The Geometry of Non-Planar On-Shell Diagrams

Daniele Galloni

Based on:

arXiv: 1607.01781 [Bourjaily, Franco, D.G., Wen]

arXiv: 1502.02034 [Franco, D.G., Penante, Wen]

arXiv: 1310.3820 [Franco, D.G., Mariotti]

arXiv: 1211.5139 [Franco, D.G., Seong]

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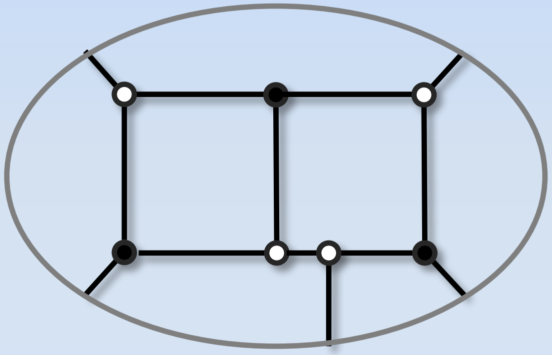
Outline

- The planar on-shell form
- Non-planar symmetries
- Non-planar on-shell diagrams and on-shell form
- Planar singularity structure – very geometric
- Non-planar singularity structure – obtained through ”geometric identification”
- bipartiteSUSY package
- Summary & future directions

The *planar* on-shell form

- The on-shell form Ω is the integrand without the delta functions
- This can be written in different ways:

$$\Omega = \prod_{\substack{\text{internal} \\ \text{nodes } v}} \frac{1}{\mathrm{Vol}(\mathrm{GL}(1)_v)} \prod_{\text{edges } X_e} \frac{dX_e}{X_e}$$



$$\Omega = \frac{df_1}{f_1}\frac{df_2}{f_2}\cdots\frac{df_d}{f_d}$$

$$\Omega = \frac{d^{k\times n}C}{\mathrm{Vol}(\mathrm{GL}(k))}\frac{1}{(1\cdots k)(2\cdots k+1)\cdots(n\cdots k-1)}$$

Symmetries (non-planar)

- Conformal
- $\mathcal{N} = 4$ Supersymmetry
- Dual conformal (?)

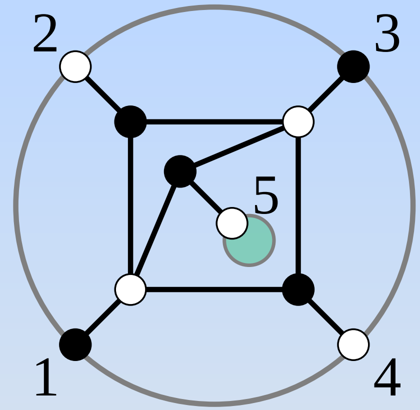
  [Bern, Herrmann, Litsey, Stankowicz, Trnka]
  [Arkani-Hamed, Bourjaily, Cachazo, Trnka]

  – Evidence at 3-loops that a related structure exists: the amplitude is still of $dlog$ form!
  – Suggests that there may be yet more hidden symmetries to find in $\mathcal{N} = 4$ SYM

Non-planar on-shell diagrams

$dlog(\mathrm{X_i})$

(?)



What do we know about non-planar on-shell diagrams?

– Which diagrams contribute to a given process? Are there any recursion relations?

– What part of the nice mathematical structure survives?

– How do we evaluate the contribution from each diagram?

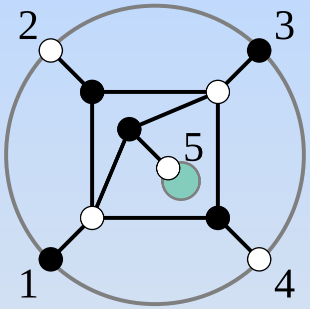
– How do we combine loop momenta from different diagrams?

– ...

The non-planar on-shell form

- Explicit $dlog$ form:
  - Directly from edges of diagram:

$$\Omega = \prod_{\substack{\text{internal} \\ \text{nodes } v}} \frac{1}{\text{Vol}(\text{GL}(1)_v)} \prod_{\text{edges } X_e} \frac{dX_e}{X_e}$$



  - Using generalized face variables (gauge-invariant):

$$\Omega = \prod_{i=1}^{F-1} \frac{df_i}{f_i} \prod_{a=1}^{B-1} \frac{db_a}{b_a} \prod_{m=1}^{g} \frac{d\alpha_m}{\alpha_m} \frac{d\beta_m}{\beta_m}$$

[Franco, D.G., Penante, Wen]

- Explicit Grassmannian:
  - Use map between d.o.f. of diagram and $C \in G(k,n)$, and do a variable transformation from face variables to d.o.f. of C:

$$\Omega = \frac{d^{k\times n}C}{\text{Vol}(\text{GL}(k))} \frac{\mathcal{F}}{(1 \cdots k)(2 \cdots k+1) \cdots (n \cdots k-1)}$$

  - There is a fast, combinatorial way of obtaining $\mathcal{F}$

[Franco, D.G., Penante, Wen]

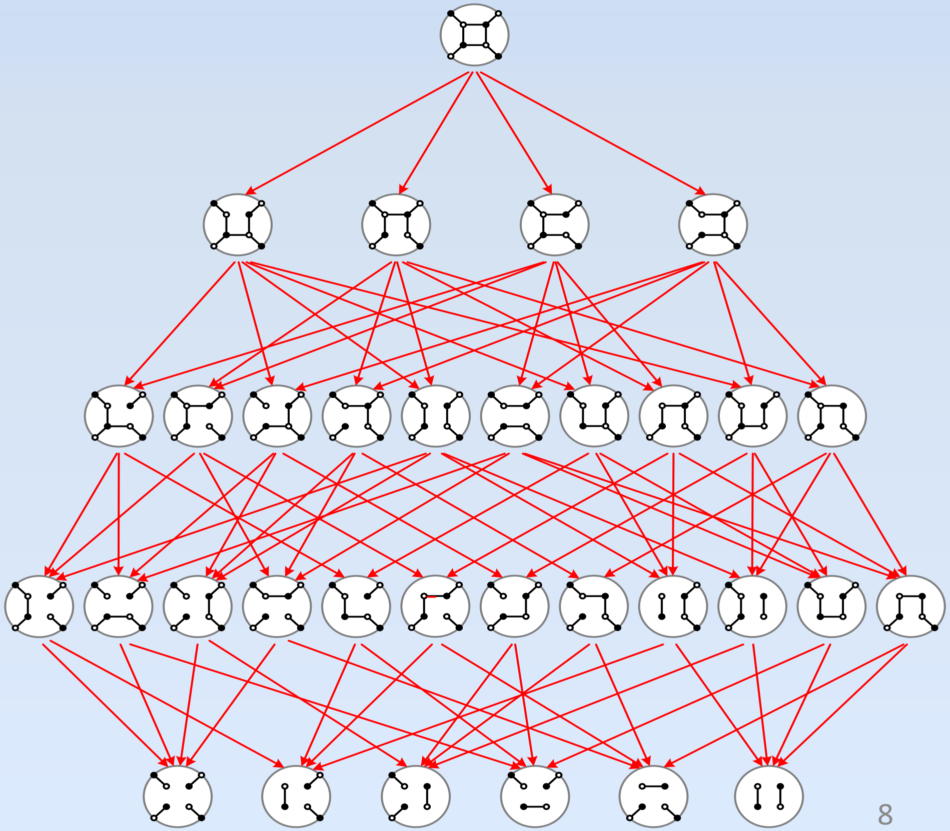
Understanding non-planar on-shell diagrams:

Singularity Structure

Singularity structure (planar)

On-shell form $\prod_{\text{edges}\, X_e} \frac{dX_e}{X_e}$ $\Leftrightarrow$ $d^{k\times n}C \, \frac{1}{(1\cdots k)\cdots(n\cdots k-1)}$

Singularity structure — Turn off X_e, i.e. remove edges from diagram $\Leftrightarrow$ Turn off minors of Grassmannian



- For planar diagrams $(i \cdots i+k) \geq 0$

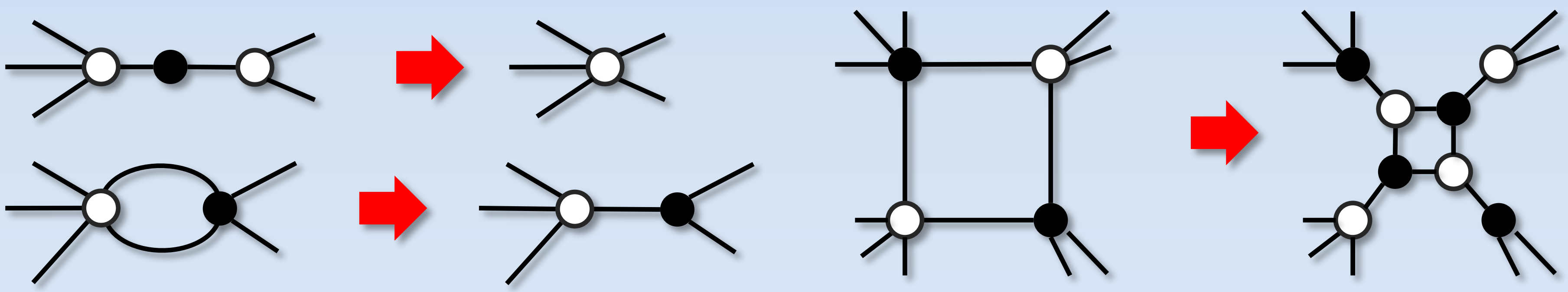
$\Rightarrow$ Turning off a minor is like going to a **geometric boundary** of the space

(→ ”singularities of the amplitude”=”boundaries of the amplituhedron”)

[Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka]

Equivalence moves

- *Question:* Sometimes turning off different sets of edges takes you to the same singularity. Why?
- Equivalence moves: certain graphical operations don't change the Grassmannian



- *Question:* Do I have to look for all possible moves, just in case two boundaries are equivalent?
- Diagnostic: It's enough to look at which minors $(i \cdots i+k)$ are > 0 and which are $= 0$.
(⇔ look at zig-zag paths on the graph)

Singularity structure (non-planar)

- Since also here the on-shell form is $\Omega = \prod \frac{dX_e}{X_e}$,
  we reach singularities by removing edges

- Diagnostic for equivalent configurations?
  - ~~It's enough to look at which minors $(i \cdots i+k)$ are > 0 and which are $= 0$.~~

    $\cancel{(\cdots) \geq 0}$
    $\Rightarrow$ no longer have same boundaries $(\cdots) \to 0$

  - ~~$\Leftrightarrow$ look at zig-zag paths on the graph~~

- "Good news": we can still by hand look for all possible equivalence moves

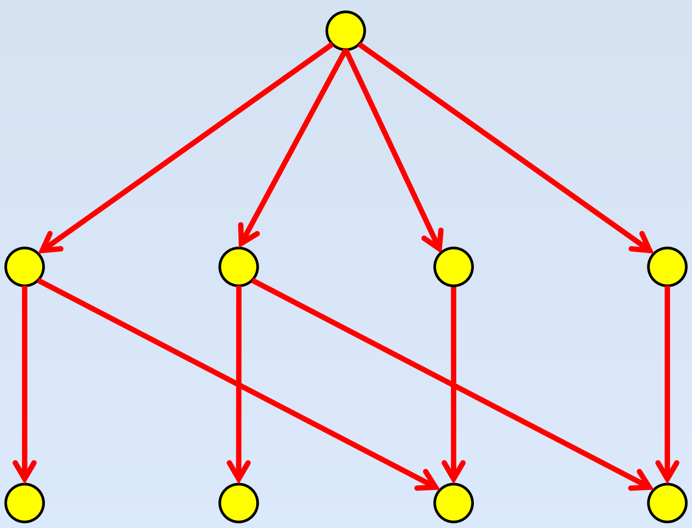
Geometric identification

- Most general equivalence identification:

If two diagrams are equivalent, their subgraphs must also be equivalent

- "Geometric identification"

- Works for planar diagrams too, but it's less efficient than looking at $(\cdots) > 0$



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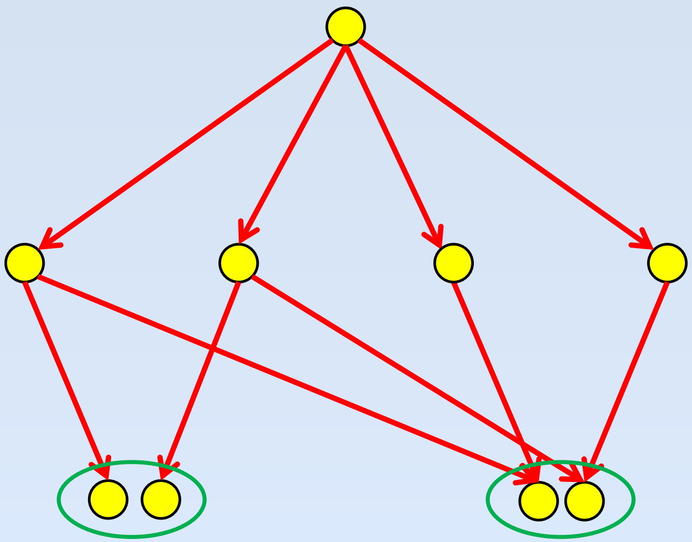

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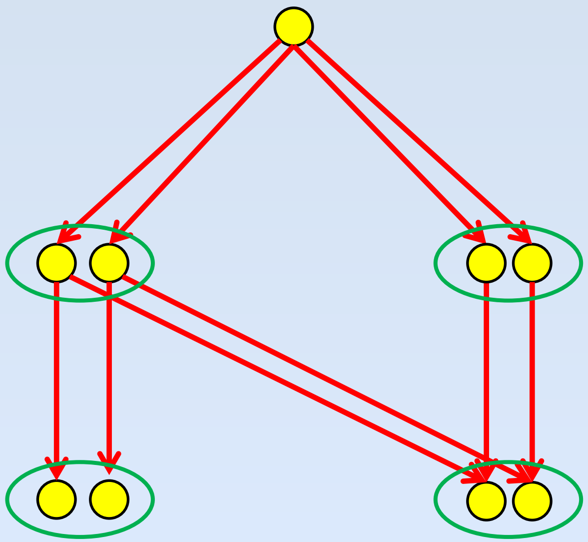

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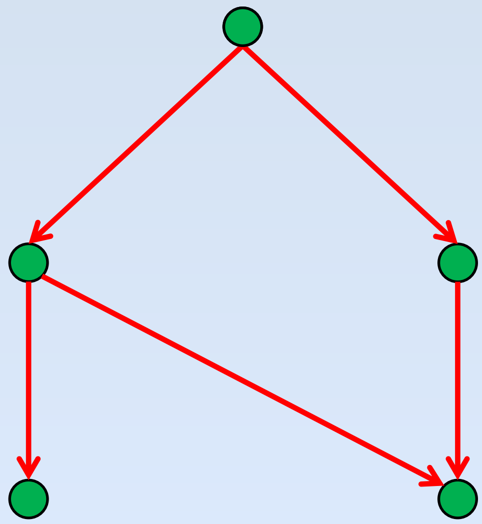
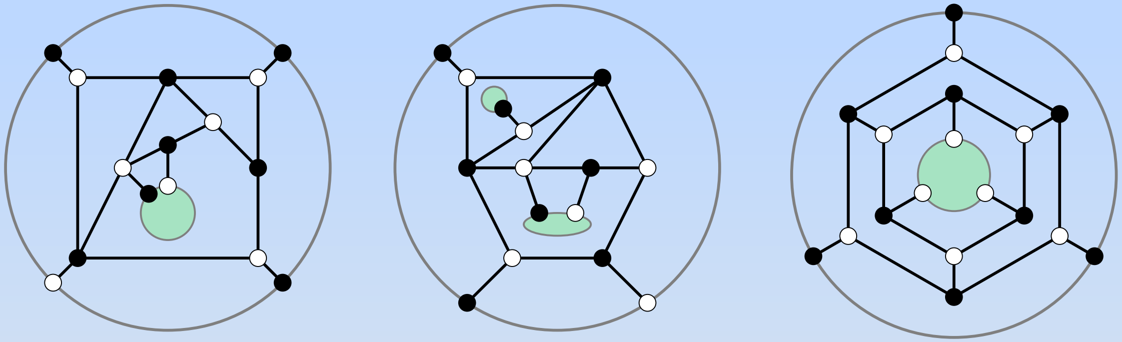


Landscaping G(3,6)



- We used our techniques to study the singularity structure of all possible on-shell diagrams in G(3,6)
- We found that there are only 24 distinct diagrams of maximal dimension (mod. color swaps and equivalence moves)
- Not all of these will be relevant to possible recursion relations – it's nonetheless important to understand our building blocks
- We also found that there are only 10 leading singularities in G(3,6) (plus their color-inverted partners!)
- We found several interesting things / surprises

Surprises in G(3,6)

- Complicated denominators:

$$\frac{1}{(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)\sqrt{\cdots}} \qquad \frac{1}{[(\cdot)(\cdot)-(\cdot)(\cdot)](\cdot)(\cdot)(\cdot)(\cdot)(\cdot)}$$

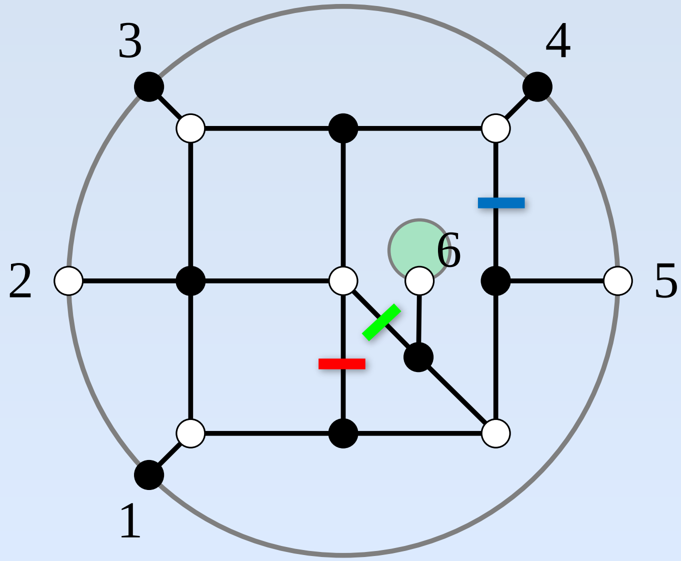
Surprises in G(3,6)

- Complicated denominators:

$$\frac{1}{(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)\sqrt{[(\cdot)(\cdot)+(\cdot)(\cdot)+(\cdot)(\cdot)]^2-4(\cdot)(\cdot)(\cdot)(\cdot)}}$$

- Top-dimensional diagram doesn't have nonstandard poles, but lower-dimensional singularities do

- Idea of geometric boundary operator not so obvious:



$\boldsymbol{X_A} \to 0 \Leftrightarrow (234) \to 0$
$\boldsymbol{X_B} \to 0 \Leftrightarrow (123) \to 0$
$\boldsymbol{X_C} \to 0 \Leftrightarrow (236) \to 0$

Any pair of these edges:
$(234) \to 0, (123) \to 0,$
$(236) \to 0, (235) \to 0$

ε	Stratification structure (# of singus of each dimension)		
1	{1,6,21,56,114,180,215,180,90,20}	13	{1,9,40,147,271,332,294,201,90,20}
6	{1,7,27,83,166,239,249,190,90,20}	10	{1,7,25,93,186,259,260,193,90,20}
8	{1,8,30,98,198,274,268,195,90,20}	9	{1,7,29,103,206,281,272,196,90,20}
11	{1,8,34,116,215,282,271,196,90,20}	7	{1,5,26,94,187,259,260,193,90,20}
16	{1,9,36,138,252,315,288,201,90,20}	10	{1,6,31,116,220,288,275,197,90,20}
9	{1,9,38,122,236,309,285,199,90,20}	12	{1,7,39,151,280,341,299,202,90,20}
5	{1,8,36,102,189,256,257,192,90,20}	8	{1,8,36,117,223,293,277,197,90,20}
8	{1,10,45,142,267,334,297,202,90,20}	4	{1,3,15,66,153,231,246,189,90,20}
5	{1,6,25,78,158,231,245,189,90,20}	9	{1,6,28,108,216,289,276,197,90,20}
10	{1,7,29,107,209,280,271,196,90,20}	13	{1,9,38,132,236,298,279,198,90,20}
7	{1,7,33,104,194,261,260,193,90,20}	-4	{1,12,54,166,348,420,339,210,90,20}
8	{1,8,35,120,231,299,279,197,90,20}	6	{1,6,21,74,157,232,246,189,90,20}

BipartiteSUSY package

(coming soon)

Summary

- On-shell diagrams make dlog form manifest
- There is intriguing evidence ($dlog$ form up to 3 loops) that such a description exists for non-planar amplitudes
- Must understand non-planar on-shell diagrams before using them
- We found several ways of obtaining the integrand contribution from a non-planar on-shell diagram
- We can now efficiently obtain the full singularity structure, and hence the geometric data, of any on-shell diagram
- We landscaped all on-shell diagrams in G(3,6)
- Found many surprises along the way