Amplitudes and Correlators to Ten Loops Using Simple, Graphical Bootstraps

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ASC Munich

based on: arXiv:1609.00007 with Bourjaily, Tran

Outline

Four-point stress-energy multiplet correlation function integrands in planar $\mathcal{N} = 4$ SYM

- to 10 loops

$\Rightarrow$ 10 loop 4-pt amplitude, 9 loop 5-point (parity even) amplitude, 8 loop 5-point (full) amplitude

Method (Bootstrappy): Symmetries (extra symmetry of correlators), analytic properties, planarity $\Rightarrow$ basis of planar graphs

- Fix coefficients of these graphs using simple graphical rules: the triangle, square and pentagon rules

Discussion of results to 10 loops

[higher point correlators + correlahedron speculations]

Correlators in $\mathcal{N} = 4$ SYM

(Correlation functions of gauge invariant operators)

- *Gauge invariant operators:* gauge invariant products (ie traces) of the fundamental fields
- Simplest operator $\mathcal{O}(x) \equiv \mathrm{Tr}(\phi(x)^2)$ (ϕ one of the six scalars)
- The simplest *non-trivial* correlation function is

$$\mathcal{G}_4(x_1, x_2, x_3, x_4) \equiv \langle \mathcal{O}(x_1)\overline{\mathcal{O}}(x_2)\mathcal{O}(x_3)\overline{\mathcal{O}}(x_4)\rangle$$

- $\mathcal{O}(x) \in$ stress energy supermultiplet. (We can consider correlators of all operators in this multiplet using superspace.)

Correlators in $\mathcal{N} = 4$

AdS/CFT

Supergravity/String theory on $AdS_5 \times S^5 \quad = \quad \mathcal{N}$=4 super Yang-Mills

- Correlation functions of gauge invariant operators in SYM $\leftrightarrow$ string scattering in *AdS*
- Contain data about anomalous dimensions of operators and 3 point functions via OPE $\rightarrow$integrability / bootstrap
- Finite
- Big Bonus more recently: Correlators give scattering amplitudes

Method for computing correlation functions at loop level

1. Rational conformally covariant integrands
2. Superconformal symmetry/ hidden symmetry
3. Analytic properties (OPE) single poles
4. Planarity

$\Rightarrow$ f-graphs

Note: f-graphs give a well-defined (small(ish)) independent basis for the result
Note 2: graph basis has no spurious poles

5. three graphical rules: triangle rule, square rule, pentagon rule fix coefficients of the f graphs

Superconformal $\Rightarrow$ hidden symmetry

Define

$$f^{(\ell)}(x_1,\ldots,x_4,x_5,\ldots,x_{4+\ell}) \equiv \frac{1}{2}\left(\frac{\mathcal{G}_4^{(\ell)}(x_1,x_2,x_3,x_4)}{\mathcal{G}_4^{(0)}(x_1,x_2,x_3,x_4)}\right)/\xi^{(4)},$$

where $\xi^{(4)} := x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2 (x_{13}^2 x_{24}^2)^2$.

Hidden symmetry (inherited from crossing symmetry) [Eden Korchemsky Sokatchev PH]:

$$f^{(\ell)}(x_1,\ldots,x_{4+\ell}) = f^{(\ell)}(x_{\sigma_1},\ldots,x_{\sigma_{4+\ell}}) \qquad \forall\,\sigma\in S_{4+\ell}$$

- Correlator of four complete supermultiplets determined entirely in terms of this single function [Eden Schubert Sokatchev]
  - (cf MHV/ $\overline{\text{MHV}}$ amplitudes - in fact close analogy with $\overline{\text{MHV}}$ amplitudes)
- NB, the symmetry mixes external variables $x_1,\ldots x_4$ with integration variables $x_5\ldots x_{4+\ell}$

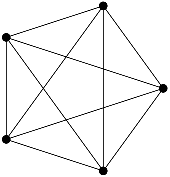
1-, 2- and 3-loop integrands

- Entire correlator defined (perturbatively) via $f^{(\ell)}$
  - conformal weight 4 at each point
  - permutation invariant
  - No double poles (from OPE)
- naively equivalent to: degree (valency) 4 graphs on $4{+}\ell$ points

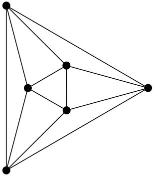
$$\text{graph edge} = \frac{1}{x_{ij}^2}$$

- (But: we are also allowed numerator lines $\Rightarrow$ degree $\geq$ 4 graphs).
- Don't need to label graph sum over permutations $\Rightarrow$ sum over all labellings
- f graphs : (always equivalent to the edges and vertices of 3d polytopes)

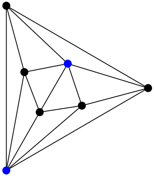
$$f^{(1)} = \frac{1}{\prod_{1\le i<j\le 5} x_{ij}^2}$$



$$f^{(2)} = \frac{x_{12}^2 x_{34}^2 x_{56}^2 \ + \ S_6 \text{ perms}}{\prod_{1\le i<j\le 6} x_{ij}^2}$$

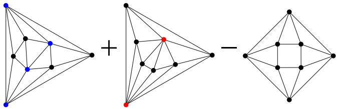


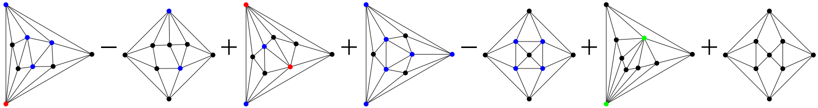
$$f^{(3)} = \frac{(x_{12}^4)(x_{34}^2 x_{45}^2 x_{56}^2 x_{67}^2 x_{73}^2) \ + \ S_7 \text{ perms}}{\prod_{1\le i<j\le 7} x_{ij}^2}$$



(Unique (planar) possibilities) $f^{(3)} =$

Four- and five-loops

$f^{(4)} =$ 

$f^{(5)} =$ 

- Very compact writing (each f-graph represents a number of different integrals which you can read off from the f-graph)
- Hidden (permutation) symmetry uniquely fixes the four-point planar correlator to 3 loops
- Fixes 4 loops planar to 3 constants
- 5 loops planar to 7 constants
- 6 loops planar to 36 constants etc.

Graph counting table

ℓ	number of plane graphs	number of graphs admitting decoration	number of decorated plane graphs (f-graphs)	number of planar DCI integrands
1	0	0	0	1
2	1	1	1	1
3	1	1	1	2
4	4	3	3	8
5	14	7	7	34
6	69	31	36	284
7	446	164	220	3,239
8	3,763	1,432	2,709	52,033
9	34,662	13,972	43,017	1,025,970
10	342,832	153,252	900,145	24,081,425
11	3,483,075	1,727,655	22,097,035	651,278,237

Further motivation: Relation to amplitudes

- At large N_c, in the polygonal lightlike limit correlators become amplitudes [Alday Eden Korchemsky Maldacena Sokatchev, Eden Korchemsky Sokatchev, Eden Korchemsky Sokatchev PH, Adamo Bullimore Mason Skinner]

Relation to amplitudes in polygon limit: eg

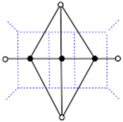
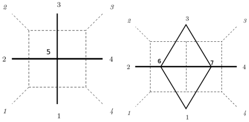
$$\lim_{\substack{\text{4-point}\\\text{light-like}}} \left(\frac{\mathcal{G}_4(x_1, x_2, x_3, x_4)}{\mathcal{G}_4^{(0)}(x_1, x_2, x_3, x_4)} \right) = \mathcal{A}_4(x_1, x_2, x_3, x_4)^2,$$

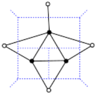
- For the amplitude x_i are "dual momenta" with $p_i = x_i - x_{i+1}$ (dual conformal symmetry [Drummond Korchemsky Sokatchev])
- Correlators are completely finite for generic x_i
- Diverge as $x_{i\,i+1}^2 \to 0$
- Amplitudes IR divergent (use dim reg)
- Integrands perfectly well-defined though

f-graphs to 4-pnt amplitudes graphically

- Amplitude limit can be seen graphically:
- Taking the lightlike limit is equivalent to projecting onto terms containing a 4-cycle
- 4-cycles on the surface (faces) $\to\ \mathcal{A}_4^{(\ell)}$
- 4-cycles "inside" $\to$ product graphs $\mathcal{A}_4^{(k)}\mathcal{A}_4^{(\ell-k)}$

Eg at 3-loops lightlike limit gives: $2\mathcal{A}_4^{(3)} + \mathcal{A}_4^{(1)}\mathcal{A}_4^{(2)}$

$\to$  $\to$ 

$\to$ 

Higher point amplitudes from 4-point correlator

- Taking a higher point lightlike limit of the four-point correlator gives special combinations of higher point amplitudes

$$\lim_{\substack{\text{n-point}\\ \text{light-like}}} \left(\xi^{(n)} f(a)\right) = \frac{1}{2}\sum_{k=0}^{n-4} \mathcal{A}_n^k \mathcal{A}_n^{n-4-k} / (\mathcal{A}_n^{n-4,(0)}).$$

- $\xi^{(n)} \equiv \prod_{a=1}^{n} x_{aa+1}^2 x_{aa+2}^2,$
- "cross-section-like combinations"
- Pentagonal lightlike limit with $\mathcal{M}_5 \equiv \mathcal{A}_5^0 / \mathcal{A}_5^{0,(0)}$ and $\overline{\mathcal{M}}_5 \equiv \mathcal{A}_5^1 / \mathcal{A}_5^{1,(0)}$

$$\lim_{\substack{\text{5-point}\\ \text{light-like}}} \left(\xi^{(5)} f^{(\ell+1)}\right) = \sum_{m=0}^{\ell} \mathcal{M}_5^{(m)} \overline{\mathcal{M}}_5^{(\ell-m)}.$$

- At 5-points, easily disentangle $\mathcal{M}_5$ from $\mathcal{M}_5 \bar{\mathcal{M}}_5$: extract parity even part at ℓ-loops, parity odd at $\ell - 1$loops.

[Ambrosio Eden Goddard Taylor PH]

Five-point amplitude from four-point correlator

- define parity even and parity odd pieces

$$\mathcal{M}_{\rm even}^{(\ell)} \equiv \frac{1}{2}\left(\mathcal{M}_5^{(\ell)} + \overline{\mathcal{M}}_5^{(\ell)}\right) \quad \text{and} \quad \mathcal{M}_{\rm odd}^{(\ell)} \equiv \frac{1}{2}\left(\mathcal{M}_5^{(\ell)} - \overline{\mathcal{M}}_5^{(\ell)}\right).$$

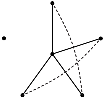
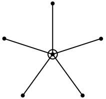
- observe parity odd piece can be represented:

$$\mathcal{M}_{\rm odd} \equiv i\epsilon_{12345\ell}\,\widehat{\mathcal{M}}_{\rm odd},$$

where $\epsilon_{abcdef} \equiv \det\{X_a, \ldots, X_f\}$ in 6d (Klein Quadric) version of 4d (dual) momentum space then:

$$\lim_{\substack{\text{5-point}\\ \text{light-like}}} \left(\xi^{(5)} f^{(\ell+1)}\right) = \sum_{m=0}^{\ell} \left(\mathcal{M}_{\rm even}^{(m)} \mathcal{M}_{\rm even}^{(\ell-m)} + \epsilon_{123456}\epsilon_{12345(m+6)} \widehat{\mathcal{M}}_{\rm odd}^{(m)} \widehat{\mathcal{M}}_{\rm odd}^{(\ell-m)}\right).$$

- The one loop amplitudes are displayed graphically as

$\mathcal{M}_{\rm even}^{(1)} \equiv$  $\equiv \dfrac{x_{13}^2 x_{24}^2}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2} + \text{cyclic}$, $\mathcal{M}_{\rm odd}^{(1)} \equiv$  $\equiv \dfrac{i\epsilon_{123456}}{x_{16}^2 x_{26}^2 x_{36}^2 x_{46}^2 x_{56}^2}$

Extracting the five point amplitude from the four point correlator

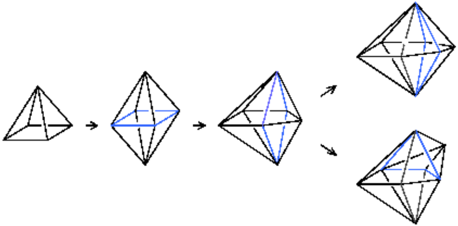
- Extraction of 5 point amplitude similar to 4 points.
- 5-cycles on the surface (faces) of $f^{(\ell+1)} \to \mathcal{M}^{(\ell)}_{\text{even}}$
- pentawheels give $\mathcal{M}^{(\ell-1)}_{\text{even}}$ (one loop lower since $\mathcal{M}^{(0)}_{\text{odd}} = 0$)
- Open question: How much info can we extract about higher point amplitudes from the 4 point correlator?

Coefficient fixing: graphical rules

- Previous methods of fixing coefficients were algebraic (eg soft/collinear divergence criterion [Bourjaily DiRe Shaikh Spradlin Volovich, Eden Korchemsky Sokatchev PH, Bourjaily Tran PH])
- Problematic at high loop order (beyond eight loops) simply due to the size of resulting algebraic expressions
- Graphical operations, directly on the f-graphs are far more efficient (as well as being conceptually appealing)
- Following three graphical rules completely fix the correlator to 10 loops

Fixing coeffs 1: The Square rule [Eden Korchemsky Sokatchev PH]

- Gluing pyramids together to obtain higher loop *f*-graphs = "square rule".
- Implies the "rung rule" first observed for amplitudes in [Bern Rozowsky Yan]



Derived from amplitude/correlator duality [Eden Korchemsky Sokatchev PH]

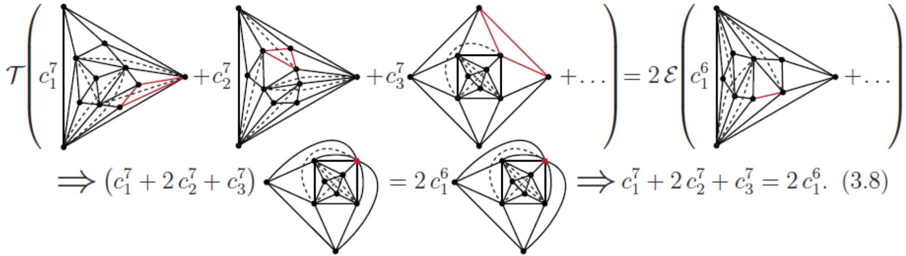
Expanding the duality: $\lim_{x_{ij}^2 \to 0} \to 2\mathcal{A}_4^{(\ell)} + 2\mathcal{A}_4^{(\ell-1)}\mathcal{A}_4^{(1)} + \ldots$ So the correlator contains $\mathcal{A}_4^{(\ell-1)} \times \mathcal{A}^{(1)}$

Fixing coeffs 2: the triangle rule (new)

$\ell+1$ loop triangle shrinks = ℓ-loop edge shrinks

- Take the ℓ+1 loop correlator as a sum of f-graphs with yet-to-be-determined coefficients.
- Shrink all inequivalent triangular faces
- This equals (2×) the result of taking the (known) ℓ-loop result and shrinking all inequivalent edges

6-loop to 7-loop example



2- to 3-loop Example

Two loop *f*-graph Edge Shrink:

Three loop *f*-graph Triangle Shrink:

3- to 4-loop

- Three-loop edge shrinks $\rightarrow g_1$ and g_2

$$\rightarrow \qquad \rightarrow$$

- Four-loop triangle shrinks:

$$\rightarrow \; g_2 \qquad \rightarrow \; g_1$$

$$\rightarrow \; g_1 \qquad \rightarrow \; g_1$$

$$\Rightarrow (g_1 + g_2) = b_1(g_1 + g_2) + b_2 g_1 + b_3 g_1 \Rightarrow b_1 = 1, b_2 + b_3 = 0$$

Proof of the triangle/edge shrink rule

Origin: Exponentiation of divergence in the limit $x_2 \to x_1$

OPE:

$$\lim_{x_2\to x_1} \log\Big(1 + \sum_{\ell\geq 1} a^\ell F^{(\ell)}\Big) = \gamma(a) \lim_{x_2\to x_1} F^{(1)} + \dots ,$$

'a is the coupling and F is (essentially) the correlator

$$F^{(\ell)} \equiv 3\, \frac{\mathcal{G}_4^{(\ell)}(x_1,x_2,x_3,x_4)}{\mathcal{G}_4^{(0)}(x_1,x_2,x_3,x_4)} ,$$

So *log* of the correlator has same divergence as the one loop correlator at all loops.

integrand of log: $\sum_{\ell\geq 1} a^\ell g^{(\ell)} \equiv \log\Big(1 + \sum_{\ell\geq 1} a^\ell F^{(\ell)}\Big)$, satisfies

$$\lim_{x_5,x_2\to x_1} \left(\frac{g^{(\ell)}(x_1,\dots,x_{4+\ell})}{g^{(1)}(x_1,\dots,x_5)} \right) = 0, \qquad \ell > 1$$

[Eden Korchemsky Sokatchev PH]

Log integrand can be nicely rewritten as (symmetrised on x_5; wider applicability):

$$g^{(\ell)} = F^{(\ell)} - \frac{1}{\ell} g^{(1)}(x_5) F^{(\ell-1)} - \sum_{m=2}^{\ell-1} \frac{m}{\ell} g^{(m)}(x_5) F^{(\ell-m)}$$

from which we get

$$\lim_{x_2, x_5 \to x_1} \frac{F^{(\ell)}(x_1, \ldots, x_{4+\ell})}{g^{(1)}(x_1, x_2, x_3, x_4, x_5)} = \frac{1}{\ell} \lim_{x_2 \to x_1} F^{(\ell-1)}(x_1, \ldots, \hat{x}_5, \ldots, x_{4+\ell}),$$

$$\Downarrow$$

$$\lim_{x_2, x_{4+\ell} \to x_1} (x_{12}^2 x_{14+\ell}^2 x_{24+\ell}^2) f^{(\ell)}(x_1, ., x_{4+\ell}) = 6 \lim_{x_2 \to x_1} (x_{12}^2) f^{(\ell-1)}(x_1, ., x_{3+\ell})$$

- (similar rephrasing for soft-collinear conjectured in [Golden Spradlin])
- Now reinterpreted graphically as the triangle rule

Results using triangle and square rule

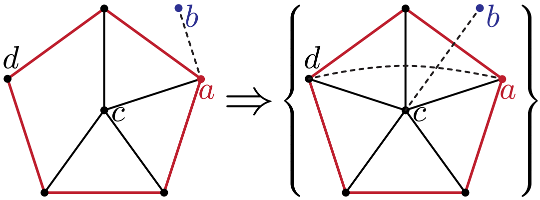
[Bourjaily Tran PH]

$\ell =$	2	3	4	5	6	7	8	9	10
number of f-graph coefficients:	1	1	3	7	36	220	2,709	43,017	900,145
unknowns remaining after square rule:	0	0	1	1	5	22	293	2,900	52,475
unknowns after square & triangle rules:	0	0	0	0	0	0	22	3	1,570

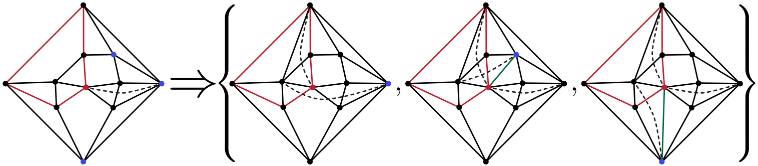
- Completely fixed to 7 loops
- 8 loops: leaves 22 free coefficients (in previous work we had fixed this using soft/collinear bootstrap plus hidden symmetry [Bourjaily Tran PH])
- But continuing to 9 and 10 loops fixes all 22 as well as fully fixing the 9 loop result!
- Correlator at ten loops needs another rule (or we need to go to higher loops)

Fixing coeffcients 3: Pentagon Rule

- Relates the following two topologies at the same loop order (with a minus sign)



- For example



- Rule implies $c_1^7 + c_2^7 + c_3^7 + c_4^7 = 0$.
- Arises from considering two seperate contributions of $\mathcal{M}_{odd}^{(\ell-1)}$ to $f^{(\ell+1)}$ (many contributions from multiplication of $\epsilon_{123456}\epsilon_{12345(m+6)}$)

Correlator to 10 loops

- Just the square and pentagon rules determine nearly everything to 7 loops:

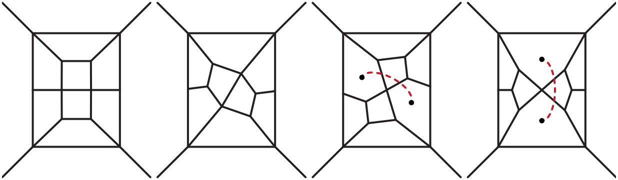
$\ell =$	2	3	4	5	6	7	8	9
number of f-graph coefficients:	1	1	3	7	36	220	2,709	43,017
unknowns remaining after square rule:	0	0	1	1	5	22	293	2,900
unknowns after square & pentagon rules:	0	0	0	0	1	0	17	64

- Square, triangle and pentagon rules in combination determine everything to 10 loops

We thus have the full 10 loop 4-point correlator

New features at high loops: finite graphs

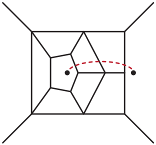
- Appearance of finite (elliptic?) integrals contributing to the amplitude from 8 loops
  eg: (in momentum space)



- Signals potential Non-polylog/MZV contribution to 4-pnt amplitude?
  - Consistent with BDS (elliptic contribution to the constant part) but
  - contradiction with the suggestion in [Arkani-Hamed Bourjaily Cachazo Goncharov Postnikov Trnka] that MHV and NMHV amplitudes are purely polylogs (only have dlog integrands)
- Of course non-polylog contribution could be apparent or could cancel ... then no contradiction

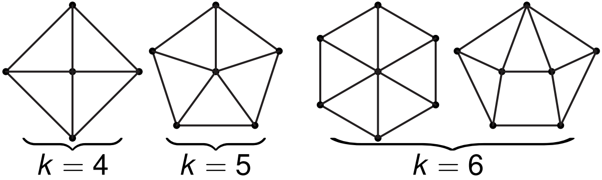
New features at high loops

- Conversely: divergent integrals contributing to the finite correlator occur from 8 loops
- eg. (again in momentum space)



- Previously it was thought they never appeared
- In fact only $k = 4$ type never appear (explained by the amplitude/correlator duality / rung rule)

Figure: Subgraphs leading to pseudoconformal divergences.



The increasing prevalence of zeros

ℓ	number of f-graphs	no. of f-graph contributions	(%)		number of DCI integrands	no. of integrand contributions	(%)
1	1	1	100		1	1	100
2	1	1	100		1	1	100
3	1	1	100		2	2	100
4	3	3	100		8	8	100
5	7	7	100		34	34	100
6	36	26	72		284	229	81
7	220	127	58		3,239	1,873	58
8	2,709	1,060	39		52,033	19,949	38
9	43,017	10,525	24		1,025,970	247,856	24
10	900,145	136,433	15		24,081,425	3,586,145	15

New features at high loops

- Appearance of half- and quarter-integer (and new integer) coefficients

number of f-graphs at ℓ loops having coefficient:

ℓ	±1	0	±2	$\pm1/2$	$\pm3/2$	±5	$\pm1/4$	$\pm3/4$	$\pm5/4$	$+7/4$	$\pm9/4$	$\pm5/2$	$+4$	$+14$
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	3	0	0	0	0	0	0	0	0	0	0	0	0	0
5	7	0	0	0	0	0	0	0	0	0	0	0	0	0
6	25	10	1	0	0	0	0	0	0	0	0	0	0	0
7	126	93	1	0	0	0	0	0	0	0	0	0	0	0
8	906	1,649	9	141	3	1	0	0	0	0	0	0	0	0
9	7,919	32,492	54	2,529	22	1	0	0	0	0	0	0	0	0
10	78,949	763,712	490	50,633	329	9	5,431	559	18	5	4	4	1	1

Patterns in new coefficients

- First new coefficient (-1) is the 4 loop “4-sided anti-prism graph”

  (first with a square face)
- Next new coefficient (+2) is the 6 loop “5-sided anti-prism graph”

  (first with a pentagonal face)
- Next new coefficient (-5) is the 8-loop “6-sided anti-prism graph”

  (first with a hexagonal face)

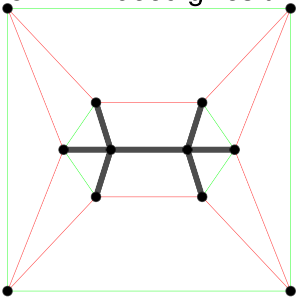
(there are also new half integer coefficients at 8 loops)

- the pattern continues: at nine loops, only new coeffs are just new sign, $+3/2$.
- 10 loops new graph with coefficient +14, “7-sided antiprism graph”

  (first seven-sided face)
- Conjecture: Catalan numbers $\Rightarrow$ 12 loop octagon anti-prism with coefficient -42

10 loop 4-point correlator byproducts

- 10-loop 4-point amplitude
- 9-loop 5-point parity even amplitude
- 8-loop 5-point full (even and odd) amplitude
- 8-loop 6-point “cross-section-like” combinations of amplitudes (extraction of amplitudes themselves?)
- etc.
- 4-loop 10-pnt
- eg 2-loop 10-point NMHV known to contain an elliptic massive double box [Caron-Huot Larsen]
- 10-pnt lightlike limit of $f^{(8)}$ indeed gives this



Conclusions and further directions

- fascinating rigid mathematical structure
- graph generating algorithm (like rung rule) without having to produce the full basis?
- Consistent with Triangle/Edge Shrink
- Higher loop correlator 11,12-loop?
- Extraction of higher point amplitudes from 4-point correlator
- Integrals? Known only to three loops so far [Drummond Duhr Eden Pennington Smirnov PH] (one integral known at 4 loops [Eden Smirnov])
- Higher charge correlators [Chicherin Drummond Sokatchev PH]
- Higher point correlators (twistors or bootstrap [Chicherin Doobary Korchemsky Mason Sokatchev PH])
- Amplituhedron for correlators / correlahedron? (with Eden, Mason)

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