics

all kinematic dependence from 1-loop amplitude

Dual conformal invariance

- Amplitude fixed, up to functions of dual conformally invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2$$

- $x_{i,i+1}^2 = k_i^2 = 0$ → no such variables for n = 4,5

$$u = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2} = \frac{s_{12}s_{45}}{s_{123}s_{345}}$$

n = 6 → precisely 3 ratios:

$$v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$$

$$w = \frac{s_{34}s_{61}}{s_{345}s_{234}}$$

Remainder function, starts at 2 loops

$$\mathcal{A}_6^{\mathrm{MHV}}(\epsilon; s_{ij}) \;=\; \mathcal{A}_6^{\mathrm{BDS}}(\epsilon; s_{ij})\exp[R_6(u,v,w)]$$

BDS-like – better than BDS!

Consider
$$\frac{\mathcal{A}_6^{\rm BDS-like}}{\mathcal{A}_6^{\rm MHV\,(0)}} = \exp\left[\sum_{L=1}^{\infty} a^L \left(f^{(L)}(\epsilon)\frac{1}{2}\hat{M}_6(L\epsilon) + C^{(L)}\right)\right]$$

where

$$\begin{aligned}\hat{M}_6(\epsilon) &= M_6^{1-\rm loop} + Y(u,v,w)\\ &= \sum_{i=1}^{6}\left[-\frac{1}{\epsilon^2}\Big(1-\epsilon\ln(-s_{i,i+1})\Big) - \ln(-s_{i,i+1})\ln(-s_{i+1,i+2}) + \frac{1}{2}\ln(-s_{i,i+1})\ln(-s_{i+3,i+4})\right]\\ &\quad + 6\,\zeta_2\,,\end{aligned}$$

Alday, Gaiotto, Maldacena, 0911.4708

It contains all the IR poles, but **no 3-particle invariants**.

Here

$$\boxed{Y(u,v,w) \equiv \mathrm{Li}_2(1-u) + \mathrm{Li}_2(1-v) + \mathrm{Li}_2(1-w) + \frac{1}{2}\Big(\ln^2 u + \ln^2 v + \ln^2 w\Big)}$$

is the dual conformally invariant part of the one-loop amplitude.

BDS-like normalized amplitude

Define

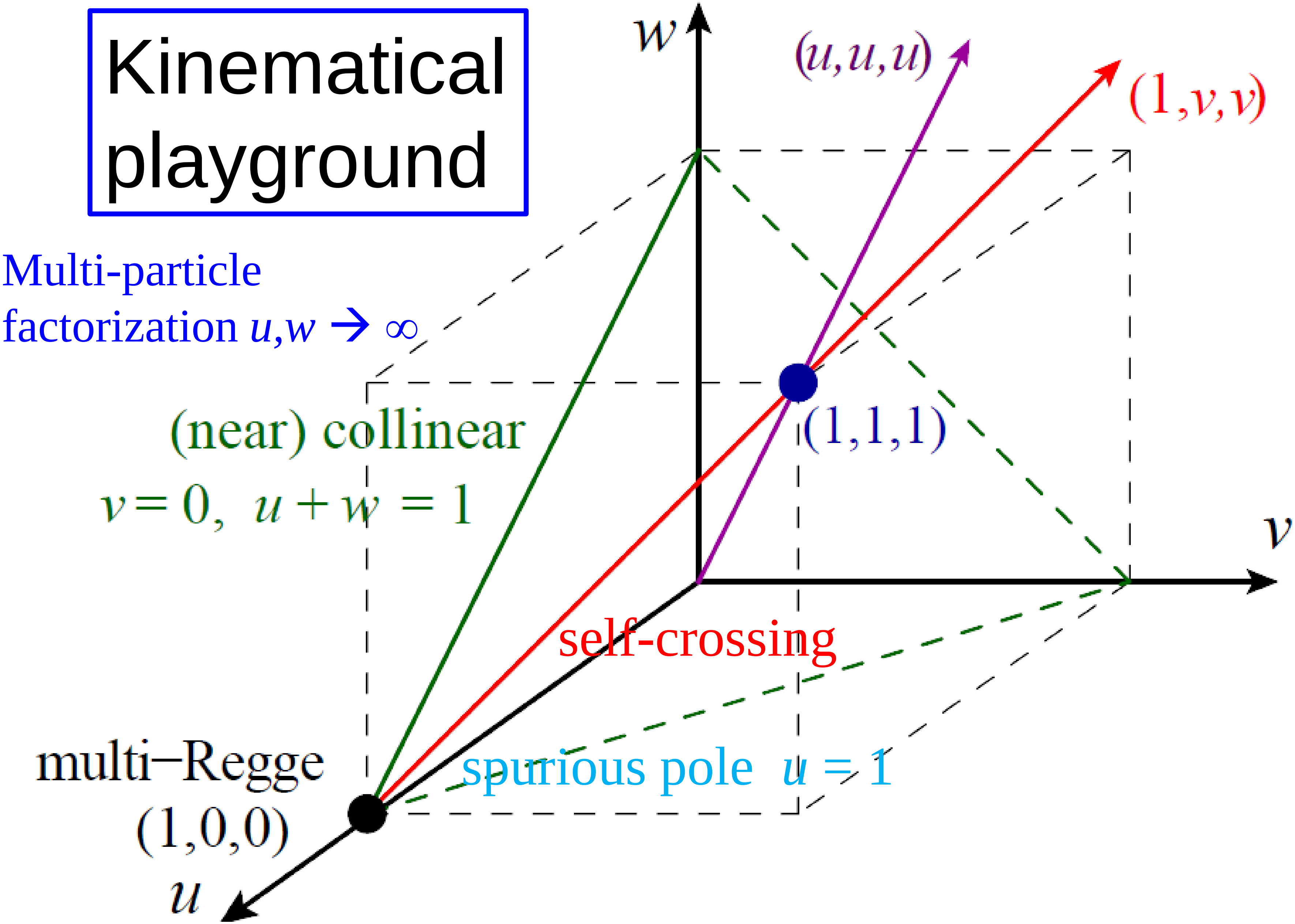
$$\boxed{\frac{\mathcal{A}_6^{\mathrm{MHV}}}{\mathcal{A}_6^{\mathrm{BDS-like}}} \equiv \mathcal{E}(u,v,w) = \exp\left[R_6 - \frac{\gamma_K(a)}{8} Y\right]}$$

where $a = \frac{\lambda}{8\pi^2}$ `t Hooft coupling

$\gamma_K(a) \ = \ 4\, f_0(a)$ cusp anomalous dimension

No 3-particle invariants in denominator of $\mathcal{E}$
→ simpler analytic behavior

Kinematical playground



Basic bootstrap assumption

- MHV: $\mathcal{E}^{(L)}(u,v,w)$ is a linear combination of weight $\mathbf{2}\boldsymbol{L}$ hexagon functions at any loop order $\boldsymbol{L}$
- NMHV: BDS-like normalized super-amplitude

$$\widehat{\mathcal{P}}_{\rm NMHV} \equiv \frac{\mathcal{A}_{\rm NMHV}}{\mathcal{A}_{\rm MHV}^{\rm BDS-like}}$$

Drummond, Henn, Korchemsky, Sokatchev, 0807.1095;
LD, von Hippel, McLeod, 1509.08127

has expansion

$$\begin{aligned}\widehat{\mathcal{P}}_{\rm NMHV} \;=\; \frac{1}{2}\Big[&[(1)+(4)]E(u,v,w)+[(2)+(5)]E(v,w,u)+[(3)+(6)]E(w,u,v)\\ &+[(1)-(4)]\tilde{E}(u,v,w)-[(2)-(5)]\tilde{E}(v,w,u)+[(3)-(6)]\tilde{E}(w,u,v)\Big]\end{aligned}$$

Grassmann-containing dual superconformal invariants, (a) = [bcdef]

$E,\widetilde{E}$ = hexagon functions

Functional interlude

Chen; Goncharov; Brown; …

- Multiple polylogarithms, or n-fold iterated integrals, or weight n pure transcendental functions f.
- Define by derivatives:

$$d\,f = \sum_{s_k \in \mathcal{S}} f^{s_k}\, d\ln s_k$$

S = finite set of rational expressions, "symbol letters", and

$$f^{s_k} \equiv \{n-1, 1\} \text{ coproduct component}$$

are also pure functions, weight n-1

- Iterate: $d\,f^{s_k} \;\Rightarrow\; f^{s_j, s_k} \equiv \{n-2, 1, 1\}$ component

Goncharov, Spradlin, Vergu, Volovich, 1006.5703

- Symbol = {1,1,…,1} component (maximally iterated)

Harmonic Polylogarithms of one variable (HPLs {0,1})

Remiddi, Vermaseren, hep-ph/9905237

- Subsector of hexagon functions.
- Generalize classical polylogs, $\mathrm{Li}_n(u) = \int_0^u \frac{dt}{t} \mathrm{Li}_{n-1}(t)$
- Define by iterated integration:

$$H_{0,\vec{w}}(u) = \int_0^u \frac{dt}{t} H_{\vec{w}}(t), \quad H_{1,\vec{w}}(u) = \int_0^u \frac{dt}{1-t} H_{\vec{w}}(t)$$

- Or by derivatives

$$dH_{0,\vec{w}}(u) = H_{\vec{w}}(u)\ d\ln u \quad dH_{1,\vec{w}}(u) = -H_{\vec{w}}(u) d\ln(1-u)$$

- Symbol letters: $\mathcal{S} = \{u, 1-u\}$

Hexagon symbol letters

- Momentum twistors $Z_i^A,\ i=1,2,\ldots,6$ transform simply under dual conformal transformations. Hodges, 0905.1473
- Construct 4-brackets $\varepsilon_{ABCD} Z_i^A Z_j^B Z_k^C Z_l^D \equiv \langle ijkl \rangle$
- 15 projectively invariant combinations of 4-brackets can be factored into 9 basic ones:

$$\boxed{\mathcal{S} \ = \ \{u, v, w, 1-u, 1-v, 1-w, y_u, y_v, y_w\}}$$

$$u = \frac{\langle 6123\rangle\langle 3456\rangle}{\langle 6134\rangle\langle 2356\rangle} \qquad 1-u = \frac{\langle 6135\rangle\langle 2346\rangle}{\langle 6134\rangle\langle 2356\rangle} \qquad y_u = \frac{\langle 1345\rangle\langle 2456\rangle\langle 1236\rangle}{\langle 1235\rangle\langle 3456\rangle\langle 1246\rangle}$$

+ cyclic

→ A_3 cluster algebra

Golden, Goncharov, Spradlin, Vergu, Volovich, 1305.1617;
Golden, Paulos, Spradlin, Volovich, 1401.6446; Golden, Spradlin, 1411.3289;
Harrington, Spradlin, 1512.07910

Hexagon function symbol letters (cont.)

- y_i not independent of u_i :

$$y_u \equiv \frac{u - z_+}{u - z_-}\ , \ \ldots$$ where

$$z_\pm = \frac{1}{2}\left[-1 + u + v + w \pm \sqrt{\Delta}\right]$$
$$\Delta = (1 - u - v - w)^2 - 4uvw$$

- Function space graded by parity:

$$\boxed{\begin{array}{rcl} i\sqrt{\Delta} & \leftrightarrow & -i\sqrt{\Delta} \\ z_+ & \leftrightarrow & z_- \\ y_i & \leftrightarrow & 1/y_i \\ u_i & \leftrightarrow & u_i \end{array}}$$

Branch cut condition

- All massless particles → all branch cuts start at origin in

$$s_{i,i+1}, \quad s_{i,i+1,i+2}$$

→ Branch cuts all start from 0 or ∞ in

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \qquad \text{or } v \text{ or } w$$

→ First symbol entry $\in \{u, v, w\}$ GMSV, 1102.0062

- Powerful constraint: At weight 8 (four loops) we would have 1,675,553 functions without it; exactly 6,916 with it.
- But this is still way too many! We know now that most of these functions are unphysical.

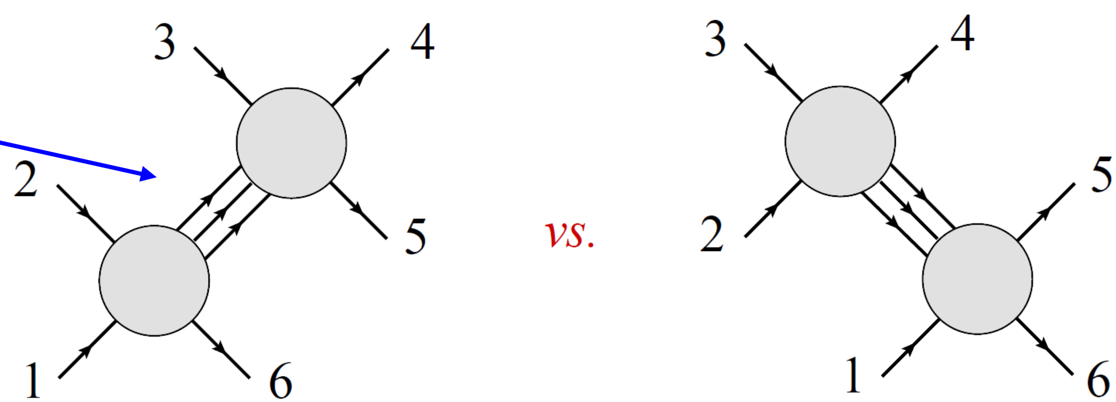
Steinmann relations

Steinmann, Helv. Phys. Acta (1960)

Bartels, Lipatov, Sabio Vera, 0802.2065

- Amplitudes should not have overlapping branch cuts.
- Cuts in 2-particle invariants subtle in generic kinematics
- Easiest to understand for cuts in 3-particle invariants using 3 → 3 scattering:

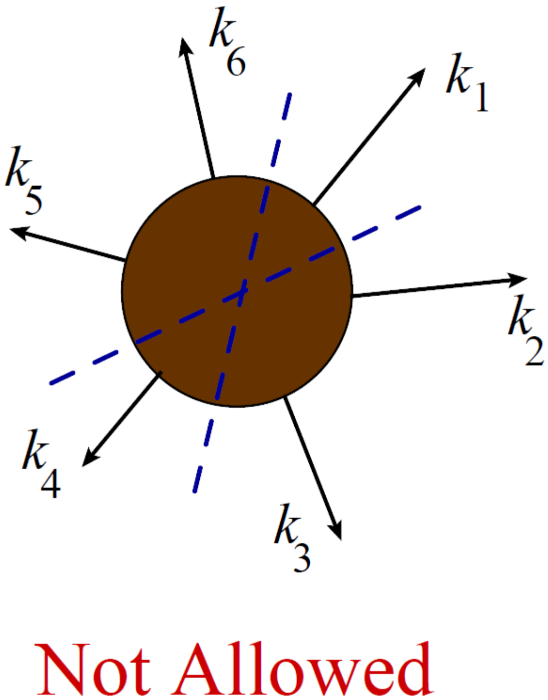
Intermediate particle flow in wrong direction for S_{234} discontinuity

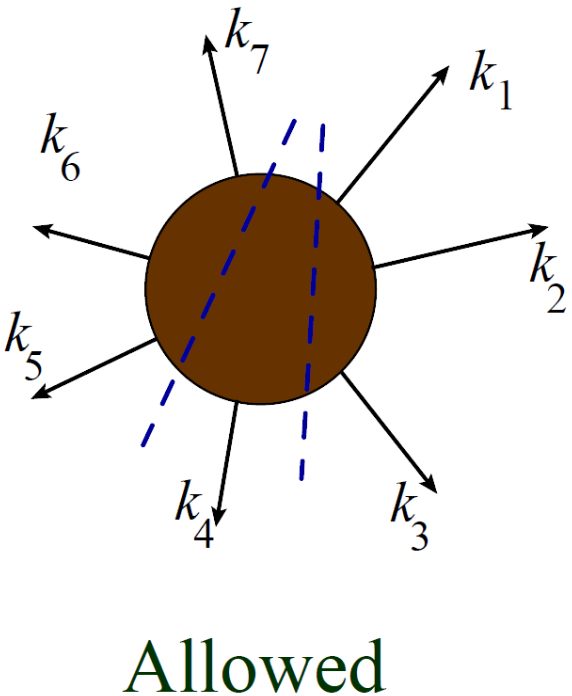


$$\boxed{\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{345}}\mathcal{E}(u,v,w)\right]=0}$$

Steinmann relations (cont.)

- Amplitudes should not have overlapping branch cuts:





$$\boxed{\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right] = 0}$$

Steinmann relations (cont.)

$$\mathrm{Disc}_{s_{234}}\left[\mathrm{Disc}_{s_{123}}\mathcal{E}(u,v,w)\right]=0$$

+ cyclic conditions

$$u=\frac{s_{12}s_{45}}{s_{123}s_{345}}\qquad v=\frac{s_{23}s_{56}}{s_{234}s_{123}}\qquad w=\frac{s_{61}s_{34}}{s_{345}s_{234}}$$

$\ln^2 u$	$\ln^2\frac{uv}{w}$
NO	**OK**

$$\frac{uv}{w}=\frac{s_{12}s_{23}s_{45}s_{56}}{s_{34}s_{61}s_{123}^2}$$

First two entries restricted to 7 out of 9:

$$\mathrm{Li}_2(1-1/u)\qquad \mathrm{Li}_2(1-1/v)\qquad \mathrm{Li}_2(1-1/w)$$

$$\ln^2\frac{uv}{w}\qquad \ln^2\frac{vw}{u}\qquad \ln^2\frac{wu}{v}$$

plus ζ_2

Analogous constraints for $n=7$ [Spradlin, Amps 2016] using $A_7^{\text{BDS-like}}$

Iterative Construction of Steinmann hexagon functions

{n-1,1} coproduct F^x characterizes first derivatives, defines F up to overall constant (a multiple zeta value).

$$\boxed{\left.\frac{\partial F}{\partial u}\right|_{v,w} = \frac{F^u}{u} - \frac{F^{1-u}}{1-u} + \frac{1-u-v-w}{u\sqrt{\Delta}} F^{y_u} + \frac{1-u-v+w}{(1-u)\sqrt{\Delta}} F^{y_v} + \frac{1-u+v-w}{(1-u)\sqrt{\Delta}} F^{y_w}}$$

$$\frac{\partial \ln y_u}{\partial u}$$

1. Insert general linear combinations for F^x
2. Apply "integrability" constraint that mixed-partial derivatives are equal
3. Stay in space of functions with good branch cuts and obeying Steinmann by imposing a few more "zeta-valued" conditions in each iteration.

Very important:

The space of Steinmann hexagon functions is not a ring

The original hexagon function space was a ring:

(good branch cuts) * (good branch cuts) = (good branch cuts)

- But:

(branch cut in s_{234}) * (branch cut in s_{345}) = **[not Steinmann]**

- This fact accounts for the relative paucity of Steinmann functions – very good for bootstrapping!
- In a ring, (crap) * (crap) = (more crap)

A subclass of Steinmann functions

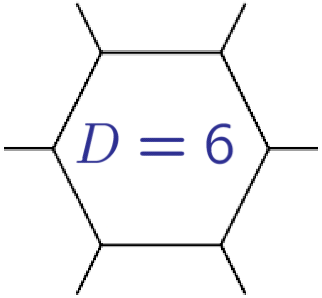
Logarithmic seeds:

$$K_k^u(u, \tfrac{v}{w}) \equiv \frac{1}{2 \cdot k!} \left[\log^k\left(\frac{v}{uw}\right) - \log^k\left(\frac{uv}{w}\right) \right]$$
$$L_k^u(u, \tfrac{v}{w}) \equiv \frac{1}{2 \cdot k!} \left[\log^k\left(\frac{v}{uw}\right) + \log^k\left(\frac{uv}{w}\right) \right]$$

$$\boxed{K_{i,\ldots}^u(u, \tfrac{v}{w}) \equiv \sum_j c_j L_j^u + \int_0^{1/u} \frac{dx}{1-x} \frac{\log^{i-1}(\frac{1}{ux})}{(i-1)!} K_{\ldots}^u(\tfrac{1}{x}, \tfrac{v}{w})}$$

- Similar to definition of HPLs.
- $u = \infty$ base point preserves Steinmann condition
- c_j constants chosen so functions vanish at u=1,
  → no u=1 branch cuts generated in next step.
- K functions exhaust non-y Steinmann hexagon functions

First true (y-containing) hexagon function



$$\Rightarrow \quad \tilde{\Phi}_6(u,v,w)$$

A real integral so it must be Steinmann

- Weight 3, totally symmetric in $\{u,v,w\}$ (secretly Li_3's)
- First parity odd function, so:

$$\tilde{\Phi}_6^u = \tilde{\Phi}_6^v = \tilde{\Phi}_6^w = \tilde{\Phi}_6^{1-u} = \tilde{\Phi}_6^{1-v} = \tilde{\Phi}_6^{1-w} = 0$$

- Only independent {2,1} coproduct:

$$\tilde{\Phi}_6^{y_u} = -\Omega^{(1)}(v,w,u) = -H_2^u - H_2^v - H_2^w - \ln v \ln w + 2\,\zeta_2$$

$$H_2^u = \text{Li}_2(1-u)$$

- Encapsulates first order differential equation found earlier

LD, Drummond, Henn, 1104.2787

Back to physics

- enumerate all Steinmann hexagon functions with weight $2L$
- write most general linear combination with unkown rational-number coefficients
- impose a series of physical constraints until all coefficients uniquely determined

Simple constraints on $\mathcal{E}$ or R_6

- S_3 permutation **symmetry** in $\{u, v, w\}$
- Even under "**parity**":

$$\begin{aligned} i\sqrt{\Delta} &\leftrightarrow -i\sqrt{\Delta} \\ z_+ &\leftrightarrow z_- \\ y_i &\leftrightarrow 1/y_i \end{aligned}$$

- R_6 vanishes in **collinear** limit ($R_6 \rightarrow R_5 = 0$)

$$v \rightarrow 0 \qquad u + w \rightarrow 1$$

Dual superconformal invariance

- Dual superconformal generator $\bar{Q}$ has anomaly due to virtual collinear singularities.
- Structure of anomaly constrains first derivatives of amplitudes → $\bar{Q}$ equation

Caron-Huot, 1105.5606; Bullimore, Skinner, 1112.1056,
Caron-Huot, He, 1112.1060

- General derivative leads to "source term" from $(n+1)$-point amplitude
- For certain derivatives, source term vanishes, leading to homogeneous constraints, good to any loop order

$\bar{\mathrm{Q}}$ equation for MHV

- Constraint on first derivative of $\mathcal{E}$ has simple form
- In terms of the final entry of symbol, restricts to 6 of 9 possible letters:

$$\left\{\frac{u}{1-u}, \frac{v}{1-v}, \frac{w}{1-w}, y_u, y_v, y_w\right\}$$

- In terms of {n-1,1} coproducts, equivalent to:

$$\boxed{\mathcal{E}^{u} + \mathcal{E}^{1-u} = \mathcal{E}^{v} + \mathcal{E}^{1-v} = \mathcal{E}^{w} + \mathcal{E}^{1-w} = 0}$$

- Similar (but more intricate) constraints for NMHV
[Caron-Huot], LD, von Hippel McLeod, 1509.08127

(MHV,NMHV): parameters left in $(\mathcal{E}^{(L)},\ E^{(L)} \& \tilde{E}^{(L)})$

Constraint	$L=1$	$L=2$	$L=3$	$L=4$	$L=5$
0. Hexagon functions	(10,10)	(82,88)	(639,761)	(5153,6916)	?????
1. Steinmann	(7,7)	(37,39)	(174,190)	(758,839)	(3105,3434)
2. Symmetry	(3,5)	(11,24)	(44,106)	(174,451)	(???,???)
3. Final entry	(2,2)	(5,5)	(19,12)	(72,32)	(272,83)
4. Collinear limit	(0,0)	(0,0)	(1,1)	(3,5)	(9,15)
5. LL MRK	(0,0)	(0,0)	(0,0)	(1,1)	(3,4)
6. NLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(1,1)
7. NNLL MRK	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)

(0,0) → amplitude uniquely determined

Next-to-final entry and NMHV spurious pole conditions are **impotent** after imposing Steinmann!!

How close is Steinmann space to "optimal"?

- Want to describe, not only $(\mathcal{E}^{(L)},\, E^{(L)} \& \tilde{E}^{(L)})$ to a given loop order, but also derivatives ({w,1,1,…,1} coproducts) of even higher loop answers.

- How many functions are we likely to need?

- We take multiple derivatives/coproducts of the answers we know, and ask how much of the Steinmann space they span at each weight.

Empirically Trimmed Steinmann space

- The first surprise is already at weight 2
- The many, many {2,1,1,…,1} coproducts of the weight 10 functions $(\mathcal{E}^{(5)},\ E^{(5)} \& \tilde{E}^{(5)})$ span only a 6 dimensional subspace of the 7 dimensional Steinmann space, with basis:

$$\mathrm{Li}_2(1-1/u) \qquad \ln^2\frac{uv}{w} + 4\zeta_2 \qquad \text{plus cyclic}$$

$\boxed{\zeta_2 \text{ is not an independent element!}}$

Empirically Trimmed Steinmann space (cont.)

- At weight 3, $\zeta_2 \ln u_i$ drop out, but this is not "new"
- But also ζ_3 is not there!
- At weight 4, nothing "new" (apparently)
- At weight 5, ζ_5 **and** $\zeta_2 \zeta_3$ go missing (can be absorbed into other functions)
- At weight 7, ζ_7 , $\zeta_2 \zeta_5$ **and** $\zeta_3 \zeta_4$ go missing

Empirically Trimmed Steinmann space (cont.)

Weight	1	2	3	4	5	6	7	8	9	10
P-even Steinmann	3	7	16	37	80	174	365	758	1543	3105
P-even empirical	3	6	12	25	48	99	195	???	???	????
P-odd Steinmann	0	0	1	2	7	16	38	81	167	329
P-odd empirical	0	0	1	2	6	13	30	??	??	???

Almost a factor of 2 smaller at high weights

But, up to the mystery of the missing zeta's, the Steinmann hexagon space appears to be "just right" for the problem of 6 point scattering in planar N=4 super-Yang-Mills theory!

Another mystery

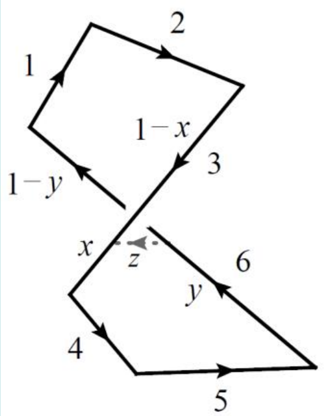
A particular linear combination of {2L-2,1,1} MHV coproducts gives 2 * NMHV – MHV at one lower loop order:

$$g^2\,(2E-\mathcal{E}) = \mathcal{E}^{y_u,y_u}+\mathcal{E}^{y_w,y_w}-3\mathcal{E}^{y_v,y_v}-\mathcal{E}^{v,v}-\mathcal{E}^{1-v,v}$$
$$+2(\mathcal{E}^{y_u,y_v}+\mathcal{E}^{y_w,y_v})-\mathcal{E}^{y_u,y_w}-\mathcal{E}^{y_w,y_u}$$

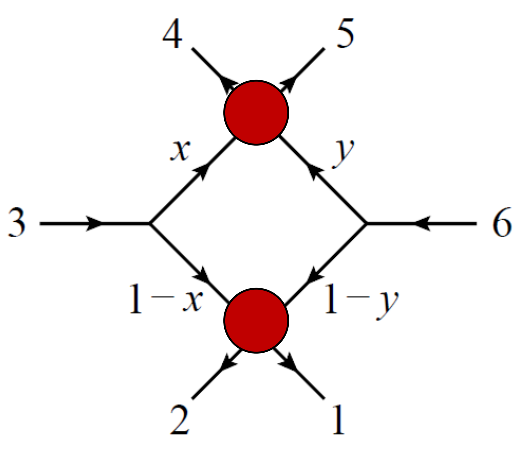
- First found at four loops LD, von Hippel, 1408.1505
- Can now check at five loops – and it looks much more natural in the BDS-like normalization.
- Resembles a second order differential equation.

Analytical behavior in new limits

- Self-crossing or "double parton scattering" limit
  Georgiou, 0904.4675; LD, Esterlis, 1602.02107



WL ← → Amp.

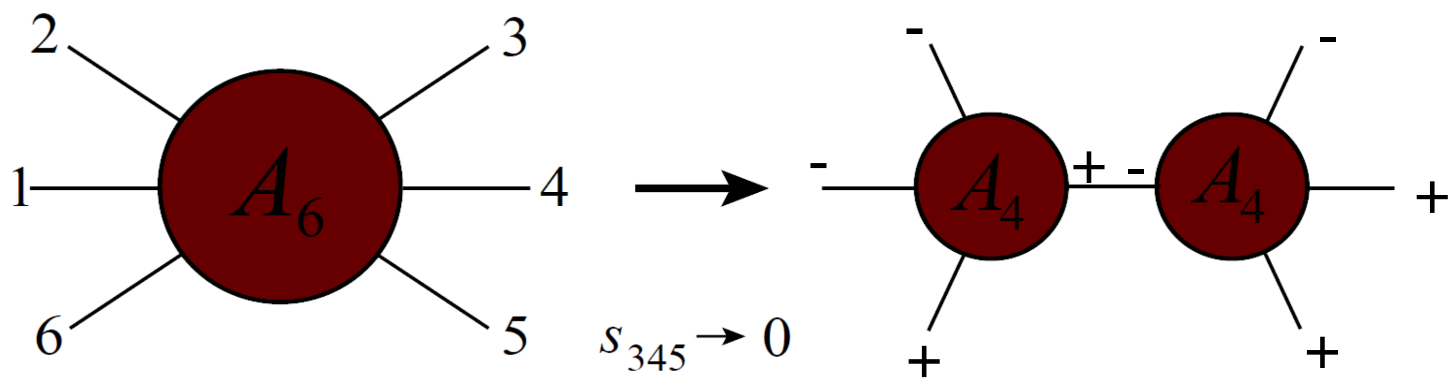


$u \to u\, e^{-2\pi i}$ then $(u,v,w) \to (1-\delta, v, v)$, $\delta \ll 1$

- Overlaps MRK limit when $v \to 0$
- In $\mathcal{E}(1-\delta, v, v)$, $\ln \delta$ terms independent of v
- Can derive using Wilson Loop RGE a la
  Korchemsky and Korchemskaya hep-ph/9409446

NMHV Multi-Particle Factorization

Bern, Chalmers, hep-ph/9503236; LD, von Hippel, 1408.1505



$$A_6^{\rm NMHV}(k_i) \stackrel{s_{345}\to 0}{\longrightarrow} A_4(k_6,k_1,k_2,K)\,\frac{F_6(K^2,s_{i,i+1})}{K^2}\,A_4(-K,k_3,k_4,k_5)$$

Only interesting for NMHV: MHV tree has no pole $\mathcal{A}_{\rm MHV}^{(0)} = i\,\frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}$

$$u = \frac{s_{12}s_{45}}{s_{123}s_{345}} \to \infty \qquad w = \frac{s_{61}s_{34}}{s_{345}s_{234}} \to \infty$$

u/w and $v = \frac{s_{23}s_{56}}{s_{234}s_{123}}$ fixed

Multi-Particle Factorization (cont.)

$$(1) = (4) \to \infty, \text{ rest finite}$$

→ look at $E(u,v,w)$

Or rather at $U(u,v,w) = \ln E(u,v,w)$

$$\frac{\mathcal{A}_{\text{NMHV}}}{\mathcal{A}_{\text{BDS-like}}} \approx e^{U}[(1) + (4)]$$

Factorization limit of U

$$U^{(1)}(u,v,w) = -\frac{1}{4}\ln^2(uw/v) - \zeta_2$$

$$U^{(2)}(u,v,w)|_{u,w\to\infty} = \frac{3}{4}\zeta_2 \ln^2(uw/v) - \frac{1}{2}\zeta_3 \ln(uw/v) + \frac{71}{8}\zeta_4$$

$$U^{(3)}(u,v,w)|_{u,w\to\infty} = \frac{1}{3}\zeta_3 \ln^3(uw/v) - \frac{75}{8}\zeta_4 \ln^2(uw/v) + (7\zeta_5 + 8\zeta_2\zeta_3)\ln(uw/v) - \frac{721}{8}\zeta_6 - 3(\zeta_3)^2$$

$$U^{(4)}(u,v,w)|_{u,w\to\infty} = \frac{1}{4}\zeta_4 \ln^4(uw/v) - (4\zeta_5 + 3\zeta_2\zeta_3)\ln^3(uw/v) + \left(\frac{3769}{32}\zeta_6 + \frac{21}{4}\zeta_3^2\right)\ln^2(uw/v) - \left(\frac{785}{8}\zeta_7 + \frac{641}{4}\zeta_3\zeta_4 + \frac{191}{2}\zeta_2\zeta_5\right)\ln(uw/v) + \frac{62629}{64}\zeta_8 + \frac{133}{4}\zeta_2\zeta_3^2 + \frac{289}{4}\zeta_3\zeta_5$$

$$\boxed{\frac{uw}{v} = \frac{s_{12}s_{34}}{s_{56}} \cdot \frac{s_{45}s_{61}}{s_{23}} \cdot \frac{1}{s_{345}^2}}$$

Simple polynomial in $\ln(uw/v)$!

Sudakov logs due to on-shell intermediate state

At $(u,v,w) = (1,1,1)$, multiple zeta values

$$R_6^{(2)}(1,1,1) = -(\zeta_2)^2 = -\frac{5}{2}\zeta_4$$

$$R_6^{(3)}(1,1,1) = \frac{413}{24}\zeta_6 + (\zeta_3)^2$$

First irreducible MZV

$$R_6^{(4)}(1,1,1) = -\frac{471}{4}\zeta_8 - \frac{3}{2}\zeta_2(\zeta_3)^2 - \frac{5}{2}\zeta_3\zeta_5 + \frac{3}{2}\zeta_{5,3}$$

$$\begin{aligned} R_6^{(5)}(1,1,1) = {} & \frac{8389}{10}\zeta_{10} + 12\zeta_2\zeta_3\zeta_5 + 17\zeta_4(\zeta_3)^2 \\ & - \frac{63}{2}\zeta_3\zeta_7 - \frac{111}{8}(\zeta_5)^2 - \frac{3}{2}\zeta_2\zeta_{5,3} - 6\zeta_{7,3} \end{aligned}$$

Numerical results

- Plot perturbative coefficients on various lines and surfaces
- Instructive to take ratios of successive loop orders: $R_6^{(L)}/R_6^{(L-1)} = \bar{R}_6^{(L)}$
  - Planar N=4 SYM has no instantons and no renormalons.
  - Its perturbative expansion has a finite radius of convergence, 1/8
  - For "asymptotically large orders", $R_6^{(L)}/R_6^{(L-1)}$ should approach -8

Cusp anomalous dimension $\gamma_K(\lambda)$

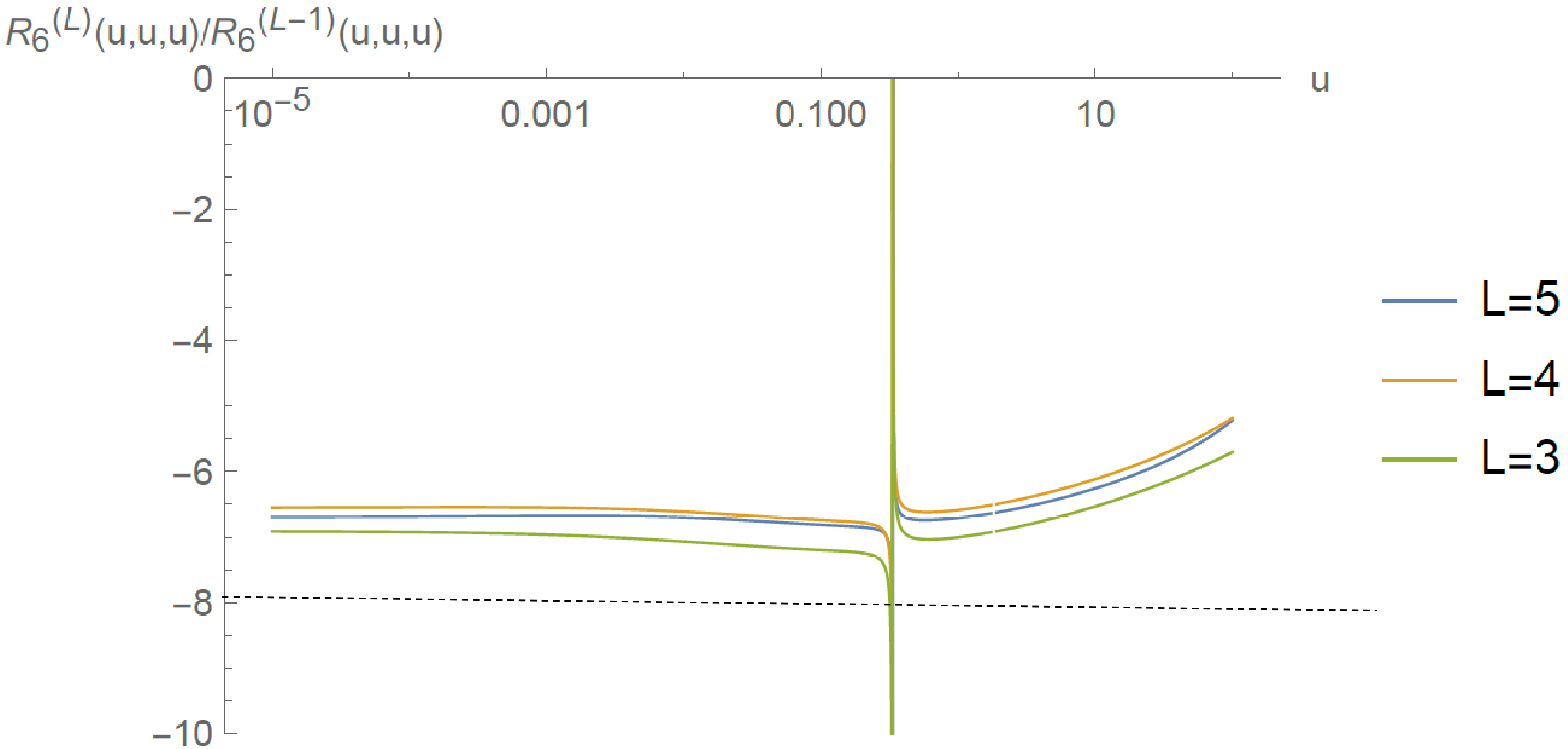
- Known to all orders, Beisert, Eden, Staudacher [hep-th/0610251]
- Closely related to amplitude/Wilson loop
- Use as benchmark for approach to large orders:

L	$\gamma_K^{(L)}/\gamma_K^{(L-1)}$	$\bar{R}_6^{(L)}(1,1,1)$	$\overline{\ln \mathcal{W}}_{\text{hex}}^{(L)}(\frac{3}{4},\frac{3}{4},\frac{3}{4})$	$\overline{\ln \mathcal{W}}_{\text{hex}}^{(L)}(\frac{1}{4},\frac{1}{4},\frac{1}{4})$
2	-1.6449340	∞	-2.7697175	-2.8015275
3	-3.6188549	-7.0040885	-5.0036164	-5.1380714
4	-4.9211827	-6.5880519	-5.8860842	-6.0359857
5	-5.6547494	-6.7092373	-6.3453695	-6.4658887
6	-6.0801089	–	–	–
7	-6.3589220	–	–	–
8	-6.5608621	–	–	–

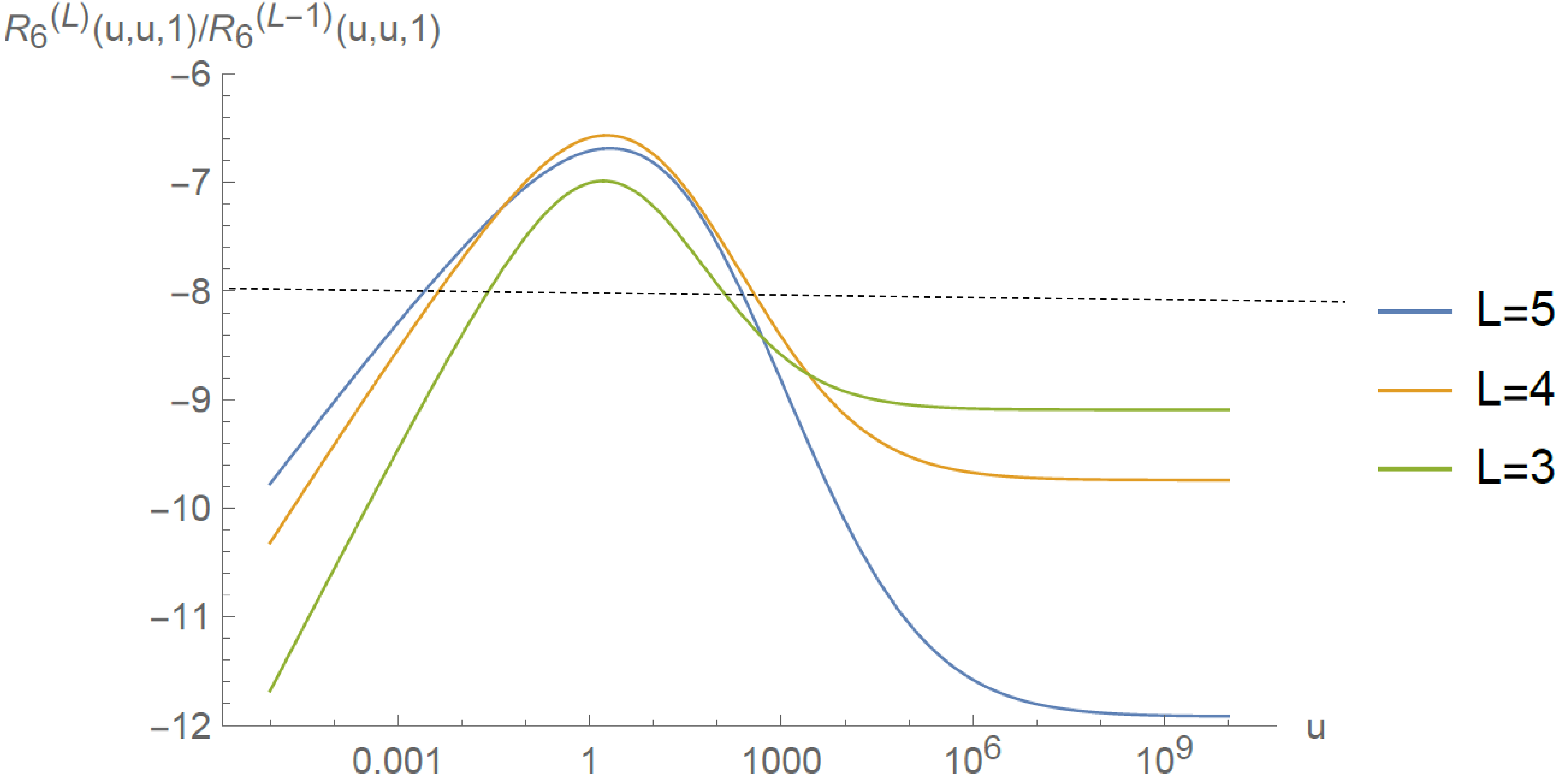
$\downarrow$

-8

On (u,u,u), remarkably constancy for $u < 1$



On $(u,u,1)$, everything collapses to HPLs of u



Beyond 6 gluons

- Cluster Algebras provide strong clues to "the right functions"

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289, Spradlin talk at Amplitudes 2016

- Power seen particularly in symbol of 3-loop MHV 7-point amplitude
  Drummond, Papathanasiou, Spradlin 1412.3763
- Can now apply Steinmann relations, and turn symbols into "heptagon" functions

Summary & Outlook

- Hexagon function ansatz → planar N=4 SYM amplitudes over full kinematical phase space, for 6 gluons, both MHV and NMHV, to high loop orders
- Steinmann + $\bar{Q}$ equation = powerful constraints → No need for loop-momentum integrands
- Only need very little additional information from multi-Regge (or OPE) limits
- Numerical and analytical results intriguing!
- → finite coupling for generic kinematics?