Near collinear limit of gluon amplitudes and novel relations to Einstein-Yang-Mills amplitudes

Jan Plefka

Humboldt-Universität zu Berlin

with

Dhritiman Nandan and Wadim Wormsbecher, 1608.04730

Dhritiman Nandan, Oliver Schlotterer and Congkao Wen, 1607.05701

New formulations for scattering amplitudes, ASC Munich, September 2016

Overview

In the past two years saw renewed interest in soft limits & discovery of subleading soft theorems for gluons and graviton.

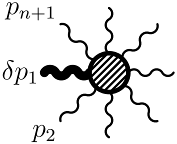
This talk:

- Study the subleading colllinear limit for gluon amplitudes
- Connected to relations between mixed gluon-graviton and gluon amplitudes.

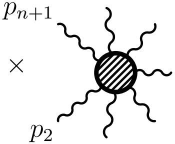
OUtline:

1. Brief review soft theorems
2. Brief review of Cachazo,He,Yuang (CHY) formalism
3. Expressing Einstein-Yang-Mills (EYM) amplitudes through YM via CHY
4. Collinear limits within CHY
5. Subleading collinear limit: Gluons, gravitons & scalars
6. Subleading collinear-soft limit for gluons
7. Summary & Outlook

Soft Theorems

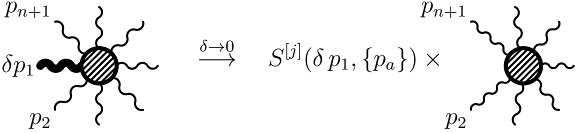


$$\xrightarrow{\delta\to 0} \quad S^{[j]}(\delta\, p_1, \{p_a\}) \times$$



Theorems of Low (1958) and Weinberg (1964)

Scattering amplitudes display universal factorization when a single photon, gluon or graviton becomes soft: Parametrize soft momentum as $\delta\, q^\mu$ and take $\delta \to 0$



$$\boxed{\mathcal{A}_{n+1}(\delta\, q, p_1, \ldots, p_n) \underset{\delta\to 0}{=} S^{[0]}(\delta\, q, \{p_a\}) \cdot \mathcal{A}_n(p_1, \ldots, p_n) + \mathcal{O}(\delta^0)}$$

At tree-level with soft leg polarization $E_{\mu(\nu)}$:

$$S^{[0]}(\delta\, q, \{p_a\}) = \begin{cases} \displaystyle\sum_{a=1}^{n} \frac{1}{\delta} \frac{E_\mu\, p_a^\mu}{p_a \cdot q} & : \text{ photon} \quad \to \text{gluon (color ordered)} \\ \displaystyle\sum_{a=1}^{n} \frac{1}{\delta} \frac{E_{\mu\nu}\, p_a^\mu\, p_a^\nu}{p_a \cdot q} & : \text{ graviton} \end{cases}$$

Proof is elementary. Tree-level exact for gravity. IR divergent loop corrections in YM.

Subleading soft theorems

Universal factorization extends to subleading order [Cachazo, Strominger][Low,Burnett,Kroll;Casali]

$$\boxed{\mathcal{A}_{n+1}(\delta\, q, p_1, \ldots, p_n) \underset{\delta\to 0}{=} S^{[j]}(\delta\, q, \{p_a\}) \cdot \mathcal{A}_n(p_1, \ldots, p_n) + \mathcal{O}(\delta^j)}$$

with soft operators

$$S^{[j]}(\delta\, q, \{p_a\}) = \begin{cases} \frac{1}{\delta} S^{(0)}_{\mathsf{YM}} + S^{(1)}_{\mathsf{YM}} & : \text{ Yang-Mills } (j = 1) \\[2em] \frac{1}{\delta} S^{(0)}_{\mathsf{G}} + S^{(1)}_{\mathsf{G}} + \delta\, S^{(2)}_{\mathsf{G}} & : \text{ Gravity } (j = 2) \end{cases}$$

Explicit constructions (using BCFW, CHY) @ tree-level yield

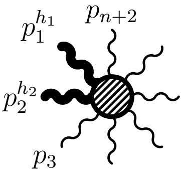
$$S^{(1)\text{tree}}_{\mathsf{YM}} = \frac{E_\mu\, q_\nu\, J_1^{\mu\nu}}{p_1 \cdot q} - \frac{E_\mu\, q_\nu\, J_n^{\mu\nu}}{p_n \cdot q} \qquad J_a^{\mu\nu} := p_a^\mu \partial_{p_a^\nu} + E_a^\mu \partial_{E_a^\nu} - \mu \leftrightarrow \nu$$

$$S^{(1)\text{tree}}_{\mathsf{G}} = \sum_{a=1}^{n} \frac{(E \cdot p_a)\, E_\mu\, q_\nu\, J_a^{\mu\nu}}{p_a \cdot q} \qquad \text{writing} \quad E_{\mu\nu} = E_\mu E_\nu$$

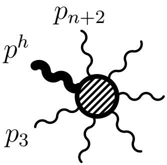
$$S^{(2)\text{tree}}_{\mathsf{G}} = \sum_{a=1}^{n} \frac{(E_\mu\, q_\nu\, J_a^{\mu\nu})^2}{p_a \cdot q}$$

$\longrightarrow$ arise from hidden BMS symmetry?

Collinear Limits

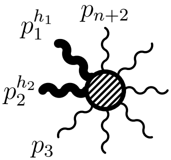


$$\xrightarrow{1\|2} \sum_{h=\pm} \mathrm{Split}_{-h}(c; 1^{h_1}, 2^{h_2}) \times$$

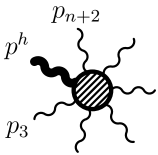


Collinear limit

- Collinear factorization is central property for gluon amplitudes



$$\xrightarrow{1\|2} \quad \sum_{h=\pm} \mathrm{Split}_{-h}(c; 1^{h_1}, 2^{h_2}) \times$$



- Parametrize the collinear limit $\epsilon \to 0$: $(c = \cos\phi\,, \quad s = \sin\phi)$ [Stieberger, Taylor]

$$|1\rangle = c\,|p\rangle - \epsilon\, s\,|r\rangle \qquad |1] = c\,|p] - \epsilon\, s\,|r]$$
$$|2\rangle = s\,|p\rangle + \epsilon\, c\,|r\rangle \qquad |2] = s\,|p] + \epsilon\, c\,|r] \qquad \Rightarrow \qquad \langle 12\rangle = \epsilon\,\langle pr\rangle$$

with reference momentum r^μ. This translates to 4-momenta

$$p_1^\mu = c^2\, p^\mu - \epsilon\, cs\, q^\mu + \epsilon^2\, s^2\, r^\mu$$
$$p_2^\mu = s^2\, p^\mu + \epsilon\, cs\, q^\mu + \epsilon^2\, c^2\, r^\mu \qquad q := |p\rangle[r| + |r\rangle[p|$$

- Momentum conservation up to order ϵ^2: $\quad p_1 + p_2 = p + \epsilon^2\, r$

Splitting functions

- Collinear factorization

$$A_{n+2}(1^{h_1}, 2^{h_2}, \ldots) \xrightarrow{1\|2} \sum_{h=\pm} \mathsf{Split}_{-h}(c; 1^{h_1}, 2^{h_2})\ A_{n+1}(p^h, \ldots) + \mathcal{O}(\epsilon^0)$$

- One has to leading order in ϵ:

$$\mathsf{Split}_+(c; 1^+, 2^+) = 0 \qquad \mathsf{Split}_-(c; 1^+, 2^+) = \frac{1}{\epsilon}\,\frac{1}{cs\,\langle pr\rangle}$$

$$\mathsf{Split}_+(c; 1^+, 2^-) = -\frac{1}{\epsilon}\,\frac{s^3}{c\,\langle pr\rangle} \qquad \mathsf{Split}_-(c; 1^+, 2^-) = \frac{1}{\epsilon}\,\frac{c^3}{s\,[pr]}$$

- Question fo this talk:

Is there universal factorization at subleading order in ϵ?

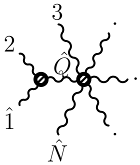
Natural question to ask in view of subleading soft theorems. However, no potential hidden symmetry at the horizon.

Study subleading collinear limit by BCFW

- Taking $1^+ \parallel 2^+$ consider a $\langle 1N]$ shift: $|\hat{1}\rangle = |1\rangle + z\,|N\rangle\,, \quad |\hat{N}] = |N] - z\,|1]$

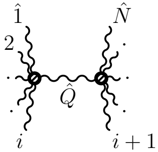
$$A_N = \sum_{i=3}^{N-1} A_i^L \frac{1}{Q_i^2} A^R_{N-i+2}$$

- 3pt-diagram yields leading order:



$\overset{1\|2}{\sim} \mathcal{O}(\frac{1}{\epsilon})$ with $z \sim \mathcal{O}(\epsilon^2)$ $\boxed{\Rightarrow A_N = \frac{1}{\epsilon}\,\text{Split} \cdot A_{N-1} + \mathcal{O}(1)}$

- Generic-diagram



$\overset{1\|2}{\sim} \mathcal{O}(1)$ with $z \sim \mathcal{O}(1)$ contributes at subleading order $\Rightarrow$ universal factorication is lost!

- But this argument only applies within "space" of gluon amplitudes!

Relation to Einstein-Yang-Mills amplitudes [Stieberger, Taylor '15]

- Consider linear combinations of collinear amplitudes s.t. leading pole cancels
  - N=5:

$$s_{5p}\, A(1^+,2^+,3,4,5) - s_{4p}\, A(1^+,2^+,3,5,4) \xrightarrow{1\|2} \frac{g^2}{\kappa\, c^2}\, A_{\mathsf{EYM}}(p^{++},3,4,5) + \mathcal{O}(\epsilon)$$

  - N=6:

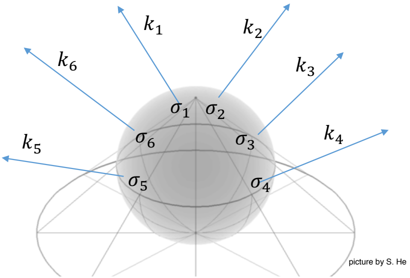
$$\begin{aligned} s_{6p}\, A(1^+,2^+,3,4,5,6) - s_{5p}\,[A(1^+,2^+,3,4,6,5) + A(1^+,2^+,3,6,4,5)] \\ + s_{4p}\, A(1^+,2^+,3,6,5,4) \xrightarrow{1\|2} \frac{g^2}{\kappa\, c^2}\, A_{\mathsf{EYM}}(p^{++},3,4,5,6) + \mathcal{O}(\epsilon) \end{aligned}$$

- **In general:** $(N-3)!/2$ constraints for the independent $(N-3)!$ gluon amplitudes in $1 \parallel 2$ limit.
- $\Rightarrow$ Suggests possible existence of subleading splitting into EYM-amplitudes

$$\boxed{A_{\mathsf{YM}}(1^+,2^+,3,\dots,N)\Big|^{\text{subleading}}_{1\|2} \overset{????}{=} S_-^{[1]}(c;1^+,2^+;3,N)\, A_{\mathsf{EYM}}(p^{++},3,\dots,N)}$$

The CHY formalism

$$\sum_{b=1, b\neq a}^{n} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0$$



Best formalism for unified description of amplitudes: CHY

- Scattering equations: $$\boxed{f_a = \sum_{\substack{b=1\\ b\neq a}}^{n} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b} = 0} \qquad \sigma_{ab} := \sigma_a - \sigma_b \in \mathbb{C}$$

  For N-particle kinematics these have $(N-3)!$ solutions

- $$\boxed{\mathcal{A}_n = \int \frac{d^n \sigma_a}{\text{vol SL}(2,\mathbb{C})} \left(\prod_{a=1}^{n}{}' \, \delta(f_a) \right) \mathcal{I}_n(\{p, E, \sigma\})}$$ [Cachazo,He,Yuan]

  with the building blocks:

$$C(1,\ldots,n) = \frac{1}{\sigma_{12}\ldots\sigma_{n,1}} \qquad \Psi_n = \begin{pmatrix} A & -C^{\mathrm{T}} \\ C & B \end{pmatrix}$$

$$A_{ab} = \begin{cases} \frac{p_a \cdot p_b}{\sigma_a - \sigma_b} \\ 0 \end{cases} \qquad B_{ab} = \begin{cases} \frac{E_a \cdot E_b}{\sigma_a - \sigma_b} \\ 0 \end{cases} \qquad C_{ab} = \begin{cases} \frac{E_a \cdot p_b}{\sigma_a - \sigma_b} & \text{for } a \neq b \\ -\sum_{c\neq a} \frac{E_a \cdot p_c}{\sigma_a - \sigma_c} & \text{for } a = b \end{cases}$$

- Integrand $\mathcal{I}_n$ defines the theory in question.

CHY: Unifying picture of gluon & graviton amplitdues

- CHY formula

$$\mathcal{A}_n = \int \frac{d^n\sigma_a}{\text{vol SL(2,}\mathbb{C}\text{)}} \left(\prod_{a=1}^{n}{}' \, \delta(f_a) \right) \mathcal{I}_n(\{p, E, \sigma\})$$

The integral completely localizes on the solutions of the scattering equations.

- Integrands:

Gravitons: $$\mathcal{I}_n^{\text{Einstein}}(1,2,\ldots,n) = \text{Pf}' \, \Psi_n^2$$

Color ordered gluons: $$\mathcal{I}_n^{\text{YM}}(1,2,\ldots,n) = C(1,2,\ldots,n) \, \text{Pf}' \, \Psi_n$$

Graviton-gluon: $$\mathcal{I}_{n+1}^{\text{EYM}}(1,2,\ldots,n;p) = \mathcal{C}(1,2,\ldots,n) \, C_{pp} \, \text{Pf}' \, \Psi_{n+1}$$

$$\mathcal{I}_{n+r}^{\text{EYM}}(1,2,\ldots,n;p_1,\ldots,p_r) = \mathcal{C}(1,2,\ldots,n) \, \text{Pf} \, \Psi_r(\{p, E_p, \sigma\}) \text{Pf}' \, \Psi_{n+r}$$

with $C_{pp} = -\sum_{b=1}^{n} \frac{E_p \cdot p_b}{\sigma_p - \sigma_b} = \text{Pf}\Psi_1(p, E, \sigma)$

- Unified description of gluon-graviton trees.

A short proof of new EYM amplitude relations

Derived from string theory: $h\, g^n$-amplitude from pure YM [Stieberger,Taylor]

$$\boxed{A_{\mathsf{EYM}}(1,2,\ldots,n;p^{\pm\pm}) = \frac{\kappa}{g}\sum_{l=1}^{n-1} E_p^{\pm}\cdot x_l\, A_{\mathsf{YM}}(1,2,\ldots,l,p,l+1,\ldots,n)}$$

with region momentum $x_l = \sum_{i=1}^{l} p_i$.

- Follows straightforwardly from CHY representation:

$$\mathcal{I}_n^{\mathsf{YM}}(1,2,\ldots,n) = \frac{1}{\sigma_{12}\ldots\sigma_{n1}}\,\mathrm{Pf}'\,\Psi_n$$

$$\mathcal{I}_{n+1}^{\mathsf{EYM}}(1,2,\ldots,n;p^{\pm\pm}) = C_{pp}\,\frac{1}{\sigma_{12}\ldots\sigma_{n1}}\,\mathrm{Pf}'\Psi_{n+1}$$

- Using the simple identity: [Nandan,JP,Schlotterer,Wen]

$$C_{pp} = \sum_{i=1}^{n}\frac{E_p\cdot p_i}{\sigma_{i,\rho}} = \sum_{i=1}^{n-1} E_p\cdot x_i\,\frac{\sigma_{i,i+1}}{\sigma_{i,\rho}\,\sigma_{\rho,i+1}}$$

Trivially shown using $\frac{\sigma_{i,i+1}}{\sigma_{i,p}\,\sigma_{p,i+1}} = \frac{1}{\sigma_{i,p}} - \frac{1}{\sigma_{i+1,p}}$ and telescoping sum.

Higher level EYM to YM relations [Nandan,JP,Schlotterer,Wen]

- May be generalized to more gravitons. One seeks identities of the type

$$\boxed{\mathrm{Pf}\,\Psi_r(\{p, E_p, \sigma\}) = \sum_{\{i,j,a\}} F_a(\{p, E_p\})\,\frac{\sigma_{i\,j}}{\sigma_{ia}\sigma_{aj}}}$$

- Two gravitons with momenta p & q:

$$\mathrm{Pf}\,\Psi_{r=2} = C_{pp}\,C_{qq} - \frac{s_{pq}\,(\epsilon_p\cdot\epsilon_q)}{\sigma^2_{p,q}} + \frac{(\epsilon_p\cdot q)\,(\epsilon_q\cdot p)}{\sigma^2_{p,q}}\,, \qquad s_{pq} \equiv p\cdot q\,.$$

- Toolbox of identities:

Schouten's $\quad \sigma_{i,i+1}\,\sigma_{p,q} = -\sigma_{i,p}\,\sigma_{q,i+1} + \sigma_{i,q}\,\sigma_{p,i+1}$

Kleiss-Kuijf $\quad \mathcal{C}(1,A,n,B) = (-)^{|B|}\sum_{\sigma\in A\sqcup\!\sqcup B^t}\mathcal{C}(1,\sigma,n)$

with shuffles $\quad A\sqcup\!\sqcup B \equiv\{\alpha_1(\alpha_2\ldots\alpha_{|A|}\sqcup\!\sqcup B)\} + \{\beta_1(\beta_2\ldots\beta_{|B|}\sqcup\!\sqcup A)\}$

Cross-ratio identity: $\quad \dfrac{s_{pq}}{\sigma^2_{p,q}} = \sum_{i\neq a,p,q} s_{pi}\,\dfrac{\sigma_{i,a}}{\sigma_{i,p}\,\sigma_{p,q}\,\sigma_{q,a}}$ [Cardona,Feng,Gomez,Huang]

New EYM to YM relations [Nandan,JP,Schlotterer,Wen]

- Two gravitons

$$
\begin{aligned}
&\mathcal{A}_{\text{EYM}}(1,2,\ldots,n;p,q) \ = \\
&\quad \frac{\kappa^2}{g^2}\Big[\sum_{1=i\leq j}^{n-1}(\epsilon_p\cdot x_i)\,(\epsilon_q\cdot x_j)\,\mathcal{A}(1,\ldots,i,p,i{+}1,\ldots,j,q,j{+}1,\ldots,n) \\
&\qquad -(\epsilon_q\cdot p)\sum_{j=1}^{n-1}(\epsilon_p\cdot x_j)\sum_{i=1}^{j+1}\mathcal{A}(1,2,\ldots,i{-}1,q,i,\ldots,j,p,j{+}1,\ldots,n) \\
&\qquad -\frac{(\epsilon_p\cdot\epsilon_q)}{2}\sum_{l=1}^{n-1}(p\cdot k_l)\sum_{1=i\leq j}^{l}\mathcal{A}(1,2,\ldots,i{-}1,q,i,\ldots,j{-}1,p,j,\ldots,n) \\
&\qquad +(p\leftrightarrow q)\Big]
\end{aligned}
$$

- Similar yet more complicated expressions have been worked out for 3 and 4 gravitons as well as double-trace EYM amplitudes with zero or one graviton.

Some explicit EYM to YM relations

- Two gluons, two gravtions

$$\mathcal{A}_{\rm EYM}(1,2;{\color{blue}3,4}) = (\epsilon_3\cdot k_4)(\epsilon_4\cdot x_1)\mathcal{A}(1,2,3,4) + (\epsilon_4\cdot k_3)(\epsilon_3\cdot x_1)\mathcal{A}(1,2,4,3) \\ - (\epsilon_3\cdot x_1)(\epsilon_4\cdot x_1)\mathcal{A}(1,3,2,4) - s_{13}(\epsilon_3\cdot\epsilon_4)\mathcal{A}(1,2,4,3)$$

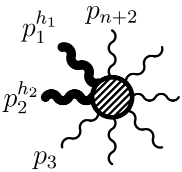
in BCJ basis:

$$= \mathcal{A}(1,2,3,4)\times\Big\{(\epsilon_3\cdot k_4)(\epsilon_4\cdot k_1) + \frac{s_{23}}{s_{13}}(\epsilon_4\cdot k_3)(\epsilon_3\cdot k_1) \\ - \frac{s_{12}}{s_{13}}(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_1) - s_{23}(\epsilon_3\cdot\epsilon_4)\Big\}$$

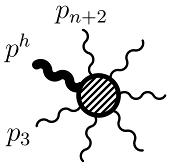
- Three gluons, two gravitons

$$\mathcal{A}_{\rm EYM}(1,2,3;{\color{blue}4,5}) = \\ (\epsilon_4\cdot x_2)(\epsilon_5\cdot k_4)\,\mathcal{A}(1,2,4,5,3) + (\epsilon_4\cdot x_2)(\epsilon_5\cdot x_1)\,\mathcal{A}(1,5,2,4,3) \\ + (\epsilon_4\cdot x_1)(\epsilon_5\cdot x_1)\,\mathcal{A}(1,4,5,2,3) + (\epsilon_4\cdot x_2)(\epsilon_5\cdot x_2)\,\mathcal{A}(1,2,4,5,3) \\ - (\epsilon_4\cdot x_1)(\epsilon_5\cdot k_4)\left[\mathcal{A}(1,5,4,2,3) + \mathcal{A}(5,1,4,2,3)\right] \\ + \tfrac{1}{2}(\epsilon_4\cdot\epsilon_5)\left[s_{24}\mathcal{A}(1,3,5,4,2) - s_{14}\mathcal{A}(1,2,3,5,4)\right] + (4\leftrightarrow 5)$$

Back to (subleading) Collinear Limits



$$\xrightarrow{1\|2} \sum_{h=\pm} \mathrm{Split}_{-h}(c; 1^{h_1}, 2^{h_2}) \times$$



The collinear scattering equations

- We take $1 \parallel 2$ with $\begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix} = \begin{pmatrix} c & -\epsilon\, s \\ s & +\epsilon\, c \end{pmatrix} \begin{pmatrix} |p\rangle \\ |r\rangle \end{pmatrix}$
- Change of variables:

$$\sigma_1 = \rho - \frac{\xi}{2} \qquad \sigma_2 = \rho + \frac{\xi}{2}$$

- In fact solutions with $\xi \to 0$ imply collinearity of $1 \parallel 2$: [Dolan,Goddard]

  Making the ansatz $\xi = \epsilon\, \xi_1 + \epsilon^2 \xi_2 + \mathcal{O}(\epsilon^3)$ the scattering eqs. for $n+2$ particles factorize to a $(n+1)$-particle problem:

$$0 = f_a\Big|_{\epsilon \to 0} = \sum_{\substack{b=1 \\ b \neq a}}^{n} \frac{p_a \cdot p_b}{\sigma_{ab}} + \frac{p_a \cdot p}{\sigma_a - \rho} + \mathcal{O}(\epsilon)$$

$$0 = f_1 + f_2\Big|_{\epsilon \to 0} = \sum_{b=1}^{n} \frac{p \cdot p_b}{\rho - \sigma_b} + \mathcal{O}(\epsilon)$$

along with the singular scattering eq.

$$0 = f_1 - f_2\Big|_{\epsilon \to 0} = -\frac{2}{\epsilon} \frac{p_1 \cdot p_2}{\xi_1} \Big(1 + \mathcal{O}(\epsilon)\Big) \quad \Rightarrow$$

enforces collinearity $1 \parallel 2$

Numerical studies of collinear scattering equations

Hence: $\boxed{\sigma_1 \to \sigma_2 \quad \Rightarrow \quad p_1 \parallel p_2}$ But is the opposite also true?

Have gained numerical insights for $N \leq 8$ with near collinear $p_1 \parallel p_2$ using the polynomial form [Dolan,Goddard;Kalousios] of the scattering eqs:

- We always find $2 \cdot (N-4)!$ degenerate $(\xi \to 0)$ solutions
- The remaining $(N-5)\,(N-4)!$ solutions are non-degenerate ($\xi =$ finite)
- Numerically the degenerate solutions are seen to scale like $\xi \sim \mathcal{O}(\epsilon)$.

Degenerate solutions are numerically seen to be dominant in the CHY integral at leading $\mathcal{O}(\frac{1}{\epsilon})$ and sub-leading $\mathcal{O}(1)$ order in the collinear limit.

$\boxed{\Rightarrow \text{ Can concentrate on degenerate solutions!}}$

Numerical studies of collinear scattering equations

Hence: $\boxed{\sigma_1 \to \sigma_2 \quad \Rightarrow \quad p_1 \parallel p_2}$ But is the opposite also true?

Have gained numerical insights for $N \leq 8$ with near collinear $p_1 \parallel p_2$ using the polynomial form [Dolan,Goddard;Kalousios] of the scattering eqs:

- We always find $2 \cdot (N-4)!$ degenerate $(\xi \to 0)$ solutions
- The remaining $(N-5)\,(N-4)!$ solutions are non-degenerate $(\xi = \text{finite})$
- Numerically the degenerate solutions are seen to scale like $\xi \sim \mathcal{O}(\epsilon)$.

Degenerate solutions are numerically seen to be dominant in the CHY integral at leading $\mathcal{O}(\frac{1}{\epsilon})$ and sub-leading $\mathcal{O}(1)$ order in the collinear limit.

$\boxed{\Rightarrow \text{Can concentrate on degenerate solutions!}}$

Finding the degenerate solutions

$$\delta(f_1)\,\delta(f_2) = 2\,\delta(f_1+f_2)\,\delta(f_1-f_2) = 2\delta(f_1+f_2)\,\delta(f_-)$$

- Degenerate solution ansatz $\xi = \epsilon\xi_1 + \epsilon^2\xi_2 + \mathcal{O}(\epsilon^3)$:

$$f_- := (f_1 - f_2) - (c^2 - s^2)\,(f_1 + f_2) = \epsilon\left[2c^2 s^2\,\xi_1\mathcal{P}_2 - 2cs\,\mathcal{Q}_1 - \frac{2(p\cdot r)}{\xi_1}\right] + \mathcal{O}(\epsilon^2)$$

with shorthands $\mathcal{Q}_i = \sum_{b=3}^{n} \frac{p_b \cdot q}{(\rho - \sigma_b)^i} \qquad \mathcal{P}_i = \sum_{b=3}^{n} \frac{p_b \cdot p}{(\sigma_b - \rho)^i} \qquad i \geq 2$

- Yields two solutions for $\xi_1 = \xi_\pm$.

$$\xi_{1,\pm} = \frac{\mathcal{Q}_1}{2cs\mathcal{P}_2} \pm \sqrt{\frac{\mathcal{Q}_1^2 + 4(p\cdot r)\mathcal{P}_2}{4(c^2 s^2)\mathcal{P}_2^2}}$$

- Solution counting:
  - Remaining $N-1$ scattering equations have $(N-4)!$ solutions
  - Total number of degenerate solutions thus constructed $2\cdot(N-4)!$
    $\Rightarrow$ Matches the numerically found number!

Finding the degenerate solutions

$$\delta(f_1)\,\delta(f_2) = 2\,\delta(f_1+f_2)\,\delta(f_1-f_2) = 2\delta(f_1+f_2)\,\delta(f_-)$$

- Degenerate solution ansatz $\xi = \epsilon\xi_1 + \epsilon^2\xi_2 + \mathcal{O}(\epsilon^3)$:

$$f_- := (f_1 - f_2) - (c^2 - s^2)\,(f_1 + f_2) = \epsilon\left[2c^2s^2\,\xi_1\mathcal{P}_2 - 2cs\,\mathcal{Q}_1 - \frac{2(p\cdot r)}{\xi_1}\right] + \mathcal{O}(\epsilon^2)$$

with shorthands $\mathcal{Q}_i = \sum_{b=3}^{n} \frac{p_b \cdot q}{(\rho - \sigma_b)^i}$ $\qquad \mathcal{P}_i = \sum_{b=3}^{n} \frac{p_b \cdot p}{(\sigma_b - \rho)^i} \qquad i \geq 2$

- Yields two solutions for $\xi_1 = \xi_\pm$.

$$\xi_{1,\pm} = \frac{\mathcal{Q}_1}{2cs\mathcal{P}_2} \pm \sqrt{\frac{\mathcal{Q}_1^2 + 4(p\cdot r)\mathcal{P}_2}{4(c^2s^2)\mathcal{P}_2^2}}$$

- Solution counting:
  - Remaining $N-1$ scattering equations have $(N-4)!$ solutions
  - Total number of degenerate solutions thus constructed $2\cdot(N-4)!$
    $\Rightarrow$ Matches the numerically found number!

Four contributions to the near collinear expansion

$$A(1,2,3,\ldots,n+2) \stackrel{1\|2}{=} \sum_{\xi_\pm} \int \prod_{a=3}^{n+2} \Big[d'\sigma_a\, \delta(f_a)\Big]\, d\rho\, \delta\Big(\underbrace{f_p + \epsilon\,(c^2 - s^2)\frac{\xi_1}{2}\mathcal{P}_2}_{f_1+f_2}\Big)$$

$$\times \Bigg[\underbrace{\frac{1}{\left|\frac{\partial f_-}{\partial \xi}\right|}}_{\text{Jacobian}} \underbrace{\frac{1}{\sigma_{1,2}\ldots\sigma_{n+2,1}}}_{\mathcal{C}_{n+2}} \,\mathsf{Pf}'\,\Psi_{n+2}\Bigg]\Bigg|_{\xi=\epsilon\xi_1+\epsilon^2\xi_2+\ldots}$$

- Jacobian:

$$\mathcal{J} = \underbrace{\frac{1}{2}\,\frac{\xi_1^2}{(p\cdot r) + c^2 s^2 \mathcal{P}_2\, \xi_1^2}}_{\mathcal{J}_0} + \epsilon\, \mathcal{J}_0^2 \left(4(p\cdot r)\frac{\xi_2}{\xi_1^3} - cs(c^2 - s^2)\mathcal{Q}_2\right) + \mathcal{O}(\epsilon^2)$$

- Parke-Taylor factor:

$$\mathcal{C}_{n+2} = -\frac{1}{\epsilon}\frac{\mathcal{C}_{n+1}}{\xi_1} + \mathcal{C}_{n+1}\left(\frac{\xi_2}{\xi_1^2} + \frac{1}{2} S_{n+2,\rho,3}\right) + \mathcal{O}(\epsilon)$$

- CHY-matrix: Needs more involved computation, but $\mathsf{Pf}'\,\Psi_{n+2} \sim \epsilon^0$

CHY matrix expansion

Back to linear algebra: Elementary matrix manipulations

$$\Psi_{n+2} = \begin{bmatrix} 0 & A_{12} & A_{1b} & -C_{11} & -C_{21} & -C_{d1} \\ A_{21} & 0 & A_{2b} & -C_{12} & -C_{22} & -C_{d2} \\ A_{a1} & A_{a2} & A_{ab} & -C_{1a} & -C_{2a} & -C_{da} \\ C_{11} & C_{12} & C_{1b} & 0 & B_{12} & B_{1d} \\ C_{21} & C_{22} & C_{2b} & B_{21} & 0 & B_{2d} \\ C_{c1} & C_{c2} & C_{cb} & B_{c1} & B_{c2} & B_{cd} \end{bmatrix}$$

1. Add row/column 1 to 2.
2. Subtract c^2 times second row/column from first row/column.

CHY matrix expansion

Back to linear algebra: Elementary matrix manipulations

$$
\begin{bmatrix}
0 & \epsilon\, A_{12} & \epsilon\, \tilde{A}_{1b} & -s^2\, C_{11} + c^2\, C_{12} & c^2\, C_{22} - s^2\, C_{21} & -\epsilon\, \tilde{C}_{d1} \\
A_{21} & 0 & \tilde{A}_{2b} & -C_{12} - C_{11} & -C_{22} - C_{21} & -\tilde{C}_{d2} \\
\tilde{A}_{a1} & \tilde{A}_{a2} & A_{ab} & -C_{1a} & -C_{2a} & -C_{da} \\
s^2\, C_{11} - c^2\, C_{12} & C_{12} + C_{11} & C_{1b} & 0 & B_{12} & B_{1d} \\
s^2\, C_{21} - c^2 C_{22} & C_{22} + C_{21} & C_{2b} & B_{21} & 0 & B_{2d} \\
\tilde{C}_{c1} & \tilde{C}_{c2} & C_{cb} & B_{c1} & B_{c2} & B_{cd}
\end{bmatrix}
$$

Now:

$$
B_{12} = \begin{cases} 0 & \text{same helicity: } 1^{\pm}2^{\pm} \\ \frac{1}{\epsilon\, \xi_1} - \frac{\xi_2}{\xi_1^2} + \mathcal{O}(\epsilon) & \text{opposite helicity: } 1^{\pm}2^{\mp} \end{cases}
$$

Use Pfaffian expansion of row one:

$$
\mathrm{Pf}(A) = \sqrt{\det A} = \sum_{j=2}^{2n} (-1)^j (A)_{1j}\, \mathrm{Pf}(A^{(1,j)}_{(1,j)})
$$

Same helicity $1^{\pm}2^{\pm}$: Leading order

- CHY matrix:

$$\mathrm{Pf}'(\Psi_{n+2}) = \left(C_{pp} - \frac{2}{s\,c\,\xi_1} \begin{Bmatrix} -[pr] \\ +\langle pr\rangle \end{Bmatrix} \right) \mathrm{Pf}'(\Psi_{n+1}) \qquad \text{for helicities} \quad \begin{Bmatrix} 1^{+}2^{+} \\ 1^{-}2^{-} \end{Bmatrix}$$

- Putting everything together we recover the splitting functions:

$$\mathcal{A}_{n+2} \overset{1\parallel 2}{=} \sum_{\xi_1} \int d\mu_n\, d\rho\, \delta(f_p)\, \frac{-\mathcal{J}_0}{\epsilon\,\xi_1}\, C_{n+1} \left(C_{pp} - \frac{2}{s\,c\,\xi_1} \begin{Bmatrix} -[pr] \\ +\langle pr\rangle \end{Bmatrix} \right) \mathrm{Pf}'(\Psi_{n+1})$$
$$= \mathrm{Split}^{\mathrm{tree}}_{\mp}(c;1^{\pm},2^{\pm})\, \mathcal{A}_{n+1}(p^{\pm},3,\ldots,n+2)$$

using the sum identities:

$$\sum_{\{\xi_1\}} \frac{\mathcal{J}_0}{\xi_1} = 0 \qquad \sum_{\{\xi_1\}} \frac{\mathcal{J}_0}{\xi_1^2} = \frac{1}{2\,p\cdot r}$$

Opposite helicity $1^{\pm}2^{\mp}$: Leading order

- CHY matrix: Look directly at summed expression

$$\sum_{\xi_1} \frac{-\mathcal{J}_0}{\epsilon \xi_1} \mathrm{Pf}(\Psi_{n+2}(\epsilon = 0)) = \frac{1}{\epsilon} \left(\frac{s^3}{c\langle rp \rangle} \mathrm{Pf}(\Psi^-_{n+1}) - \frac{c^3}{s[rp]} \mathrm{Pf}(\Psi^+_{n+1}) \right)$$

leading to

$$\begin{aligned} \mathcal{A}^{1||2}_{n+2}(1^+, 2^-, \ldots) \overset{1\|2}{=} & \, \mathsf{Split}^{\mathsf{tree}}_{-}(c; 1^+, 2^-)\, \mathcal{A}_{n+1}(p^+, 3, \ldots, n+2) \\ & + \mathsf{Split}^{\mathsf{tree}}_{+}(c; 1^+, 2^-)\, \mathcal{A}_{n+1}(p^-, 3, \ldots, n+2) + \mathcal{O}(1) \end{aligned}$$

- Hence the tree-level splitting functions are recovered from CHY.
- Now everything is in place to expand out to the subleading collinear limit.

Same helicity $1^{\pm}2^{\pm}$: Subleading order

Sum of Jacobian, Parke-Taylor, CHY-matrix and $\delta'(f_p)$ contributions at $\mathcal{O}(\epsilon^0)$:

$$\boxed{\begin{aligned}\mathcal{A}(1,2,3,\ldots,n+2)\Big|_{1\|2}^{\text{subleading}} &= \int d\mu_{n+1} \\ &\left(\frac{C_{pp}}{\mathcal{P}_2}\left(\frac{1}{c^2}\,\frac{1}{\sigma_{n+2,\rho}}+\frac{1}{s^2}\,\frac{1}{\sigma_{\rho,3}}\right)+\frac{c^2-s^2}{c^2\,s^2\,\mathcal{P}_2}\left(C_{pp}^{(2)}-\frac{C_{pp}\mathcal{P}_3}{\mathcal{P}_2}\right)\right)C_{n+1}\ \mathrm{Pf}'(\Psi_{n+1}) \\ &+\frac{c^2-s^2}{2\,c^2\,s^2}\int\frac{d^n\sigma_a\,d\rho}{\text{vol SL}(2,\mathbb{C})}\,\frac{\partial}{\partial\rho}\left[\left({\prod_{a=1}^{n}}'\,\delta(f_a)\right)\delta(f_p)\,\frac{C_{pp}}{\mathcal{P}_2}\,C_{n+1}\ \mathrm{Pf}'(\Psi_{n+1})\right]+\mathcal{O}(\epsilon)\end{aligned}}$$

Compare to Einstein-Yang-Mills amplitude

$$\mathcal{A}_{n+1}(p^{\pm\pm};3,\ldots,n+2)=\int d\mu_{n+1}\,C_{pp}\ C_{n+1}\ \mathrm{Pf}'(\Psi_{n+1})$$

where

$$C_{pp}^{(i)}=-\sum_{b=3}^{n+2}\frac{E_p\cdot p_b}{(\rho-\sigma_b)^i}\qquad \mathcal{P}_i=\sum_{b=3}^{n}\frac{p_b\cdot p}{(\sigma_b-\rho)^i}$$

Same helicity $1^\pm 2^\pm$: Subleading order

Final result (w/o total derivative):

$$\mathcal{A}(1,2,3,\ldots,n+2)\Big|_{1\|2}^{\text{subleading}} = \int d\mu_{n+1}$$
$$\left(\frac{C_{pp}}{\mathcal{P}_2}\left(\frac{1}{c^2}\,\frac{1}{\sigma_{n+2,\rho}}+\frac{1}{s^2}\,\frac{1}{\sigma_{\rho,3}}\right)+\frac{c^2-s^2}{c^2\,s^2\,\mathcal{P}_2}\left(C_{pp}^{(2)}-\frac{C_{pp}\mathcal{P}_3}{\mathcal{P}_2}\right)\right)C_{n+1}\ \mathrm{Pf}'(\Psi_{n+1})$$

- Taking the sum over permutations $\{4,..,n+2\}$ a la Stieberger-Taylor

Purple terms: $\displaystyle\sum_{i_s\in ST} s_{i_s p}\, C_{n+1} = \mathcal{P}_1 = f_p \quad \Rightarrow$ zero due to $\delta(f_p)$

Red terms: $\displaystyle\sum_{i_s\in ST} s_{i_s p}\,\frac{C_{n+1}}{\sigma_{n+2,\rho}} = \mathcal{P}_2$

- We recover S-T result (e.g. $N=5$):

$$s_{5p}\,A(1^+,2^+,3,4,5)^{(1)} - s_{4p}\,A(1^+,2^+,3,5,4)^{(1)} =$$
$$\frac{1}{c^2}\int d\mu_{n+1} C_{pp}\, C_{n+1}\mathrm{Pf}'\Psi_{n+1} = \frac{1}{c^2}\,A_{\text{EYM}}(p^{++},3,4,5)$$

A curious identity for $1^{\pm}2^{\pm}$: Subleading order

Final result (w/o total derivative):

$$\mathcal{A}(1,2,3,\ldots,n+2)\Big|_{1\|2}^{\text{subleading}} = \int d\mu_{n+1}$$

$$\left(\frac{C_{pp}}{\mathcal{P}_2}\left(\frac{1}{c^2}\,\frac{1}{\sigma_{n+2,\rho}} + \frac{1}{s^2}\,\frac{1}{\sigma_{\rho,3}}\right) + \frac{c^2-s^2}{c^2\,s^2\,\mathcal{P}_2}\left(C_{pp}^{(2)} - \frac{C_{pp}\mathcal{P}_3}{\mathcal{P}_2}\right)\right) C_{n+1}\ \mathrm{Pf}'(\Psi_{n+1})$$

- Consider the differential operator $\boxed{p\cdot\partial_{E_p}}$ (gauge transf. in eff. coll. leg)

$$C_{pp}^{(i)} = -\sum_{b=3}^{n+2}\frac{E_p\cdot p_b}{(\sigma_p-\sigma_b)^i} \quad\Rightarrow\quad p\cdot\partial_{E_p}\,C_{pp}^{(i)} = \mathcal{P}_i \qquad p\cdot\partial_{E_p}\ \mathrm{Pf}'(\Psi_{n+1}) = 0$$

- As $\mathcal{P}_1 = 0$ by scattering equations we find

[Nandan,JP,Wormsbecher;unpublished]

$$\boxed{p\cdot\partial_{E_p}\,\mathcal{A}^{\mathsf{YM}}(1,2,3,\ldots,n+2)\Big|_{1\|2}^{\text{subleading}} = \frac{s^2-c^2}{c^2s^2}\,\mathcal{A}^{\mathsf{YM}}(p,3,\ldots,n+2)}$$

Factorization at subleading collinear order?

- **Final result** in democratic collinear limit $c = s$

$$\boxed{\begin{aligned}&\mathcal{A}(1^{\pm}, 2^{\pm}, 3, \ldots, n+2)\Big|_{1\|2, c=s}^{\text{subleading}} = \\ &\qquad\qquad \int d\mu_{n+1} \frac{1}{\mathcal{P}_2} \frac{\sigma_{n+2,3}}{\sigma_{n+2,\rho}\, \sigma_{\rho,3}} C_{pp}\, C_{n+1}\, \mathrm{Pf}'(\Psi_{n+1})\end{aligned}}$$

where $\mathcal{P}_2 = \sum_{b=3}^{n+2} \frac{p_b \cdot p}{(\sigma_b - \rho)^2} = \frac{\partial}{\partial \rho} f_p$, derivative of scattering equation.

- Still, have not (yet) been able to write this in factorized form!!

$$\mathcal{A}(1^{\pm}, 2^{\pm}, 3, \ldots, n+2)\Big|_{1\|2, c=s}^{\mathcal{O}(\epsilon^0)} \neq \mathsf{Split}^{(1)}(p_a, E_a, \partial_{p_a}, \ldots)\, \mathcal{A}(p^{\pm\pm}, 3, \ldots, n+2)$$

- $\Rightarrow$ Absence of a subleading collinear theorem for gluons.

Collinear gravitons

With the collinear expansion of CHY building blocks in place, can deduce collinear limits of scalar and gravitons:

- Gravitons: $\mathcal{A}_n = \int d\mu_n \, \mathrm{Pf}'(\Psi_n) \, \mathrm{Pf}'(\Psi_n)$
- In the collinear expansion this yields the leading behavior

$$\mathcal{A}_n^{1^{++},2^{++}} \overset{1\|2}{=} 2 \sum_{\xi_1} \int d\mu_{n-1} \, \mathcal{J}_0 \left(C_{pp} + \frac{2\,[pr]}{c\,s\,\xi_1} \right)^2 \mathrm{Pf}'(\Psi_{n-1}) \, \mathrm{Pf}'(\Psi_{n-1})$$

with the result [Bern,Dixon,Perelstein,Rozowsky]

$$\boxed{\mathcal{A}_n \overset{1\|2}{=} \frac{[p\,r]}{c^2\,s^2\,\langle r\,p\rangle} \mathcal{A}_{n-1} + \frac{1}{c^2\,s^2} \int d\mu_{n-1} \, \frac{C_{pp}^2}{\mathcal{P}_2} \, \mathrm{Pf}'(\Psi_{n-1}) \, \mathrm{Pf}'(\Psi_{n-1})}$$

- This result is universal. Identical behaviour for scattering of m gravitons and k gluons, $\mathcal{A}_n = \int d\mu_n \, C_k \, \mathrm{Pf}(\Psi_m) \, \mathrm{Pf}'(\Psi_n)$. Collinear graviton limit:

$$\mathcal{A}_n \overset{1\|2}{=} \frac{[p\,r]}{c^2\,s^2\,\langle r\,p\rangle} \mathcal{A}_{n-1} + \frac{1}{c^2\,s^2} \int d\mu_{n-1} \, C_k \, \frac{C_{pp}^2}{\mathcal{P}_2} \, \mathrm{Pf}'(\Psi_{m-1}) \, \mathrm{Pf}'(\Psi_{n-1}) \, .$$

Collinear gravitons

With the collinear expansion of CHY building blocks in place, can deduce collinear limits of scalar and gravitons:

- Gravitons: $\mathcal{A}_n = \int d\mu_n \, \mathrm{Pf}'(\Psi_n) \, \mathrm{Pf}'(\Psi_n)$
- In the collinear expansion this yields the leading behavior

$$\mathcal{A}_n^{1^{++},2^{++}} \overset{1\|2}{=} 2 \sum_{\xi_1} \int d\mu_{n-1} \, \mathcal{J}_0 \left(C_{pp} + \frac{2\,[pr]}{c\,s\,\xi_1} \right)^2 \mathrm{Pf}'(\Psi_{n-1}) \, \mathrm{Pf}'(\Psi_{n-1})$$

with the result [Bern,Dixon,Perelstein,Rozowsky]

$$\boxed{\mathcal{A}_n \overset{1\|2}{=} \frac{[p\,r]}{c^2\,s^2\,\langle r\,p\rangle} \mathcal{A}_{n-1} + \frac{1}{c^2\,s^2} \int d\mu_{n-1} \, \frac{C_{pp}^2}{\mathcal{P}_2} \, \mathrm{Pf}'(\Psi_{n-1}) \, \mathrm{Pf}'(\Psi_{n-1})}$$

- This result is universal. Identical behaviour for scattering of m gravitons and k gluons, $\mathcal{A}_n = \int d\mu_n \, C_k \, \mathrm{Pf}(\Psi_m) \, \mathrm{Pf}'(\Psi_n)$. Collinear graviton limit:

$$\mathcal{A}_n \overset{1\|2}{=} \frac{[p\,r]}{c^2\,s^2\,\langle r\,p\rangle} \mathcal{A}_{n-1} + \frac{1}{c^2\,s^2} \int d\mu_{n-1} \, C_k \, \frac{C_{pp}^2}{\mathcal{P}_2} \, \mathrm{Pf}'(\Psi_{m-1}) \, \mathrm{Pf}'(\Psi_{n-1}) \, .$$

Collinear scalars

- Pure scalar amplitudes

$$\mathcal{A}_{n+2} = \int d\mu_{n+2}\, \mathcal{C}_{n+2}^2$$

- Working out the leading and subleading collinear limit $1 \parallel 2$ one finds

$$\mathcal{A}_{n+2}(1,2,3,\ldots,n+2) \overset{1\|2}{=} \frac{1}{\epsilon^2\, 2p\cdot r}\mathcal{A}_{n+1}(p,3,\ldots,n+2)$$
$$-\frac{1}{\epsilon}\int d\mu_{n+1}\Big(\underbrace{c}_{\text{Jacobian}} - \underbrace{c}_{\text{Parke-Taylor}} + \underbrace{0}_{\delta'(f_+)}\Big)\mathcal{C}_{n+1}^2 + \mathcal{O}(1)$$

Scalars have vanishing subleading collinear behaviour!

Subleading collinear-soft gluon limit

- Further spinoff: Take a soft limit $p \to \tau\, p$ with $\tau \to 0$ of the subleading collinear gluon result:

$$\mathcal{A}_n^{1||2} = \frac{1}{2\,\pi\, i} \int d\mu_{n-2} \oint \frac{d\rho}{\tau\, f_p} \left(\frac{C_{pp}}{\tau \mathcal{P}_2} \left(\frac{1}{c^2}\, \frac{1}{\sigma_{n\rho}} + \frac{1}{s^2}\, \frac{1}{\sigma_{\rho 3}} \right) \right.$$
$$\left. + \frac{c^2 - s^2}{c^2\, s^2\, \tau\, \mathcal{P}_2} \left(C_{pp}^{(2)} - \frac{C_{pp} \mathcal{P}_3}{\mathcal{P}_2} \right) \right) \frac{\sigma_{n3}}{\sigma_{n\rho}\, \sigma_{\rho 3}}\, C_{n-2}\, C_{pp}\ \mathrm{Pf}'(\Psi_{n-2})$$

- Performing the ρ integral yields universal factorized expression:

$$\mathcal{A}_n^{1||2, p \to \tau p}\Big|_{\mathcal{O}(\epsilon^0)} = \frac{1}{\tau^2} \left[\frac{1}{c^2} \left(\frac{E_p \cdot p_n}{p \cdot p_n} \right)^2 + \frac{1}{s^2} \left(\frac{E_p \cdot p_3}{p \cdot p_3} \right)^2 \right] \mathcal{A}_n(3, ..., n)$$

Summary: Understanding the subleading collinear limit

- Question: Is there a subleading collinear theorem for gluons?
- Intriguing relations:
  Linear combinations of subleading collinear gluon amplitudes = Einstein-Yang Mills amplitudes [Stieberger, Taylor]
- Results: Reproduced tree-level splitting function from collinear limit of CHY.
- Gluons: We do not see factorization in the subleading collinear limit for pure glue. Stieberger-Taylor relations proven. Curious identity between gauge transformation of subleading collinear limit and gluon amplitude.
- Gravitons: Leading factorized and non-factorized contributions to collinear graviton limit established
- Scalars: Vanishing subleading collinear behaviour.