On-Shell Diagrams for N=8 Supergravity

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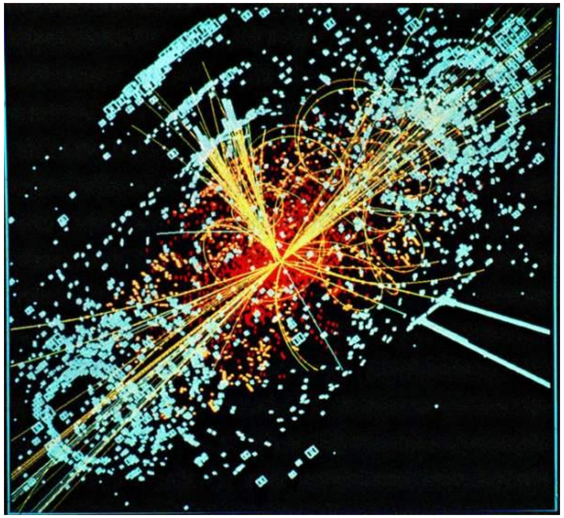
Based on 1604.03046 (Heslop,Lipstein)

see also 1604.3479 (Trnka,Herrmann)

Introduction

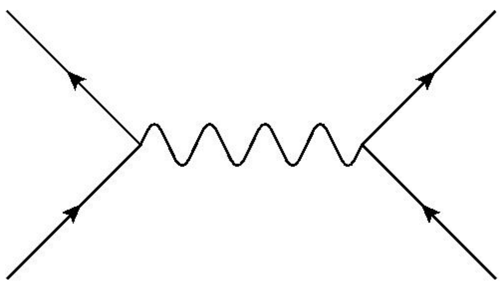
- Scattering amplitudes are the basic quantities used to compare theory with experiment.



- They also have a rich mathematical structure which is interesting in its own right.

Feynman Diagrams

- The traditional method for computing scattering amplitudes uses Feynman diagrams:



- As the number of legs increases, the number of Feynman diagrams quickly gets out of hand, even though the final answer is often surprisingly simple.

- One reason for the complexity of Feynman diagrams is that they contain off-shell states in the internal lines, whereas amplitudes only know about on-shell states.

- These difficulties can be overcome by using the analytic properties of amplitudes in order to compute them using only on-shell states.

Spinor-Helicity

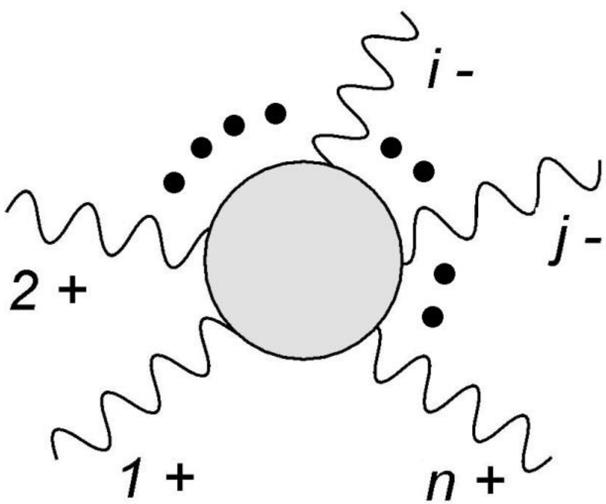
- Massless on-shell momentum in 4d:

$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

- Expressing amplitudes in terms of these spinors leads to dramatic simplifications.

MHV Amplitudes

At tree-level:



$$A_n = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \ldots \langle n1 \rangle}$$

(Parke,Taylor)

where $\langle ij \rangle = \epsilon_{\alpha\beta} \lambda_i^\alpha \lambda_j^\beta$

BCFW Recursion

- Deform two external momenta by a complex parameter which preserves on-shell properties:

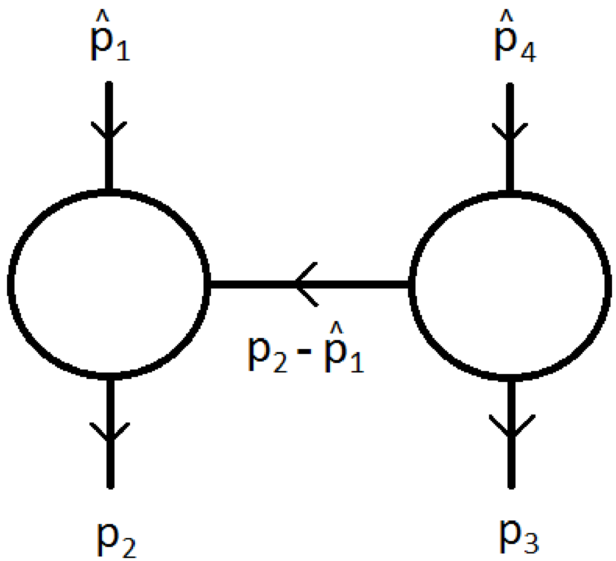
$$\lambda_1 \tilde{\lambda}_1 \to \lambda_1 \left(\tilde{\lambda}_1 - \alpha \tilde{\lambda}_n \right)$$

$$\lambda_n \tilde{\lambda}_n \to \left(\lambda_n + \alpha \lambda_1 \right) \tilde{\lambda}_n$$

- Tree amplitudes become rational functions of α, which can be reconstructed from their poles and residues. (Britto,Cachazo,Feng,Witten)

Example

- Consider deforming a 4-pt amplitude:

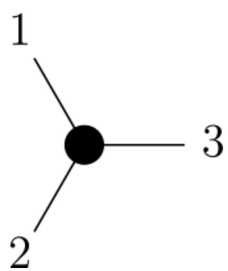


- The pole in α corresponds to $(p_2 - \hat{p}_1)^2 = 0$, and the residue corresponds to the product of two 3-point amplitudes.

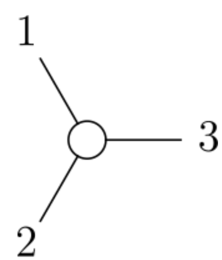
On-Shell Diagrams

- BCFW recursion can be implemented using on-shell diagrams, first developed for planar N=4 super-Yang-Mills theory by Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, Trnka.

- The building blocks are 3-point amplitudes:



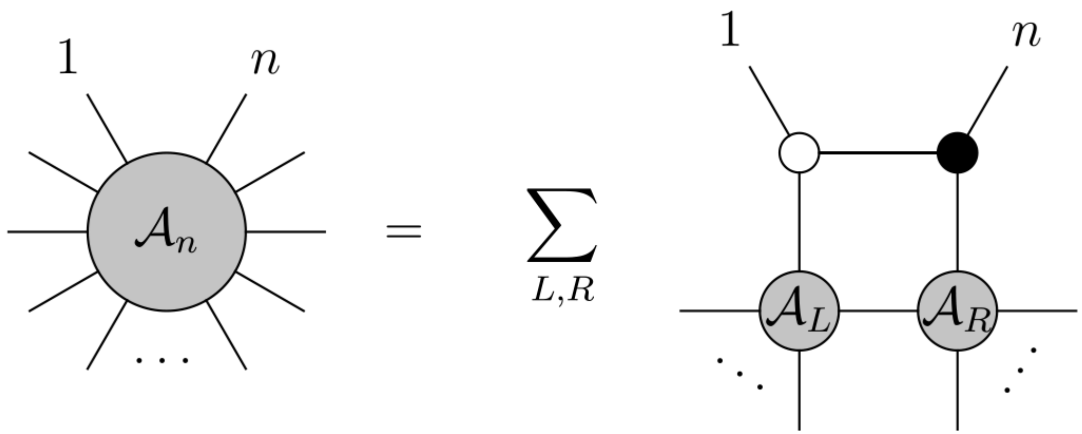
$\tilde{\lambda}_1 \propto \tilde{\lambda}_2 \propto \tilde{\lambda}_3$



$\lambda_1 \propto \lambda_2 \propto \lambda_3$

Tree-Level Recursion

- In terms of on-shell diagrams, BCFW corresponds to

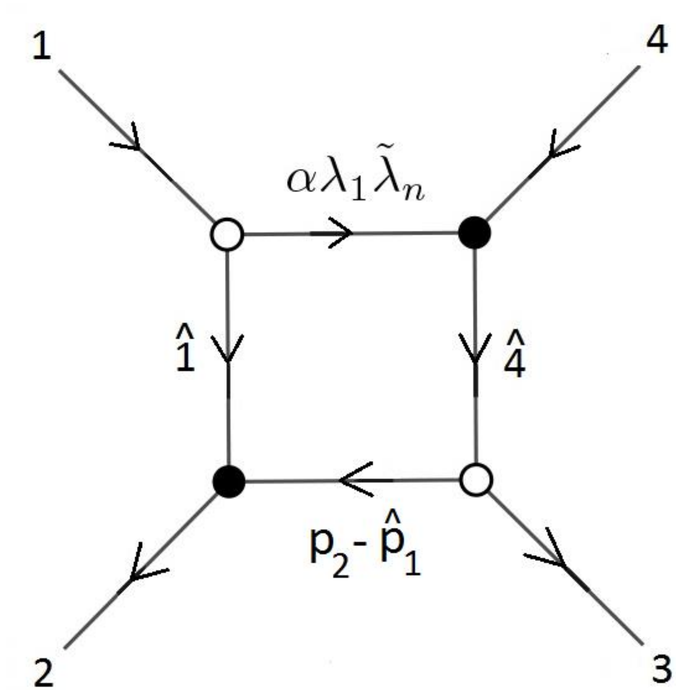


where internal lines correspond to integrals over on-shell states (no virtual particles!):

$$\int \frac{\mathrm{d}^4\tilde{\eta}\,\mathrm{d}^2\lambda\,\mathrm{d}^2\tilde{\lambda}}{\mathrm{Vol}\,\mathrm{GL}(1)}$$

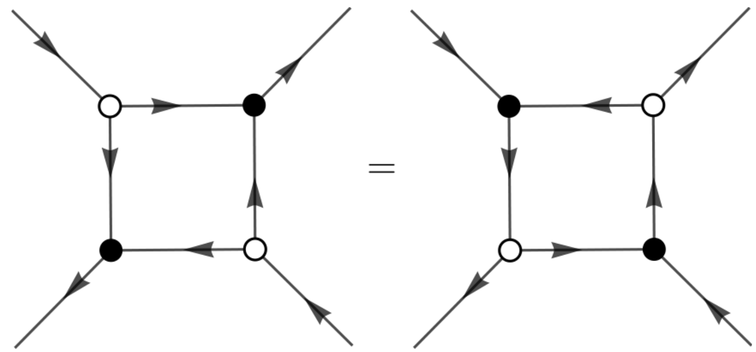
Example

- 4-point tree amplitude:

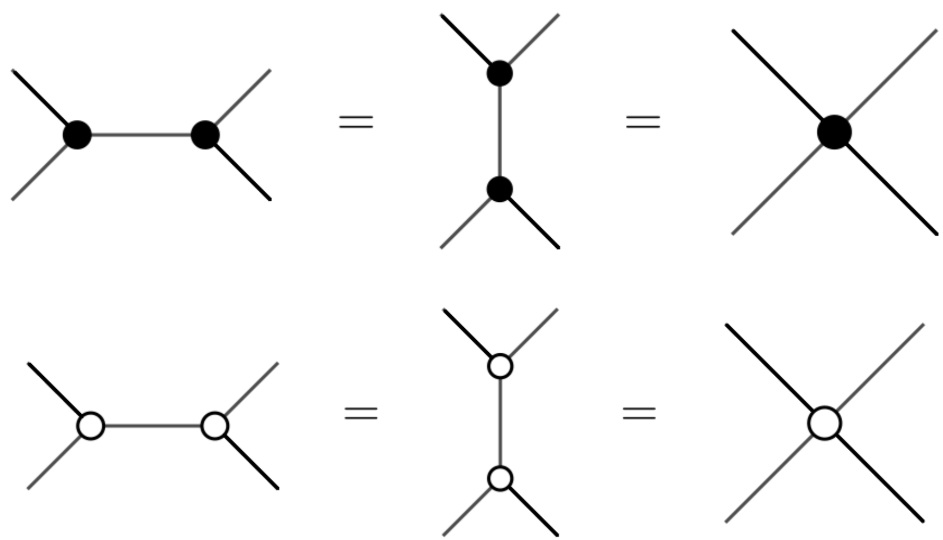


Equivalence Relations

- Square move:



- Mergers:



Positivity

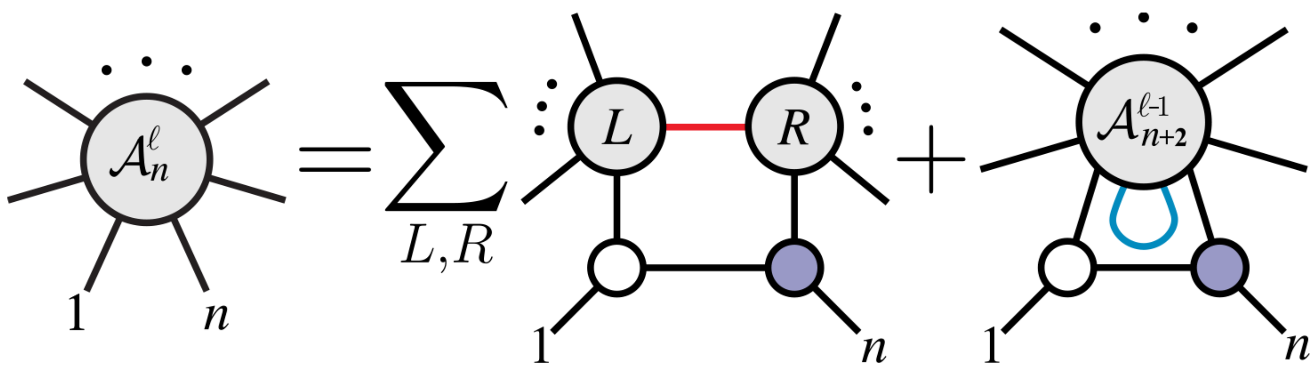
- On-shell diagrams are in one-to-one correspondence with permutations.

- They are also in one-to-one correspondence with cells of the positive Grassmannian.

- This suggests a new interpretation of scattering amplitudes as the volume of an object known as the Amplituhedron (Arkani-Hamed, Trnka).

Loop-Level Recursion

- For planar N=4 SYM there is a canonical definition for the loop integrand, making it possible to extend BCFW recursion to loop level:



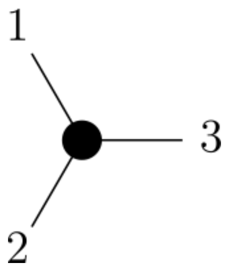
N=8 SUGRA

- An important question is how to generalize these ideas beyond planar N=4 SYM.
- In this talk, I will describe on-shell diagrams for N=8 supergravity, which has been argued to be the simplest QFT in four dimensions (Arkani-Hamed, Cachazo, Kaplan).
- Perturbative finiteness of N=8 SUGRA is an important open problem. (Green, Russo, Vanhove / Bern, Carrasco, Dixon, Johansson, Kosower, Roiban)
- Fantasy: Use on-shell diagrams to deduce
  - an all-loop integrand for N=8 SUGRA
  - a gravitational “Amplituhedron”

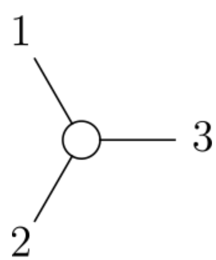
Overview

- I will primarily focus on tree-level amplitudes. Reformulating BCFW recursion in terms of on-shell diagrams will reveal interesting new relations to N=4 SYM such as:
  - non-planar identities
  - equivalence relations
  - Grassmannians
- Moreover, I will describe a simple algorithm for reading off formulae for on-shell diagrams.
- Finally, I will briefly speculate on the extension of these ideas to loop level.

Building Blocks



$$= \frac{\delta^{16}\left(\lambda_1\tilde{\eta}_1 + \lambda_2\tilde{\eta}_2 + \lambda_3\tilde{\eta}_3\right)\delta^4\left(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3\right)}{\langle 12\rangle^2\,\langle 23\rangle^2\,\langle 31\rangle^2}$$

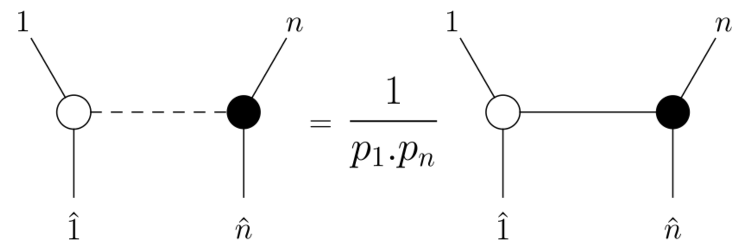


$$= \frac{\delta^{8}\left([12]\,\tilde{\eta}_3 + [23]\tilde{\eta}_1 + [31]\tilde{\eta}_2\right)\delta^4\left(\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3\right)}{[12]^2[23]^2[31]^2}$$

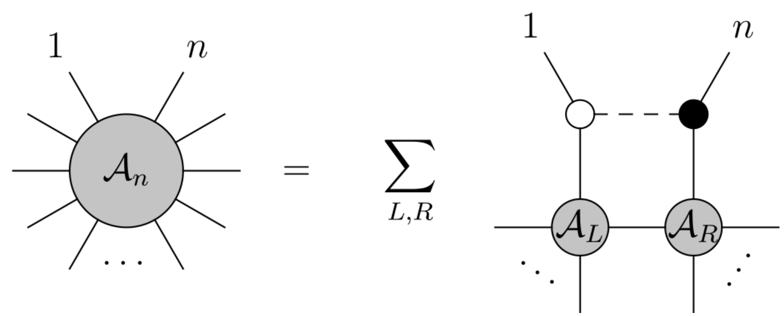
$$\int \frac{\mathrm{d}^8\tilde{\eta}\,\mathrm{d}^2\lambda\,\mathrm{d}^2\tilde{\lambda}}{\mathrm{VolGL}(1)}$$

Tree-Level Recursion

- Naïve BCFW bridge doesn't work; need to decorate it!



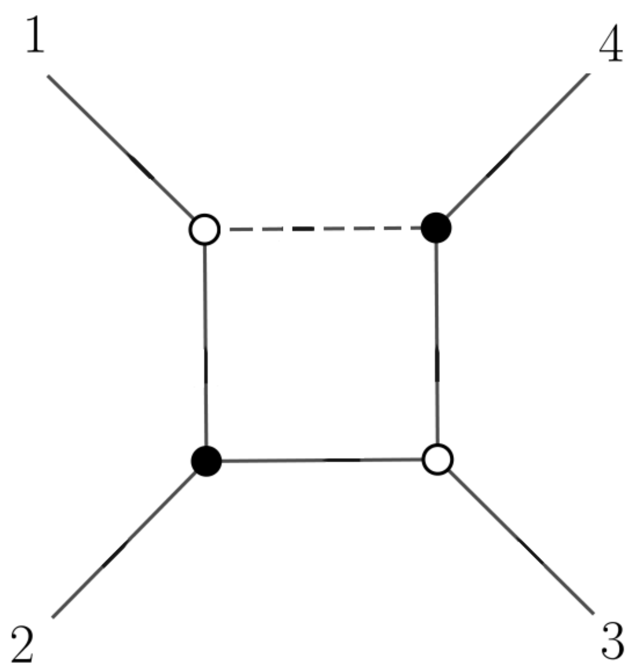
- In terms of the decorated bridge, BCFW is given by



where the sum is over all partitions of particles {2,…,n-1} into two sets L,R.

Examples

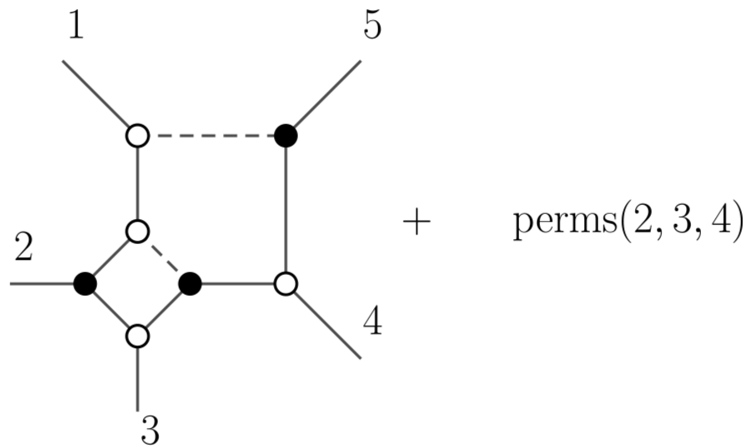
4-point:



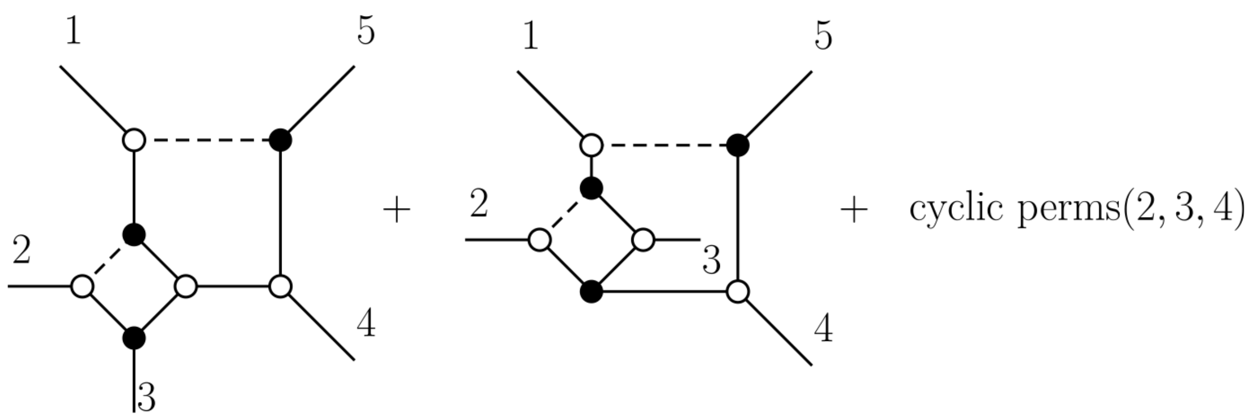
$+ \quad 2 \leftrightarrow 3$

Examples

5-point:



Alternatively:



Planar Sector

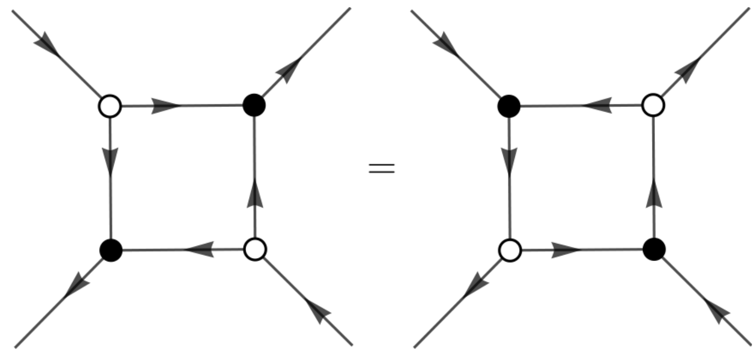
- If we always insert the fixed legs of each subdiagram into the recursion relation to obtain higher-point amplitudes, the result will always be a sum over planar diagrams which are exactly the same as those appearing in planar N=4 SYM.

- In summary, one can obtain N=8 SUGRA amplitudes simply by decorating on-shell diagrams of the corresponding amplitude in planar N=4 SYM and summing over permutations of the unshifted legs!

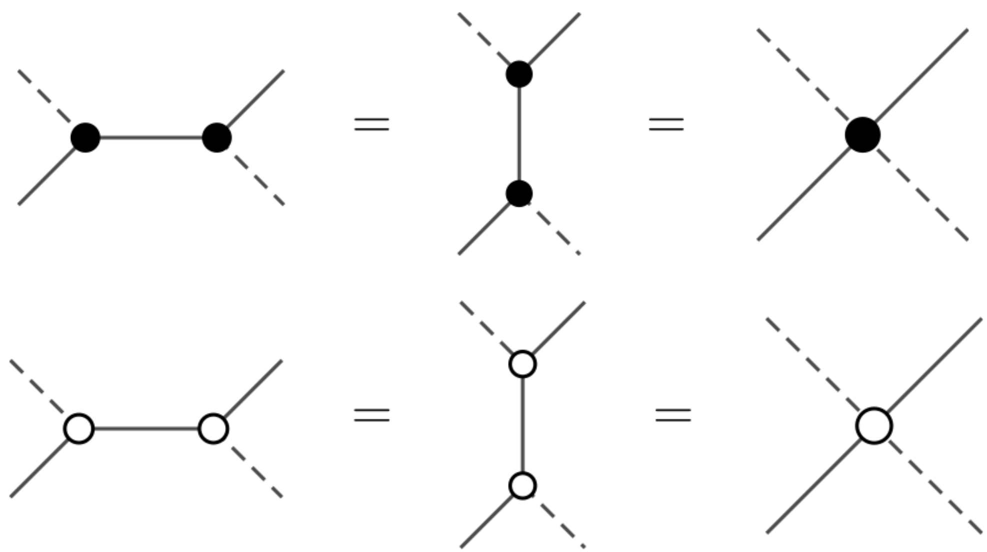
- On the other hand, if we choose to carry out the recursion in a different way this will generically give non-planar diagrams, implying remarkable new identities.

Equivalence Relations

- Square move:



- Mergers:



Grassmannians

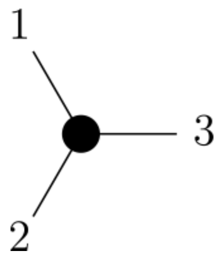
- I will now describe an algorithm for reading off expressions for on-shell diagrams in the form of Grassmannian integral formulae, which also play a prominent role in the scattering amplitudes of planar N=4 SYM.

- The Grassmannian Gr(k,n) is the space of k-planes in n-dimensions, or equivalently the set of kxn matrices modulo the left action of GL(k). In the context of amplitudes, k refers to the MHV degree and n refers to the number of external legs.

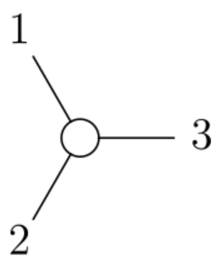
- Given a k-plane C, define $C^{\perp}$ to be the orthogonal (n-k) plane whose minors obey

$$(i_{k+1} \ldots i_n)^{\perp} = (i_1 i_2 \ldots i_k)\epsilon^{i_1 i_2 \ldots i_k}{}_{i_{k+1} \ldots i_n}$$

3-point Amplitudes



$$= \int \frac{d^{2\times 3}C}{GL(2)} \frac{\delta^{4|16}\left(C\cdot\tilde{\lambda}|C\cdot\tilde{\eta}\right)\delta^{2}\left(\lambda\cdot C^{\perp}\right)}{(12)^{2}\,(23)^{2}\,(31)^{2}} \frac{\langle ij\rangle}{(ij)}$$

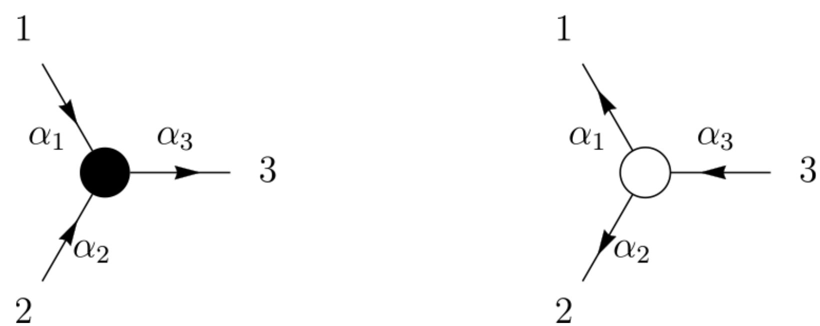


$$= \int \frac{d^{1\times 3}C}{GL(1)} \frac{\delta^{2|8}\left(C\cdot\tilde{\lambda}|C\cdot\tilde{\eta}\right)\delta^{4}\left(\lambda\cdot C^{\perp}\right)}{(1)^{2}\,(2)^{2}\,(3)^{2}} \frac{[ij]}{(ij)^{\perp}}$$

To see that these formulae reduce to the standard ones, use GL(2) to set C=(λ_1 λ_2 λ_3) in the first case, and use GL(1) symmetry to set C=([23] [31] [12]) in the second case.

Canonical Coordinates

- There is a canonical way to choose coordinates of Grassmannian by assigning arrows and variables to the edges of the on-shell diagram. For 3-point amplitudes, this choice can be displayed as follows:



- In terms of these variables, C and $C^{\perp}$ are then determined by the following rules:

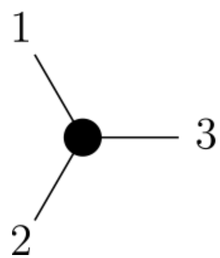
$$\tilde{\lambda}_i = \sum_{\substack{\text{paths} \\ i \to j}} \Big(\prod_{\substack{\text{edges} \\ \text{in path:}e}} \alpha_e \Big) \tilde{\lambda}_j \qquad \lambda_i = \sum_{\substack{\text{paths} \\ i \leftarrow j}} \Big(\prod_{\substack{\text{edges} \\ \text{in path:}e}} \alpha_e \Big) \lambda_j$$

- For the black and white vertices, this gives respectively

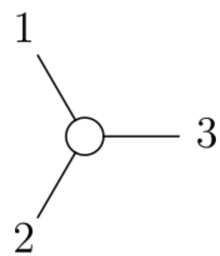
$$C_{\mathrm{MHV}} = \begin{pmatrix} 1 & 0 & -\alpha_1\alpha_3 \\ 0 & 1 & -\alpha_2\alpha_3 \end{pmatrix} \qquad C^{\perp}_{\mathrm{MHV}} = \begin{pmatrix} -\alpha_1\alpha_3 & -\alpha_2\alpha_3 & 1 \end{pmatrix}$$

$$C_{\overline{\mathrm{MHV}}} = \begin{pmatrix} -\alpha_1\alpha_3 & -\alpha_2\alpha_3 & 1 \end{pmatrix} \qquad C^{\perp}_{\overline{\mathrm{MHV}}} = \begin{pmatrix} 1 & 0 & -\alpha_1\alpha_3 \\ 0 & 1 & -\alpha_2\alpha_3 \end{pmatrix}$$

- Plugging this into the Grassmannian integral formulae presented earlier then gives the following expressions:



$$= \langle 12 \rangle \int d\left(\alpha_1\alpha_3\right) d\left(\alpha_2\alpha_3\right) \frac{\delta^{4|16}\left(C \cdot \tilde{\lambda} | C \cdot \tilde{\eta}\right) \delta^{2}\left(\lambda \cdot C^{\perp}\right)}{\alpha_1^2 \alpha_2^2 \alpha_3^4}$$



$$= [12] \int d\left(\alpha_1\alpha_3\right) d\left(\alpha_2\alpha_3\right) \frac{\delta^{2|8}\left(C \cdot \tilde{\lambda} | C \cdot \tilde{\eta}\right) \delta^{4}\left(\lambda \cdot C^{\perp}\right)}{\alpha_1^2 \alpha_2^2 \alpha_3^4}$$

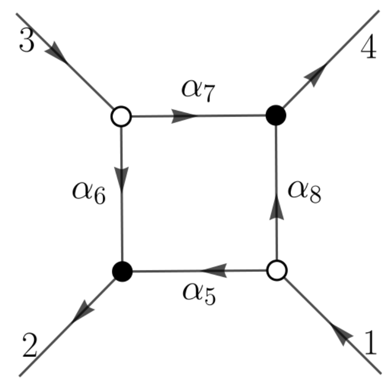
- Remarkably, these expressions can be generalized to any on-shell diagram!

Algorithm

- Draw arrows on each edge such that there are two arrows entering/one arrow leaving every black node and two arrows leaving/one arrow entering every white node.
- Label every edge with a variable α, and set one variable associated to each vertex to unity.
- Associate $d\alpha/\alpha^2$ with each edge variable leaving a white vertex or entering a black vertex and $d\alpha/\alpha^3$ with each edge variable entering a white vertex or leaving a black vertex.
- For each black vertex associate the bracket $\langle ij \rangle$ where i,j are the two edges with ingoing arrows. For each white vertex associate the bracket [ij] where i,j are the two edges with outgoing arrows.
- Determine C and $C^{\perp}$ using the rules on a previous slide.

Example

- First consider the following undecorated 4-point diagram:



- Using the rules on the previous slide, we obtain

$$\mathcal{A}_4 = \int \frac{d\alpha_5 d\alpha_6 d\alpha_7 d\alpha_8}{\alpha_5^2 \alpha_6^2 \alpha_7^2 \alpha_8^2} \langle 56 \rangle \langle 78 \rangle [67] [58] \, \delta^{4|16} \left(C \cdot \tilde{\lambda} | C \cdot \tilde{\eta} \right) \delta^4 \left(\lambda \cdot C^{\perp} \right)$$

where $C = \begin{pmatrix} 1 & -\alpha_5 & 0 & -\alpha_8 \\ 0 & -\alpha_6 & 1 & -\alpha_7 \end{pmatrix}$ $\quad C^{\perp} = \begin{pmatrix} -\alpha_5 & 1 & -\alpha_6 & 0 \\ -\alpha_8 & 0 & -\alpha_7 & 1 \end{pmatrix}$

- We then use the path prescription to rewrite the internal brackets as external ones:

$$\tilde{\lambda}_5 = \tilde{\lambda}_2 \qquad \tilde{\lambda}_6 = \tilde{\lambda}_2 \qquad \tilde{\lambda}_7 = \tilde{\lambda}_4 \qquad \tilde{\lambda}_8 = \tilde{\lambda}_4$$

$$\lambda_5 = \alpha_5 \lambda_1 \qquad \lambda_6 = \alpha_6 \lambda_3 \qquad \lambda_7 = \alpha_7 \lambda_3 \qquad \lambda_8 = \alpha_8 \lambda_1$$

- Plugging this into the expression on the previous slide gives

$$\mathcal{A}_4 = \int \frac{d\alpha_5 d\alpha_6 d\alpha_7 d\alpha_8}{\alpha_5 \alpha_6 \alpha_7 \alpha_8} \langle 13 \rangle^2 [24]^2 \delta^{4|16} \left(C \cdot \tilde{\lambda} | C \cdot \tilde{\eta} \right) \delta^4 \left(\lambda \cdot C^{\perp} \right)$$

which uplifts to the following covariant formula:

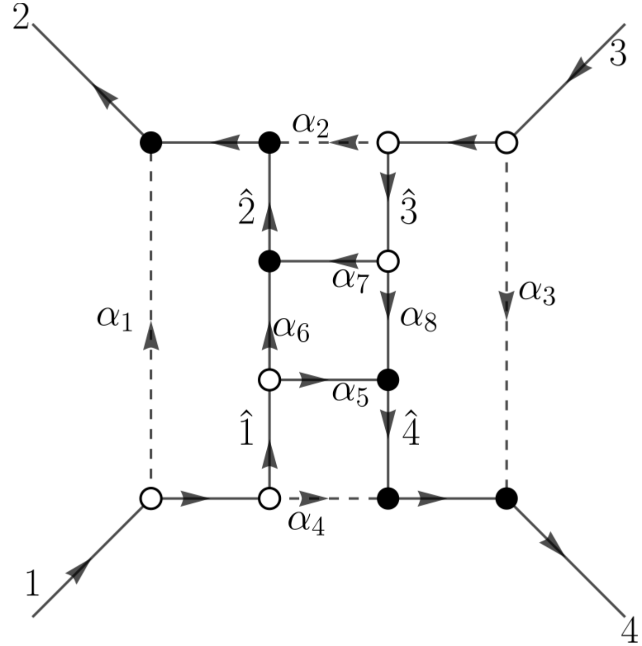
$$\mathcal{A}_4 = \int \frac{d^{2\times 4} C}{GL(2)} \frac{\delta^{4|16} \left(C \cdot \tilde{\lambda} | C \cdot \tilde{\eta} \right) \delta^4 \left(\lambda \cdot C^{\perp} \right)}{(12)(23)(34)(41)} \frac{\langle kl \rangle^2}{(kl)^2} \frac{[pq]^2}{(p^{\perp} q^{\perp})^2}$$

- Dividing by the bridge factor <12>[12] and summing over the permutations of legs 3 and 4 finally gives the following Grassmannian integral formula for the 4-point amplitude:

$$\mathcal{M}_4 = \int \frac{d^{2\times 4}C}{GL(2)} \frac{\delta^{4|16}\left(C\cdot\tilde{\lambda}|C\cdot\tilde{\eta}\right)\delta^4\left(\lambda\cdot C^{\perp}\right)}{\Pi_{i<j}(ij)} \frac{\langle kl\rangle}{(kl)} \frac{[pq]}{(p^{\perp}q^{\perp})}$$

Loops

Remarkably, decorating the on-shell diagram corresponding to the 4-point 1-loop amplitude in planar N=4 SYM and summing over permutations of the external legs gives the 1-loop 4-point amplitude of N=8 SUGRA!



Future Directions

- Loop-level recursion?
- Relation to ambitwistor string theory?
- Integrability?

Thank You