

# Ambitwistor strings

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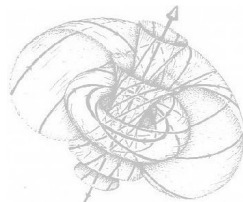
ASC Munich, 6 Sept. 2016

With David Skinner. arxiv:1311.2564, and collaborations with Tim Adamo, Eduardo Casali, Yvonne Geyer, Arthur Lipstein, Ricardo Monteiro, Kai Roehrig, & Piotr Tourkine, 1312.3828, 1404.6219, 1405.5122, 1406.1462, 1506.08771, 1507.00321, 1511.06315.

[Cf. also Cachazo, He, Yuan arxiv:1306.2962, 1306.6575, 1307.2199, 1309.0885, 1412.3479]

**Ambitwistor spaces:** spaces of complex null geodesics.

- Extends Penrose/Ward's gravity/Yang-Mills twistor constructions to non-self-dual fields.
- Yang-Mills Witten and Isenberg, et. al. 1978, 1985.
- Conformal and Einstein gravity LeBrun [1983,1991]  
Baston & M. [1987] .



**Ambitwistor Strings:**

- Tree S-Matrices in all dimensions for gravity, YM etc. [CHY]
- From strings in ambitwistor space [M. & Skinner 1311.2564]
- New models for Einstein-YM, DBI, BI, NLS, etc. [Casali, Geyer, M.,  
Monteiro, Roehrig 1506.08771].
- Loop integrands from the Riemann sphere [Geyer, M., Monteiro,  
Tourkine, 1507.00321].

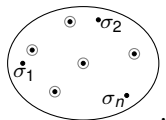
Provide string theories at  $\alpha' = 0$  for field theory amplitudes.

# The scattering equations

Take  $n$  null momenta  $k_i \in \mathbb{R}^d$ ,  $i = 1, \dots, n$ ,  $k_i^2 = 0$ ,  $\sum_i k_i = 0$ ,

- define  $P : \mathbb{CP}^1 \rightarrow \mathbb{C}^d$

$$P(\sigma) := \sum_{i=1}^n \frac{k_i}{\sigma - \sigma_i}, \quad \sigma, \sigma_i \in \mathbb{CP}^1$$



- Solve for  $\sigma_i \in \mathbb{CP}^1$  with the  $n$  scattering equations [Fairlie 1972]

$$\text{Res}_{\sigma_i} (P^2) = k_i \cdot P(\sigma_i) = \sum_{j=1}^n \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0.$$

$$\Rightarrow P^2 = 0 \quad \forall \sigma.$$

- For Möbius invariance  $\Rightarrow P \in \mathbb{C}^d \otimes K$ ,  $K = \Omega^{1,0} \mathbb{CP}^1$
- There are  $(n-3)!$  solutions.

Arise in large  $\alpha'$  strings [Gross-Mende 1988] & twistor-strings [Roiban, Spradlin,

# Amplitude formulae for massless theories.

## Proposition (Cachazo, He, Yuan 2013,2014)

*Tree-level massless amplitudes in  $d$ -dims are integrals/sums*

$$\mathcal{M}_n = \delta^d \left( \sum_i k_i \right) \int_{(\mathbb{CP}^1)^n} \frac{\mathcal{I}^l \mathcal{I}^r \prod_i \bar{\delta}(k_i \cdot P(\sigma_i))}{\text{Vol SL}(2, \mathbb{C}) \times \mathbb{C}^3}$$

where  $\mathcal{I}^{l/r} = \mathcal{I}^{l/r}(\epsilon_i^{l/r}, k_i, \sigma_i)$  depend on the theory.

- polarizations  $\epsilon_i^l$  for spin 1,  $\epsilon_i^l \otimes \epsilon_i^r$  for spin-2 ( $k_i \cdot \epsilon_i = 0 \dots$ ).
- Introduce skew  $2n \times 2n$  matrices  $M = \begin{pmatrix} A & C \\ -C^t & B \end{pmatrix}$ ,

$$A_{ij} = \frac{k_i \cdot k_j}{\sigma_i - \sigma_j}, \quad B_{ij} = \frac{\epsilon_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad C_{ij} = \frac{k_i \cdot \epsilon_j}{\sigma_i - \sigma_j}, \quad \text{for } i \neq j$$

and  $A_{ii} = B_{ii} = 0$ ,  $C_{ii} = \epsilon_i \cdot P(\sigma_i)$ .

- For YM,  $\mathcal{I}^l = Pf'(M)$ ,  $\mathcal{I}^r = \prod_i \frac{1}{\sigma_i - \sigma_{i-1}}$ .
- For GR  $\mathcal{I}^l = Pf'(M^l)$ ,  $\mathcal{I}^r = Pf'(M^r)$ .

# More CHY formulae:

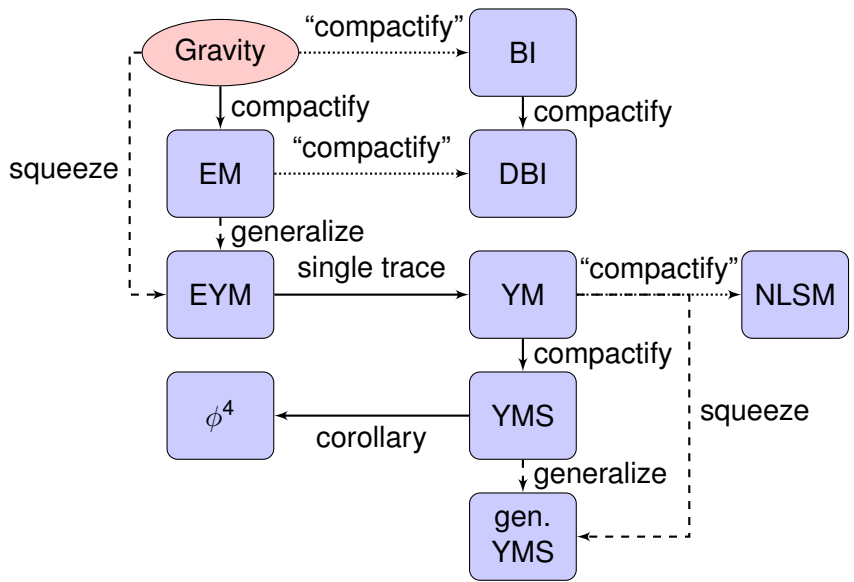


Figure: Theories studied by CHY and operations relating them.

# Ambitwistors from chiral bosonic strings at $\alpha' = 0$

## Bosonic ambitwistor string action:

- $\Sigma$  Riemann surface, coordinate  $\sigma \in \mathbb{C}$
- Complexify space-time  $(M, g)$ , coords  $X \in \mathbb{C}^d$ ,  $g$  hol.
- $(X, P) : \Sigma \rightarrow T^*M$ ,  $P \in K$ , holomorphic 1-forms on  $\Sigma$ .

$$S_B = \int_{\Sigma} P_{\mu} \bar{\partial} X^{\mu} - e P^2 / 2.$$

## Underlying geometry:

- $e$  enforces  $P^2 = 0$ ,
- $P^2$  generates gauge freedom:  $\delta(X, P, e) = (\alpha P, 0, 2\bar{\partial}\alpha)$ .

So target is

$$\mathbb{A} = T^*M|_{P^2=0} / \{\text{gauge}\}.$$

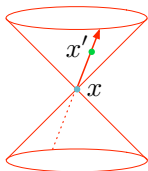
This is *Ambitwistor space*, space of complexified light rays.

# The geometry of the space of light rays

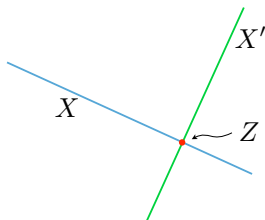
Ambitwistor space  $\mathbb{A}$  is space of complexified light rays.

- Light rays primary, events determined by lightcones  $X \subset \mathbb{A}$  of light rays incident with  $x$ .
- Space-time  $M =$  space of such  $X \subset \mathbb{A}$ .

Space-time



Twistor Space



Space-time geometry is encoded in complex structure of  $\mathbb{A}$ .

**Theorem (LeBrun 1983 following Penrose 1976)**

*Complex structure of  $\mathbb{A}$  determines  $(M, [g])$ . Correspondence stable under deformations of  $P\mathbb{A}$  that preserve  $\theta = P_\mu dX^\mu$ .*

## Quantize bosonic ambitwistor string:

- $(X, P) : \Sigma \rightarrow T^*M,$

$$S_B = \int_{\Sigma} P_{\mu}(\bar{\partial} + \tilde{e}\partial)X^{\mu} - e P^2/2.$$

- Gauge fix  $\tilde{e} = e = 0, \rightsquigarrow$  ghosts & BRST  $Q$
- Introduce vertex operators  $V_i \leftrightarrow$  field perturbations.

Amplitudes are computed as correlators of vertex ops

$$\mathcal{M}_n = \langle V_1 \dots V_n \rangle$$

For gravity add type II worldsheet susy  $S_{\psi_1} + S_{\psi_2}$  where

$$S_{\psi} = \int_{\Sigma} \psi_{\mu} \bar{\partial} \psi^{\mu} + \chi P \cdot \psi.$$



# From deformations of $\mathbb{A}$ to the scattering equations

Gravitons  $\leftrightarrow$  vertex operators  $V_i = \text{def'm of action } \delta S = \int_{\Sigma} \delta\theta.$

- $\theta$  determines complex structure on  $P\mathbb{A}$  via  $\theta \wedge d\theta^{d-2}$ . So:
- Deformations of complex structure  $\leftrightarrow [\delta\theta] \in H^1_{\partial}(P\mathbb{A}, L)$ .

## Proposition

*For perturbation  $\delta g_{\mu\nu} = e^{ik \cdot X} \epsilon_{\mu} \epsilon_{\nu}$  of flat space-time*

$$\delta\theta = \bar{\delta}(k \cdot P) e^{ik \cdot X} (\epsilon \cdot P)^2$$

**Proof:** Penrose transform.

**Ambitwistor repr**  $\Rightarrow \bar{\delta}(k \cdot P) \Rightarrow$  scattering equs.

## Proposition

*CHY formulae for massless tree amplitudes e.g. YM & gravity arise from appropriate choices of worldsheet matter.*

- Take  $e^{ik_i \cdot X(\sigma_i)}$  factors into action to give

$$S = \frac{1}{2\pi} \int_{\Sigma} P \cdot \bar{\partial} X + 2\pi \sum_i ik \cdot X(\sigma_i).$$

- Gives field equations  $\bar{\partial} X = 0$  and,

$$\bar{\partial} P = 2\pi \sum_i ik \delta^2(\sigma - \sigma_i).$$

- Solutions  $X(\sigma) = X = \text{const.}$ ,  $P(\sigma) = \sum_i \frac{k_i}{\sigma - \sigma_i} d\sigma$ .

Thus path-integral reduces to

$$\mathcal{M}_n = \delta^d \left( \sum_i k_i \right) \int_{(\mathbb{CP}^1)^{n-3}} \frac{\prod_i (\epsilon_i \cdot P(\sigma_i))^2 \bar{\delta}(k_i \cdot P)}{\text{Vol } G}$$

We see  $P(\sigma)$  appearing and scattering equations.

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We see  $P(\sigma)$  appearing and scattering equations.

**Unfortunately:** amplitudes for  $\sim \int_M R + R^3$ , cf [Hohm, Segal, Zwieback].

- Decorate null geodesics with spin vectors, vectors for internal degrees of freedom & other holomorphic CFTs.
- Take

$$S = S_B + S^l + S^r$$

where  $S^l, S^r$  are some worksheet matter CFTs.

- Total vertex operators given by

$$v^l v^r \bar{\delta}(k \cdot P) e^{ik \cdot X}$$

with  $v^l, v^r$  worksheet currents from  $S^l, S^r$  resp..

- Amplitudes become

$$\mathcal{M}_n = \delta^d \left( \sum_i k_i \right) \int_{(\mathbb{CP}^1)^n} \frac{\mathcal{I}^l \mathcal{I}^r \prod_i \bar{\delta}(k_i \cdot P)}{\text{Vol Gauge}}$$

where  $\mathcal{I}^l, \mathcal{I}^r$  are worksheet correlators of  $v^l$ s,  $v^r$ s resp..

- Q-invariance and discrete symmetries (GSO) rule out unwanted vertex operators in good situations.

- **Worksheet SUSY:**  $S_\Psi = \int g_{\mu\nu} \Psi^\mu \bar{\partial} \Psi^\nu - \chi P_\mu \Psi^\mu$  gives CHY Pfaffians from worldsheet correlator

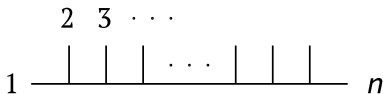
$$\mathcal{I}^{l/r} = \langle u_1 u_2 v_3 \dots v_n \rangle = Pf'(M).$$

- **Free fermions and current algebras:** gives 'Parke-Taylor' correlators + unwanted multi-trace terms

$$\langle v_1 \dots v_n \rangle = \frac{\text{tr}(t_1 \dots t_n)}{\sigma_{12} \sigma_{23} \dots \sigma_{n1}} + \dots \quad \text{where } \sigma_{ij} = \sigma_i - \sigma_j.$$

- **Comb system:** [Casali-Skinner]

Combines level zero current algebra & spin 3/2 gauging. Gives Parke Taylor *without* unwanted multitrace terms, and colour structure as *comb*:



# The 2013 CHY formulae & ambitwistor models

Above lead essentially to original models & formulae:

- $(S', S^r) = (S_{\tilde{\psi}}, S_{\psi}) \rightsquigarrow$  type II gravity,
- $(S', S^r) = (S_{CS}, S_{\psi}) \rightsquigarrow$  heterotic with YM,
- $(S', S^r) = (S_{CS}, S_{CS}) \rightsquigarrow$  bi-adjoint scalar.

The latter two come with unphysical gravity.

$S_{CS}$  improves on current algebras in avoiding multi-trace terms and all models critical in 10d.

$$S_{\Psi_1, \Psi_2} = S_{\Psi_1} + S_{\Psi_2}$$

two worldsheet susy's for  $S^l$  or  $S^r$  (this is maximum).

$$S_{\Psi, \rho} = S_{\Psi} + S_{\rho}$$

combines 'real' Fermions with susy.

$$S_{\Psi, CS} = S_{\Psi} + S_{CS}$$

but with same spin 3/2 gauge field (worldsheet Rarita-Schwinger) in both.

GSO now reverses signs of all fields in matter system.

# Ambitwistor strings with combinations of matter

CGMMRS 1506.08771

$S' \backslash S^r$	$S_\Psi$	$S_{\Psi_1, \Psi_2}$	$S_{\rho, \Psi}^{(\tilde{m})}$	$S_{CS, \Psi}^{(\tilde{N})}$	$S_{CS}^{(\tilde{N})}$
$S_\Psi$	E				
$S_{\Psi_1, \Psi_2}$	BI	Galileon			
$S_{\rho, \Psi}^{(m)}$	$EM$ $U(1)^m$	DBI	$EMS$ $U(1)^m \times U(1)^{\tilde{m}}$		
$S_{CS, \Psi}^{(N)}$	EYM	ext. DBI	$EYMS$ $SU(N) \times U(1)^{\tilde{m}}$	$EYMS$ $SU(N) \times SU(\tilde{N})$	
$S_{CS}^{(N)}$	YM	Nonlinear $\sigma$	$EYMS$ $SU(N) \times U(1)^{\tilde{m}}$	<i>gen. YMS</i> $SU(N) \times SU(\tilde{N})$	<i>Biadjoint Scalar</i> $SU(N) \times SU(\tilde{N})$

**Table:** Theories arising from the different choices of matter models.



# Models from different geometric realizations of $\mathbb{A}$

Can start with other formulations of null superparticles

- Pure spinor version (Berkovits)  $S = \int P \cdot \bar{\partial} X + p_\alpha \bar{\partial} \theta^\alpha + \dots$
- In  $d = 4$  have (super) Twistor space  $\mathbb{T} := \mathbb{C}^{4|\mathcal{N}}$

$$\mathbb{A} = T^*\mathbb{PT} := \{(Z, W) \in \mathbb{T} \times \mathbb{T}^* \mid Z \cdot W = 0\} / \{Z \cdot \partial_Z - W \cdot \partial_W\}$$

$$S = \int_{\Sigma} W \cdot \bar{\partial} Z + a Z \cdot W$$

$\rightsquigarrow$  Twistor-strings [Witten, Berkovits & Skinner].

- In 4d have full ambitwistor representation [Geyer, Lipstein, M. 1404.6219]

$$S = \int_{\Sigma} Z \cdot \bar{\partial} W - W \cdot \bar{\partial} Z + a Z \cdot W$$

Not twistor string:  $(Z, W) \in K^{1/2}$  gives simpler 4d formulae with no moduli. Nonchiral, working with no supersymmetry.

- Adapts to null infinity  $\mathcal{I}$ :  $\mathbb{A} = T^*\mathcal{I}$ , admits BMS symmetries and Ward identity proof of soft theorems.

The string paradigm gives

$$\mathcal{M}_n = \text{[disk with 6 dots]} + \text{[torus with 6 dots]} + \dots + \text{[genus-2 surface with 6 dots]} + \dots$$

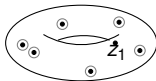
Can we make sense of this at 1 loop, i.e., on a torus?

Need critical model with all anomalies cancelling, i.e., type II super-gravity.

# 1-loop: the scattering equations on a torus

[Adamo, Casali, Skinner 2013, Casali Tourkine 2014 Geyer, M., Monteiro, Tourkine 2015]

On torus  $\Sigma_q = \mathbb{C}/\{\mathbb{Z} \oplus \mathbb{Z}\tau\}$ ,  $q = e^{2\pi i\tau}$ , solve



$$\bar{\partial}P = 2\pi i \sum_i k_i \bar{\delta}(z - z_i) dz \quad \text{with}$$

$$P = 2\pi i \ell dz + \sum_i k_i \left( \frac{\theta'_1(z - z_i)}{\theta_1(z - z_i)} + \frac{\theta'_1(z_i - z_0)}{\theta_1(z_i - z_0)} \right) dz.$$

zero-modes  $\ell \in \mathbb{R}^d \leftrightarrow$  loop momenta ( $z_0$  some fixed basepoint).

**Scattering eqs:**

$$\text{Res}_{z_i} P^2 := k_i \cdot P(z_i) = 0, \quad i = 2, \dots, n, \quad P(z_0)^2 = 0.$$

Gives amplitude formula

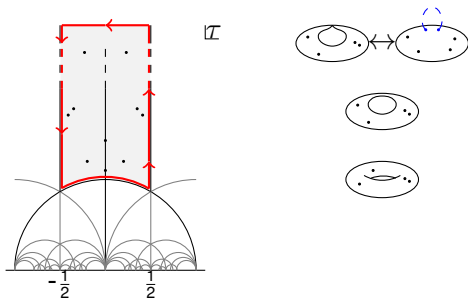
$$\mathcal{M}_{\text{SG}}^{(1)} = \int \mathcal{I}_q d^d \ell d\tau \bar{\delta}(P^2(z_0)) \prod_{i=2}^n \bar{\delta}(k_i \cdot P(z_i)) dz_i.$$

Localizes on discrete set of solutions to scattering eqs.

With  $\mathcal{I}_q = 1$ , conjectured to be permutations sum of  $n$ -gons.

# From the elliptic curve to the Riemann sphere

[Geyer, M., Monteiro, Tourkine 1507.00321]



$\sum \{\text{residues at } P^2(z_0) = 0\} = \{\text{residue at } q = 0\}$  so

$$\begin{aligned} \mathcal{M}_n^{(1)} &= \int \mathcal{I}_q d^d \ell \frac{dq}{q} \bar{\partial} \frac{1}{P^2(z_0)} \prod_{i=2}^n \bar{\delta}(k_i \cdot P(z_i)) dz_i, \\ &= - \int \mathcal{I}_0 d^d \ell \frac{1}{\ell^2} \prod_{i=2}^n \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2}, \end{aligned}$$

# Off-shell scattering eqs and $n$ -gon conjecture

At  $q = 0$

$$P(z) = P(\sigma) = \ell \frac{d\sigma}{\sigma} + \sum_{i=1}^n \frac{k_i d\sigma}{\sigma - \sigma_i}.$$

Set  $S = P^2 - \ell^2 d\sigma^2/\sigma^2$ , gives *off-shell scattering equations*:

$$0 = \text{Res}_{\sigma_i} S = k_i \cdot P(\sigma_i) = \frac{k_i \cdot \ell}{\sigma_i} + \sum_{j \neq i} \frac{k_j \cdot k_i}{\sigma_i - \sigma_j}.$$

The  $n$ -gon conjecture becomes

$$\mathcal{M}_{n\text{-gon}}^{(1)} = - \int d^{2n+2} \ell \frac{1}{\ell^2} \prod_{i=2}^n \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2},$$

which yields

$$\mathcal{M}_n^{(1)} = \frac{(-1)^n}{\ell^2} \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n-1} \frac{1}{\ell \cdot K_{\sigma_i} + \frac{1}{2} K_{\sigma_i}^2}, \quad K_{\sigma_i} = \sum_{j=1}^i k_{\sigma_i(j)}$$

Partial fractions + shifts in  $\ell$  gives permutation sum of  $n$ -gons.

# (Super-) gravity 1-loop integrand

For supergravity  $\mathcal{I}_q = \mathcal{I}_q^L \mathcal{I}_q^R$  with  $\mathcal{I}^{L/R} \equiv \mathcal{I}^{L/R}(k_i, \epsilon_i^{L/R}, z_i | q)$ .  
At  $q = 0$

$$\mathcal{I}_0^{L/R} = 16 \left( \text{Pf}(M_2^{L/R}) - \text{Pf}(M_3^{L/R}) \right) - 2 \partial_{q^{1/2}} \text{Pf}(M_3^{L/R}),$$

So the 1-loop supergravity integrand is

$$\mathcal{M}_n^{(1)} = - \int \mathcal{I}_0^L \mathcal{I}_0^R \frac{1}{\ell^2} \prod_{i=2}^n \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i^2}.$$

Checked at 4 points algebraically and 5 points numerically.

$M_2 \leftrightarrow$  Ramond,  $M_3 \leftrightarrow$  Neveu-Schwartz:  
so dropping  $M_2$  gives non-supersymmetric theories.

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# (Super) Yang-Mills 1-loop integrand

Conjecture for super Yang-Mills at 1 loop

$$\mathcal{M}_n^{(1)} = \int \mathcal{I}_0^L PT_n \prod_{i=2}^n \bar{\delta}(k_i \cdot P(\sigma_i)) \frac{d\sigma_i}{\sigma_i},$$

i.e. replace  $\mathcal{I}_0^R$  by cyclic sum of Parke-Taylor's running through loop,

$$PT_n = \sum_{i=1, i \bmod n}^n \frac{\sigma_{0\infty}}{\sigma_{0i} \sigma_{ii+1} \sigma_{i+1i+2} \cdots \sigma_{i+n\infty}}.$$

Checked algebraically at 4 and numerically at 5 points.

Again, dropping  $M_2$  gives non-susy YM.

$PT_n^2$  integrand also work for bi-adjoint scalar [Bjerrum-Bohr, Bourjaily, Damsgaard] & [He & Yuan].  $\rightsquigarrow$  KLT at 1-loop.



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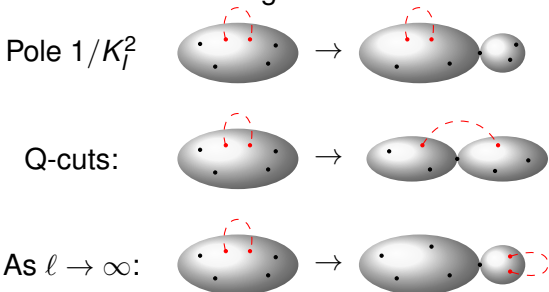
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# Proof for non-supersymmetric theories at 1-loop

Q-cuts [Bjerrum-Bohr, Bourjaily, Caron-Huot, Damsgaard, Feng] yield:

$$\mathcal{M}^{(1)} \Big|_{\text{Q-cut}} = \sum_I \text{Diagram} = \sum_I \frac{\mathcal{M}_I^{(0)} \mathcal{M}_{\bar{I}}^{(0)}}{\ell^2 (2\ell \cdot K_I + K_I^2)}$$

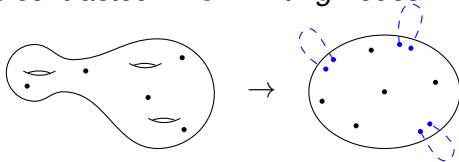
Poles arise from worldsheet degenerations:



Identification with Q-cuts follows from Liouville thm.

# Outlook: All-loop Scattering equations on $\mathbb{CP}^1$

Residue thms localize genus  $g$  moduli integrals to bdy cpt with the  $g$   $a$ -cycles contracted  $\rightsquigarrow \mathbb{CP}^1$  with  $g$  nodes.



Fixes  $g$  moduli, remaining  $2g - 3 \leftrightarrow 2g$  new marked points.  
1-form  $P$  becomes

$$P = \sum_{r=1}^g \ell_r \omega_r + \sum_i k_i \frac{d\sigma}{\sigma - \sigma_i},$$

$\omega_r$  is basis of  $g$  global holomorphic 1-forms on nodal  $\mathbb{CP}^1$ .

Now have choice of coefficients  $a_{rs}$ :

$$S(\sigma) := P^2 - \sum_{r=1}^g \ell_r^2 \omega_r^2 + \sum_{r < s} a_{rs} \omega_r \omega_s,$$

so scattering equations are  $\text{Res}_{\sigma_i} S = 0, i = 1, \dots, n + 2g.$

- **Proposal:** all-loop integrand is

$$\mathcal{M}_n^{(g)} = \int_{(\mathbb{CP}^1)^{n+2g}} d^{dg} \ell \frac{\mathcal{I}_0^L \mathcal{I}_0^R}{\text{Vol } G} \prod_{r=1}^g \frac{1}{\ell_r^2} \prod_{i=1}^{n+2g} \bar{\delta}(\text{Res}_{\sigma_i} \mathcal{S}(\sigma_i)),$$

$$\text{where } \mathcal{I}_0 = \begin{cases} \mathcal{I}_0^L \mathcal{I}_0^R, & \text{gravity} \\ \mathcal{I}_0^L PT_n, & \text{Yang-Mills} \\ PT_n PT'_n & \text{biadjoint scalar} \end{cases}.$$

Here the PT factors run through each loop.

- At two loops, factorization fixes  $a_{12} = \pm(\ell_1^2 + \ell_2^2)$ .
- Factorization also implies need for additional cross-ratios.
- **Checked numerically:** Supergravity and SYM at two loops 4 points.

Chiral  $\alpha' = 0$  ambitwistor strings use LeBrun's correspondence to give theories generalizing twistor-strings to CHY formulae.

- Gives strings whose SFT is precisely field theory.
- Incorporates colour/kinematics Yang-Mills/gravity ideas.
- Extends to many theories from DBI to Nonlinear  $\sigma$ -models.
- Critical models extend to loops on a Riemann surface.
- Higher genus formulae reduce to  $\mathbb{CP}^1$  with *off-shell scattering eqs*  $\rightsquigarrow$  loop integrands for non-critical theories.
- Gives similar complexity for  $n$ -point  $g$ -loop integrands and  $n + 2g$ -point tree amplitudes.

## Outlook

- Proof uses CHY tree formulae, but these don't exist for Ramond sector and are needed to prove susy formulae.
- Higher loops require more work.
- Origins: Siegel's chiral string or better Schild's Null strings

Thank You