Amplitudes and form factors from N=4 super Yang-Mills to QCD

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based on

Brandhuber, Kostacińska, Penante, GT, Young 1606.08682 [hep-th]
& earlier work also with Bill Spence, Congkao Wen and Gang Yang

Brandhuber, Hughes, Panerai, Spence, GT 1608.083277 [hep-th]

New formulations for scattering amplitudes, LMU Munich, 5th September 2016

- Scattering amplitudes
  - fully on-shell
- Form factors
  - partially on-shell
- Correlation functions
  - off-shell

progressively less on shell

Why form factors?

- They share the beautiful simplicity of amplitudes
  - calculation with textbook (i.e. Feynman diagrams) methods cumbersome, however final results are often strikingly simple
- Important applications
  - phenomenology
  - dilatation operator
- Work in N=4 SYM, but with QCD in mind….
  - we like models…
  - …though QCD has non-zero beta function, is not superconformal, (anti)quarks in (anti)-fundamental representation, no scalars

- Example: supersymmetric decomposition of one-loop amplitudes in pur Yang-Mills (Bern, Dixon, Dunbar, Kosower `94)
    - decomposes the calculation of a one-loop amplitude in pure YM into three simpler calculations, two of which are performed in N=4 and N=1 SYM
    - remaining N=0 calculation simpler than the original one

- Apply this kind of ideas to form factors
    - conceptual motivation: explore simplicity of off-shell quantities
    - practical application: surprising connection to Higgs + multi-gluon amplitudes in QCD (no supersymmetry!)

Plan

- Three form factor calculations in N=4 SYM, towards QCD

  1. Half-BPS quadratic operators $\mathrm{Tr}\,(\phi_{12})^2$ & connection to Higgs amplitudes
     - Leading term in the effective action for Higgs+multi-gluon processes
  2. Half-BPS operators of the form $\mathrm{Tr}\,(\phi_{12})^3$ (more in general $\mathrm{Tr}\,(\phi_{12})^k$)
  3. Non-BPS operators, operators of the form $\mathrm{Tr}(\,X[Y\,,Z])$ $(SU(2|3)$ sector)
     - subleading terms in $1/m^2_{\mathrm{top}}$ in the Higgs + multi-gluon effective action ?

- Long-term goal
  - Understand better the connection to Higgs+multi-gluon amplitudes
  - N=4 super Yang-Mills as a tool to compute Higgs amplitudes in QCD?
  - Dilatation operator, Yangian symmetry

What are form factors ?

- Less on-shell (i.e. partially off-shell) quantities

a gauge-invariant operator in the theory

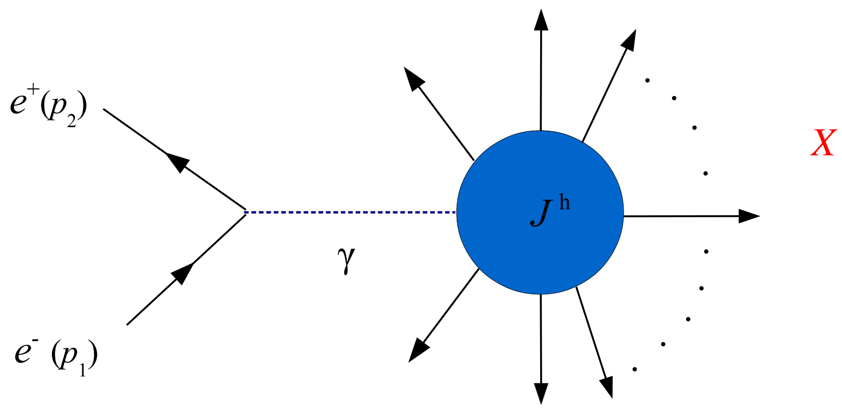
$$F_{\mathcal{O}} := \int d^4x \, e^{-iqx} \langle state | \, \mathcal{O}(x) \, |0\rangle = \delta^{(4)}(q - p_{state}) \langle state | \, \mathcal{O}(0) \, |0\rangle$$

  - momentum q carried by the operator is off shell

- Form factors appear in many important contexts:
  - electromagnetic form factor, or g–2
  - deep inelastic scattering ($e^- + p \rightarrow e^- +$ hadrons)
  - $e^+ e^- \rightarrow$ hadrons (X)

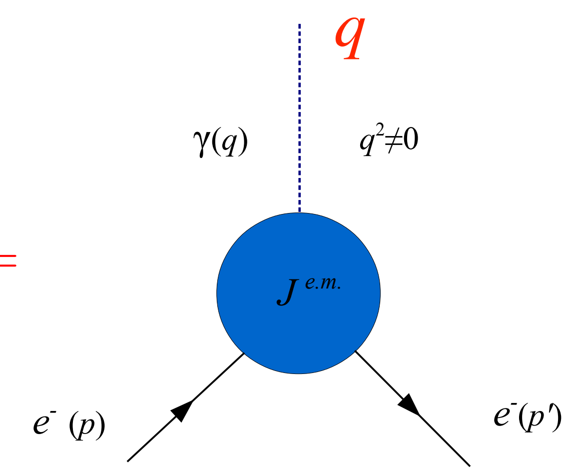
- $e^+ e^- \rightarrow$ hadrons (X), all orders in $\alpha_{\rm strong}$, first order in $\alpha_{\rm e.m.}$

hadronic electromagnetic current



$$= \bar{v}(p_2)\gamma_\mu u(p_1)\frac{\eta^{\mu\nu}}{(p_1+p_2)^2}(-e)\langle X|\, J^{\rm h}_\nu(0)\,|0\rangle$$

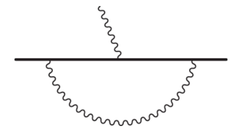
- electron g–2: $\langle e^-(p')|J^{\rm e.m.}_\mu(0)|e^-(p)\rangle =$



  - $J^{\rm e.m.}_\mu = \bar{\psi}\gamma_\mu\psi$
  - $p^2 = m_e^2$ on shell, but $q = p - p'$ off shell

Simplicity of the g–2

- one loop: $$\frac{g-2}{2} = \frac{\alpha}{2\pi} + \mathcal{O}(\alpha^2)$$

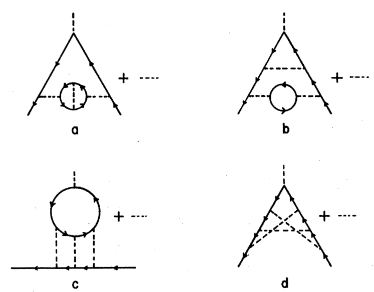


(Schwinger 1948)

  - $\alpha = \dfrac{e^2}{4\pi\epsilon_0\hbar c} \sim \dfrac{1}{137}$ fine structure constant

- Three loops:

72 diagrams like



$$= (1.181241456...)(\alpha/\pi)^3$$

(Cvitanovic & Kinoshita '74; Laporta & Remiddi '96)

  - numerical values of each diagram oscillate wildly…
  - … but final result is $O(1)$
  - an example of surprising simplicity outside amplitudes!

- A side remark: from form factors to amplitudes

  - at $q \neq 0$: $$F_{\mathcal{O}} := \int d^4x\, e^{-iqx} \langle state|\, \mathcal{O}(x)\, |0\rangle$$

  - at $q = 0$: $$F_{\mathcal{O}}|_{q=0} = \int d^4x\, \langle state|\, \mathcal{O}(x)\, |0\rangle$$

  - this is the same as the correction to the amplitude $\langle state \mid 0 \rangle$ due to the addition of a new coupling to the action

    $$\delta S = g_{\mathcal{O}} \int d^4x\; \mathcal{O}(x)$$

    to the first order in $g_{\mathcal{O}}$

  - a particular soft limit of the form factor…

- ## Recent interest from the CHY perspective (He, Zhang)

  - insertion of the operator represented as the sum of two auxiliary null momenta x and y
  - compact expression for the supersymmetric form factor of the (chiral part of the) stress-tensor multiplet T_2 (Brandhuber, Hughes, Panerai, Spence, GT)

$$\boxed{\begin{aligned}\mathcal{F}(\{\lambda,\tilde\lambda\}) = \langle x\,y\rangle^2 \int \frac{1}{\text{vol}\;GL(2)} \frac{\mathrm{d}^2\sigma_x\,\mathrm{d}^2\sigma_y}{(x\,y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\,a+1)} \\ \times \prod_{i\in\{+,x,y\}} \delta^{(2)}(\lambda_i - \lambda(\sigma_i)) \prod_{J\in\{-\}} \delta^{(2|4)}(\tilde\lambda_J - \tilde\lambda(\sigma_J), \eta_J - \eta(\sigma_J))\end{aligned}}$$

  - standard definition with $(ab) := \epsilon_{\alpha\beta}\,\sigma_a^\alpha \sigma_b^\beta$ $\quad \lambda(\sigma) := \sum_{J\in\mathrm{m}} \frac{\lambda_J}{(\sigma\,\sigma_J)}\,, \qquad \tilde\lambda(\sigma) := \sum_{i\in\bar{\mathrm{p}}} \frac{\tilde\lambda_i}{(\sigma_i\,\sigma)}\,, \qquad \eta(\sigma) := \sum_{i\in\bar{\mathrm{p}}} \frac{\eta_i}{(\sigma_i\,\sigma)}$
  - note: auxiliary particles x and y in the "positive-helicity" set
  - Parke-Taylor denominator $\prod_{a=1}^{n} \frac{1}{(a\,a+1)}$ does not include the auxiliary particles

- Can be re-expressed in terms of the link variables of Arkani-Hamed, Cachazo, Cheung and Kaplan

  - link variables linearise momentum conservation
  - introduced via $1 = \int \mathrm{d}c_{iJ}\, \delta\left(c_{iJ} - 1/(iJ)\right)$
  - expression in terms of link variables:

$$\mathcal{F}(\{\lambda, \tilde{\lambda}\}) = \langle x\, y\rangle^2 \int \prod_{i\in\{+,x,y\}, J\in\{-\}} \mathrm{d}c_{iJ}\, U(c_{iJ}) \prod_{i\in\{+,x,y\}} \delta^{(2)}(\lambda_i - c_{iJ}\lambda_J) \prod_{J\in\{-\}} \delta^{(2|4)}(\tilde{\lambda}_J + c_{iJ}\tilde{\lambda}_i, \eta_J + c_{iJ}\eta_i)$$

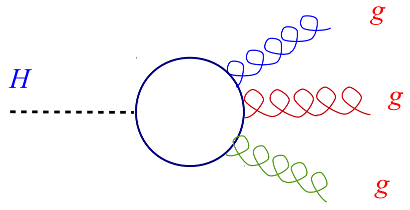
with $$U(c_{iJ}) := \int \frac{1}{\mathrm{vol}\, GL(2)} \frac{\mathrm{d}^2\sigma_x\, \mathrm{d}^2\sigma_y}{(x\, y)^2} \prod_{a=1}^{n} \frac{\mathrm{d}^2\sigma_a}{(a\, a+1)} \prod_{i\in\bar{\mathrm{p}}, J\in\mathrm{m}} \delta\Big(c_{iJ} - \frac{1}{(i\, J)}\Big)$$

  - For amplitudes, re-expressing RSV in terms of link variables leads to a direct connection with BCFW diagrams (Spradlin & Volovich)
  - Similarly, here we can relate CHY to BCFW !
    (Brandhuber, Hughes, Panerai, Spence, GT)

One (more) reason SUSY is useful even if there is no SUSY…

Higgs amplitudes and form factors

- Higgs production at the LHC
  - dominant process at low $M_{\rm H}$ is gluon fusion
  - coupling to gluons through a fermion loop
    - proportional to the mass of the quark $\Rightarrow$ top quark dominates



- Effective Lagrangian description
(Wilczek '77; Shifman, Vainshtein, Voloshin, Zakharov '79; Dawson '91; Djouadi, Graudenz, Spira, Zerwas '95)
  - for $M_{\rm H} < 2\, m_{\rm top}$, integrate out the top quark (shrink loop to a point-like effective interaction)
  - leading order: $\boxed{\mathcal{L}_{\rm eff}^{(0)} \sim H\, {\rm Tr} F^2}$, coupling independent of $m_{\rm top}$
  - efficient MHV rules (Dixon, Glover, Khoze; Badger, Glover & Risager; Boels, Schwinn)
  - How do we compute a process with one Higgs + gluons with $\mathcal{L}_{\rm eff}^{(0)}$?

- Higgs amplitudes are form factors of $\mathrm{Tr}\, F^2$!

$$F_{\mathrm{Tr}F^2}(1,\ldots,n) \;=\; \int d^4x\, e^{-iqx}\, \langle state|\, \mathrm{Tr}\, F^2(x)\, |0\rangle \qquad q^2 = M_{\mathrm{H}}^2$$

- in N=4 super Yang-Mills, the form factor of $\mathrm{Tr}\, F_{\mathrm{SD}}{}^2$ (SD = self-dual) is related to that of $\mathrm{Tr}\,(\phi_{12})^2$ (simpler!)

$$F_{\mathrm{Tr}\phi_{12}^2}(1,\ldots,n) \;=\; \int d^4x\; e^{-iqx}\; \langle state'|\, \mathrm{Tr}\, \phi_{12}^2(x)\, |0\rangle$$

  - $\mathrm{Tr}\ \phi^2{}_{12}$ and $\mathrm{Tr}\ F_{\mathrm{SD}}{}^2$ part of the same half-BPS supermultiplet
  - supersymmetric form factor of the chiral part of the stress tensor multiplet (Brandhuber, Gurdogan, Mooney, GT, Yang)

- Note: a priori no connection between QCD and N=4 SYM form factors, however comparing them will lead to a surprise…

Higgs → 3 gluons at 2 loops

(Brandhuber, GT, Yang)

- In N=4 SYM: 2 scalars, one gluon (MHV)

$$F_3(1,2,3) \;=\; \langle\phi_{12}(p_1)\,\phi_{12}(p_2)g^+(p_3)|\,\mathrm{Tr}(\phi_{12}\phi_{12})(0)\;|0\rangle$$

- A particularly simple form factor in N=4 super Yang-Mills
  - operator is protected from quantum corrections ("1/2 BPS")
- Loops: $F_3^{(L)} \;=\; F_3^{\mathrm{tree}}\,\mathcal{G}_3^{(L)}(1,2,3)$
  - $\mathcal{G}_3^{(L)}$ helicity-blind function, totally symmetric under legs exchange
  - one loop: IR divergences + sum of finite two-mass easy box
  - two loops: result encoded in finite remainder function

The form factor remainder

- Construct the ABDK/BDS finite remainder, $\mathcal{R}$

$$\boxed{\mathcal{R}_n^{(2)} := \mathcal{G}_n^{(2)} - \frac{1}{2}\left(\mathcal{G}_n^{(1)}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\,\mathcal{G}_n^{(1)}(2\epsilon) - C^{(2)} + \mathcal{O}(\epsilon)}$$

  - particular combination introduced for amplitudes by Anastasiou, Bern Dixon & Kosower and Bern, Dixon & Smirnov

  - Ingredients:

    - two-loop form factor $\mathcal{G}_n^{(2)}$, one-loop form factor $\mathcal{G}_n^{(1)}$ in dimensional regularisation ($D = 4 - 2\,\epsilon$)

    - $f^{(2)}(\epsilon) = -2\zeta_2 - 2\zeta_3\epsilon - 2\zeta_4\epsilon^2$ contains cusp and collinear anomalous dimensions (integrability!), $C^{(2)}(\epsilon) = 4\,\zeta_4$

  - Key properties:

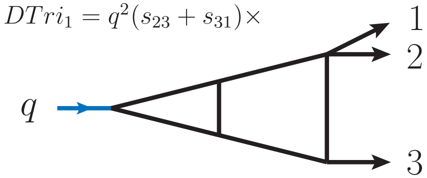
    1. finite: infrared divergences cancel (as in Bloch-Nordsiek)

    2. trivial collinear limits $\mathcal{R}_n^{(2)} \to \mathcal{R}_{n-1}^{(2)}$ (in particular: $\mathcal{R}_3^{(2)} \to 0$)

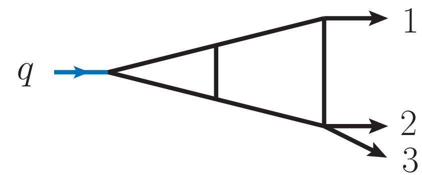
- Result of a unitarity-based two-loop calculation:

$$\boxed{\frac{F_3^{(2)}}{F_3^{\rm tree}} = \sum_{i=1}^{2}(DTri_i + DBox_i) + TriPent + NBox + NTri + \text{cyclic}}$$

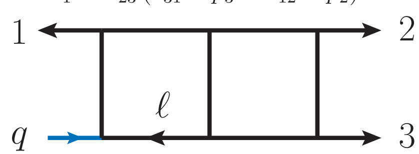
$DTri_1 = q^2(s_{23}+s_{31})\times$



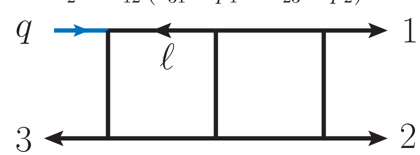
$DTri_2 = q^2(s_{12}+s_{31})\times$



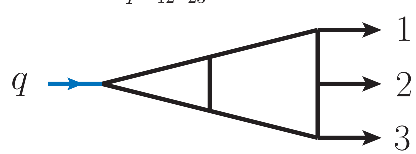
$DBox_1 = s_{23}\,(s_{31}\ell\cdot p_3 - s_{12}\ell\cdot p_2)\times$



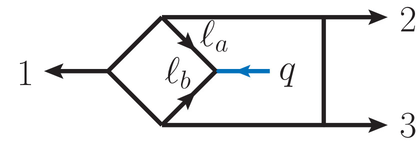
$DBox_2 = s_{12}\,(s_{31}\ell\cdot p_1 - s_{23}\ell\cdot p_2)\times$



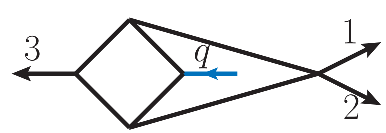
$TriPent = q^2 s_{12}s_{23}\times$



$NBox = s_{23}\left(\frac{1}{2}s_{12}s_{31} - s_{12}\ell_a\cdot p_2 - s_{31}\ell_b\cdot p_3\right)\times$



$NTri = \frac{1}{2}q^2(s_{23}+s_{31})\times$



- result expressed as rational coefficients × two-loop planar and non-planar integrals

- Some features of the result:
  - sum of transcendental functions, typically quite complicated: Goncharov's polylogarythms
  - defined recursively

$$G(a_1; z) \; := \int_0^z \frac{dt_1}{t_1 - a_1}\,, \qquad G(a_1, \vec{a}; z) \; := \int_0^z \frac{dt_1}{t_1 - a_1} G(\vec{a}; t_1)$$

  - compare to something simpler: classical polylogarithms

$$\mathrm{Li}_1(z) \; = \; -\log(1-z)\,, \qquad \mathrm{Li}_n(z) \; = \int_0^z \frac{dt}{t}\,\mathrm{Li}_{n-1}(t)$$

  - key finding: our result is a sum of functions of homogeneous degree of "transcendentality". All terms have transcendentality 4 (this will change later...)

Strategy

- Compute the symbol of the finite remainder
  - either by taking the symbol of the known (but complicated answer)…
  - or by computing it directly using symmetry properties & analyticity
    - finite, trivial/understood collinear limits
    - analiticity
    - need to know the possible letters
- "lift" it to a function
  - result might be remarkably simple, and in particular much simpler than the original expression!
  - fix "beyond-the-symbol" terms

- The unique symbol satisfying these requirements:

$$
\begin{aligned}
\mathcal{S}^{(2)} \;=\;& -2u \otimes (1-u) \otimes (1-u) \otimes \frac{1-u}{u} + u \otimes (1-u) \otimes u \otimes \frac{1-u}{u} \\
& -u \otimes (1-u) \otimes v \otimes \frac{1-v}{v} - u \otimes (1-u) \otimes w \otimes \frac{1-w}{w} \\
& -u \otimes v \otimes (1-u) \otimes \frac{1-v}{v} - u \otimes v \otimes (1-v) \otimes \frac{1-u}{u} \\
& +u \otimes v \otimes w \otimes \frac{1-u}{u} + u \otimes v \otimes w \otimes \frac{1-v}{v} \\
& +u \otimes v \otimes w \otimes \frac{1-w}{w} - u \otimes w \otimes (1-u) \otimes \frac{1-w}{w} \\
& +u \otimes w \otimes v \otimes \frac{1-u}{u} + u \otimes w \otimes v \otimes \frac{1-v}{v} \\
& +u \otimes w \otimes v \otimes \frac{1-w}{w} - u \otimes w \otimes (1-w) \otimes \frac{1-u}{u} \\
& + \text{cyclic permutations}\,.
\end{aligned}
$$

- four-fold tensor product ($2L$-fold at L loops, transcendentality $2L$)
- kinematic variables: $u_1 = u = s_{12} / q^2$, $u_2 = v = s_{23} / q^2$, $u_3 = w = s_{31} / q^2$ where $s_{ij} := (p_i + p_j)^2$ and $u_1 + u_2 + u_3 = 1$
- Note: coefficients ± 1, ± 2 (well... -2)

- How to "integrate" the symbol:
  - $\mathcal{S}^{(2)}$ satisfies a particular relation of Goncharov:

$$\mathcal{S}^{(2)}_{abcd} - \mathcal{S}^{(2)}_{bacd} - \mathcal{S}^{(2)}_{abdc} + \mathcal{S}^{(2)}_{badc} - (a \leftrightarrow c\,,\, b \leftrightarrow d) \;=\; 0$$

  - $\Rightarrow$ can re-express as a linear combination of classical polylogarithms only

$$\log x_1 \log x_2 \log x_3 \log x_4\;,\; \mathrm{Li}_2(x_1)\log x_2 \log x_3\;,\; \mathrm{Li}_2(x_1)\mathrm{Li}_2(x_2)\;,\; \mathrm{Li}_3(x_1)\log x_2 \text{ and } \mathrm{Li}_4(x_i)$$

  - we find the following arguments:

$$\left(u, v, w, 1-u, 1-v, 1-w, 1-\frac{1}{u}, 1-\frac{1}{v}, 1-\frac{1}{w}, -\frac{uv}{w}, -\frac{vw}{u}, -\frac{wu}{v}\right)$$

- Final answer is very compact

- **Final answer:** (Brandhuber, GT, Yang)

$$
\begin{aligned}
\mathcal{R}_3^{(2)} \;=\; & -2\left[\mathrm{J}_4\left(-\frac{uv}{w}\right)+\mathrm{J}_4\left(-\frac{vw}{u}\right)+\mathrm{J}_4\left(-\frac{wu}{v}\right)\right]-8\sum_{i=1}^{3}\left[\mathrm{Li}_4\left(1-u_i^{-1}\right)+\frac{\log^4 u_i}{4!}\right] \\
& -2\left[\sum_{i=1}^{3}\mathrm{Li}_2(1-u_i^{-1})\right]^2+\frac{1}{2}\left[\sum_{i=1}^{3}\log^2 u_i\right]^2-\frac{\log^4(uvw)}{4!}-\frac{23}{2}\zeta_4
\end{aligned}
$$

  - $u_1 = u = s_{12}/q^2$, $u_2 = v = s_{23}/q^2$, $u_3 = w = s_{31}/q^2$ kinematic invariants
  - $\mathrm{J}_4(z) := \mathrm{Li}_4(z) - \log(-z)\mathrm{Li}_3(z) + \frac{\log^2(-z)}{2!}\mathrm{Li}_2(z) - \frac{\log^3(-z)}{3!}\mathrm{Li}_1(z) - \frac{\log^4(-z)}{48}$.
  - Block-Wigner-Ramakrishnan(-Zagier) polylogarithmic function
  - Result is free of Goncharov polylogarithms

Next: QCD

Higgs amplitudes in QCD

- Higgs + 3 partons (Koukoutsakis 2003; Gehrmann, Glover, Jaquier & Koukoutsakis 2011)
  - $H\, g^+\, g^-\, g^-$ MHV
  - $H\, g^+\, g^+\, g^+$ maximally non-MHV
  - $H\, q\;\; \bar{q}\;\; g$ fundamental quarks

$$F^{\text{tree}}(H, g_1^-, g_2^-, g_3^+) = \frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle\,\langle 3\,1\rangle}$$

$$F^{\text{tree}}(H, g_1^+, g_2^+, g_3^+) = \frac{q^4}{[1\,2]\,[2\,3]\,[3\,1]}$$

$$q^2 = M_H^2$$

- In N=4 SYM:
  - $(H\, g^+\, g^-\, g^-)$ and $(H\, g^+\, g^+\, g^+)$ both derived from super form factor
  - from supersymmetric Ward identities: (Brandhuber, GT, Yang)

$$\frac{F^{(L)}(g_1^-, g_2^-, g_3^+)}{F^{\text{tree}}(g_1^-, g_2^-, g_3^+)} = \frac{F^{(L)}(g_1^+, g_2^+, g_3^+)}{F^{\text{tree}}(g_1^+, g_2^+, g_3^+)} = \mathcal{G}^{(L)}(u, v, w) \quad \leftarrow \text{what we computed}$$

- QCD answer from Gehrmann, Glover, Jaquier & Koukoutsakis
  - expressed in terms of several pages of Goncharov polylogarithms
  - transcendentality $4, 3, 2, 1$ and rational
  - entirely expected because of expansion as $\sum$ (coefficient x integral) !
    - each integral is separately quite complicated
- Next, compare N=4 form factors to Higgs amplitudes:
  - take maximally transcendental piece of $(H\, g^+\, g^-\, g^-)$ and $(H\, g^+\, g^+\, g^+)$

- We find a surprising connection...

$$\boxed{\mathcal{R}^{(2)}_{H\,g^-g^-g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{H\,g^+g^+g^+}\Big|_{\text{MAX TRANS}} = \mathcal{R}^{(2)}_{\mathcal{N}=4\,\text{SYM}}}$$

!

  - N=4 result is a particular part of the QCD result - in fact it is the "most complicated part"
  - all Goncharov polylogarithms in QCD results can be eliminated in favour of classical polylogarithms

- Nothing similar seems to hold for the form factor $(H, q, \bar{q}, g)$ (see also Duhr '12)

  - maximally transcendental part does not satisfy Goncharov et al criterion

Comments

- Typical presentation of the result of a calculation:
  - result = $\sum$ (coefficient x integral)
  - integrals are separately complicated, but final result is strikingly simple
  - there must be better way to present the result than Σ(coefficient x integral)

- Supersymmetry is a very useful organisational principle!
  - even if there is no supersymmetry…

What next?

- Obvious (but nontrivial) extensions:
  - different operators, more legs (Penante, Spence, GT, Wen; Brandhuber, Penante, GT, Wen)
  - further potential connections to phenomenology, e.g. in Higgs + 4 gluons

- Corrections due to the finiteness of the top mass
  - leading order term (infinite top mass limit) is the dimension-5 coupling studied earlier

$$\mathcal{L}_{\rm eff}^{(0)} \sim H \, {\rm Tr} F^2$$

  - next corrections from four dimension-7 operators, suppressed by powers of $1/m^2{}_{\rm top}$ (Buchmüller & Wyler; Neill; Harlander & Neumann)

- Look at this question with the N=4 SYM microscope…
  - identify couplings which are present also in N=4 SYM. Just two:

$$\mathcal{L}_{\rm eff}^{(1)} \sim H\, {\rm Tr} F^3 \qquad\qquad \mathcal{L}_{\rm eff}^{(2)} \sim H\, {\rm Tr}(D_\mu F_{\rho\sigma})(D^\mu F^{\rho\sigma})$$

  - compute in N=4 SYM
  - ideal plan: use Ward identities to connect to operators in the same multiplet but containing less derivatives / more scalars
  - compare to QCD

- Key questions & conjectures:
  - does the "maximal-transcendental connection" still holds?
  - any other interesting connection?

- Perform simpler "toy" calculations
  - Form factors of operators containing three fields in N=4 SYM
  - simpler than $\mathrm{Tr}\, F^3$. Operators with scalars!
  - Naturally leads to the $SU(2|3)$ sector studied by Beisert
  - Several possibilities, two broad classes:
    - protected operators (no UV divergences)
    - unprotected operators (with UV divergences)
  - interesting, unexpected connections between the two classes!

The two classes of operators:

- Protected
  - $\mathrm{Tr}\,(\phi_{12})^3$ half-BPS, form factors free of UV divergences
  - Generalisation: $\mathrm{Tr}\,(\phi_{12})^k$, also half-BPS $\forall\, k$
- Non-protected
  - Length 3: $\mathcal{O}_{\mathrm{B}} := \mathrm{Tr}\,(X\,[Y\,,Z])$ where $X = \phi_{12}\,,\ \ Y = \phi_{23}, \quad Z = \phi_{31}$
    - same one-loop anomalous dimension as $\mathrm{Tr}\,F^3$
  - Carries along a few dimension-three friends via operator mixing:
    - $\mathcal{O}_{\mathrm{BPS}} := \mathrm{Tr}\,(X\,\{Y\,,Z\})$, which is BPS (symmetric traceless)
    - $\mathcal{O}_{\mathrm{F}} := (1/2)\,\mathrm{Tr}\,(\psi\psi)$, which mixes with $\mathcal{O}_{\mathrm{B}}$ (and $\psi := \psi_{123}$)
  - This is the $SU(2|3)$ sector! The $SU(2|3)$ "dynamic" spin chain (Beisert '03)
    - key features: 1. closed sector, 2. length changing $(\psi\psi \leftrightarrow XYZ)$

- Two distinguished combinations:
  (Bianchi, Kovacs, Rossi, Stanev; Eden; …)

  1. an additional BPS operator $\mathcal{O}'_{\mathrm{BPS}} = (1/2)\,\mathrm{Tr}\,(\psi\psi) + g\,\mathrm{Tr}\,(X\,[Y\,,Z])$
     - can also be obtained by acting with 2 susy transformations on $\mathrm{Tr}\,(\phi_{12})^2$
  2. A descendant of the Konishi operator

$$\mathcal{O}_K \;=\; \mathrm{Tr}(X[Y,Z]) - \frac{gN}{8\pi^2}\mathrm{Tr}(\psi\psi)$$

- Four interesting calculations to carry out:

$\langle\, X\,Y\,Z \mid \mathrm{Tr}\,(X\,[Y\,,Z]) \mid 0\,\rangle$	minimal	harder
$\langle\, X\,Y\,Z \mid \mathrm{Tr}\,(\psi\psi) \mid 0\,\rangle$	non-minimal	v. easy
$\langle\, \psi\psi \mid \mathrm{Tr}\,(X\,[Y\,,Z]) \mid 0\,\rangle$	sub-minimal	easy
$\langle\, \psi\psi \mid \mathrm{Tr}\,(\psi\psi) \mid 0\,\rangle$	minimal ("Sudakov")	♻

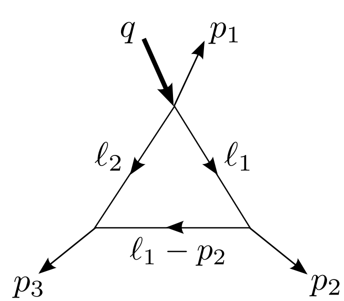
Protected operators

3-point form factor of $\mathrm{Tr}\phi^3$ at 2 loops

(Brandhuber, Penante, GT, Wen)

$$F_3(1,2,3) \;:=\; \langle \phi_{12}(p_1),\, \phi_{12}(p_2),\, \phi_{12}(p_3)\,|\,\mathrm{Tr}\,[(\phi_{12})^3](0)\,|0\rangle$$

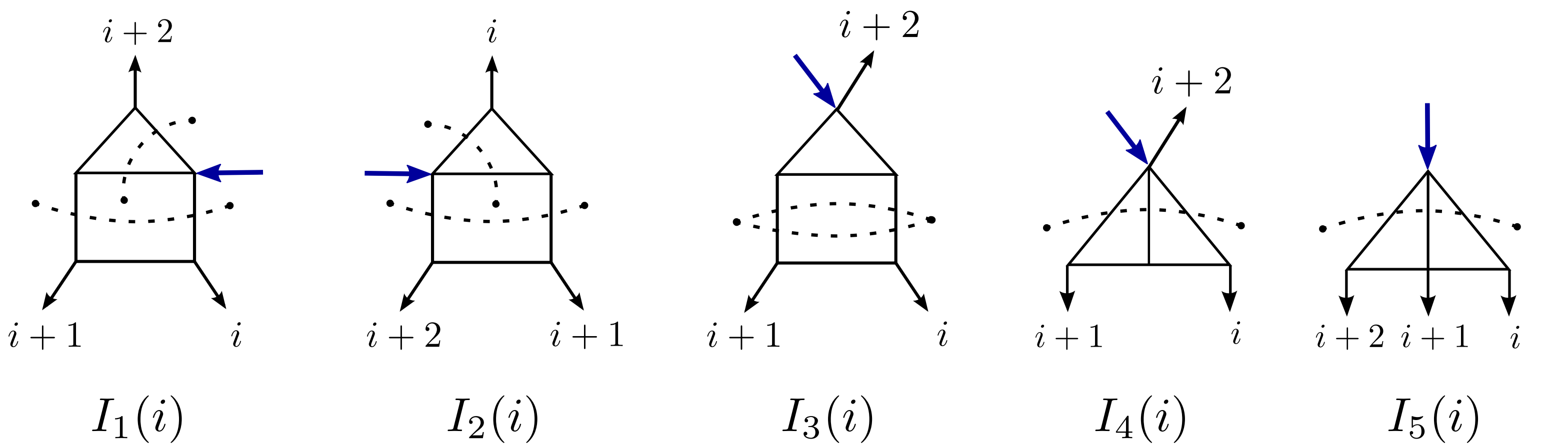
- "minimal form factor": as many particles as fields
- Tree: $F_3^{(0)}(1,2,3) \,=\, 1$
- One loop: sum of three "one-mass" triangles



\+ 2 cyclic perms

Result at two loops:

$$F^{(2)}_{\mathcal{T}_3,3} = \sum_{i=1}^{3} \Big[I_1(i) + I_2(i) + I_3(i) + I_4(i) - I_5(i) \Big]$$



- Result expressed in terms of two-loop planar integrals
- No sub-triangle and -bubble topologies on the amplitude side (no triangle theorem for N=4 SYM amplitudes)
- All integrals known from work of Gehrmann & Remiddi except 1 (and 2), decompose remaining ones using FIRE/LiteRed (Smirnov/Lee)
- Compute the symbol and lift it to a function

- The symbol of $\mathcal{R}_3$ is very simple!

☞

$$\boxed{\begin{aligned} \mathcal{S}_3^{(2)}(u,v,w) \;=\; & -\frac{3}{2}\, u\otimes(1-u)\otimes\frac{v}{w}\otimes\frac{v}{w} + \frac{1}{2}\, u\otimes u\otimes\frac{v}{w}\otimes\frac{v}{w} \\ & + u\otimes v\otimes\left(\frac{u}{w}\otimes\frac{v}{w}+\frac{v}{w}\otimes\frac{u}{w}\right) + \text{perms}\,(u,v,w) \end{aligned}}$$

- transcendentality four function ⇨ rank-four tensor
- entries: $(u, v, w, 1\text{-}u, 1\text{-}v, 1\text{-}w)$ $\quad u := \frac{s_{12}}{q^2}\,,\; v := \frac{s_{23}}{q^2}\,,\; w := \frac{s_{31}}{q^2}\,,$
- first entry: (u, v, w) for correct branch cuts (Gaiotto, Maldacena, Sever, Vieira)
  - $\mathcal{S}[\mathcal{R}^{(2)}] = \sum_{i,j} P_{i,j}^2 \otimes \mathcal{S}[\mathrm{disc}_{i,j}\mathcal{R}^{(2)}]$ with $P_{i,j} := p_i + \ldots + p_j$
- unusual second entry condition
- last entry condition: ratios of simple ratios only
- satisfies Goncharov, Spradlin, Vergu & Volovich's criterion, thus can be re-expressed in terms of classical polylogarithms only

- Table of symmetry properties from Goncharov, Spradlin, Vergu & Volovich:

Function	A $\otimes$ A	S $\otimes$ A	A $\otimes$ S	S $\otimes$ S
$\mathrm{Li}_4(x)$	$\times$	$\times$	$\checkmark$	$\checkmark$
$\mathrm{Li}_3(x)\ \log(y)$	$\times$	$\times$	$\checkmark$	$\checkmark$
$\mathrm{Li}_2(x)\,\mathrm{Li}_2(y)$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$
$\mathrm{Li}_2(x)\ \log(y)\ \log(z)$	$\times$	$\checkmark$	$\checkmark$	$\checkmark$
$\log(x)\ \log(y)\ \log(z)\ \log(w)$	$\times$	$\times$	$\times$	$\checkmark$

- Two more stringent properties of our symbol: AA[$\mathcal{S}^{(2)}$] = SA[$\mathcal{S}^{(2)}$] = 0

- Need: $\mathrm{Li}_4\,(x)$, $\mathrm{Li}_3\,(x)\log(x)$, $\log(x)\log(y)\log(z)\log(w)$ but no Li_2 !

- Entries: $\left\{u, v, w, 1-u, 1-v, 1-w, -\frac{u}{v}, -\frac{u}{w}, -\frac{v}{u}, -\frac{v}{w}, -\frac{w}{u}, -\frac{w}{v}, -\frac{uv}{w}, -\frac{uw}{v}, -\frac{vw}{u}\right\}$

- Final answer fits on a couple of lines…

- Final answer (including beyond the symbol terms):

$$\boxed{\begin{aligned}\mathcal{R}^{(2)}_{3,3} \;:=\; & -\frac{3}{2}\,\mathrm{Li}_4(u) + \frac{3}{4}\,\mathrm{Li}_4\left(-\frac{uv}{w}\right) - \frac{3}{2}\log(w)\mathrm{Li}_3\left(-\frac{u}{v}\right) + \frac{1}{16}\log^2(u)\log^2(v) \\ & + \frac{\log^2(u)}{32}\Big[\log^2(u) - 4\log(v)\log(w)\Big] + \frac{\zeta_2}{8}\log(u)[5\log(u) - 2\log(v)] \\ & + \frac{\zeta_3}{2}\log(u) + \frac{7}{16}\zeta_4 + \text{permutations}\,(u,v,w)\end{aligned}}$$

  - beyond the symbol terms: fixed using numerics (with GiNaC)
  - no Goncharov polylogarithms, no Li_2's

Non-BPS operators

Form factors in the $SU(2|3)$ sector

(Brandhuber, Kostacinska, Penante, GT, Young)

- Strategy:
  - compute the four form-factors in terms of two-loop integrals, using unitarity (two- and three-particle cuts)
  - compute the remainder functions
    - remainders are free of IR divergences; UV divergences still present
  - simplify the remainders using symbols, lift back to (simpler) functions
  - renormalise the operators, and resolve the mixing
    - eigenvalues of the mixing matrix: anomalous dimensions
    - eigenvectors: operators that diagonalise the dilatation operator

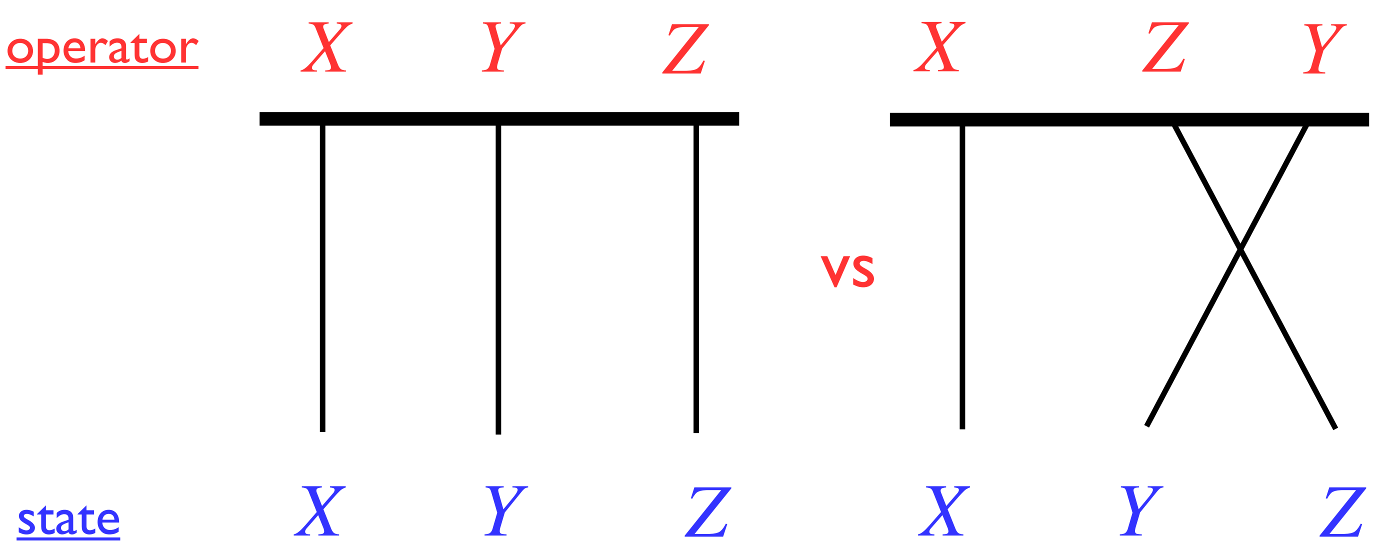
- The most interesting/complicated
  - minimal form factor $\langle\, X\, Y\, Z \,|\, \mathrm{Tr}\,(X\,[Y\,,Z])\,(0) \,|\, 0 \,\rangle$
- Key observation (very simple!)

$$\boxed{\mathrm{Tr}(X\,[Y\,,Z]) = \mathrm{Tr}(X\,\{Y\,,Z\}) - 2\,\mathrm{Tr}(X\,Z\,Y) := \mathcal{O}_{\mathrm{BPS}} + \mathcal{O}_{\mathrm{offset}}}$$

half-BPS

“one shuffling”, hence simpler

compare



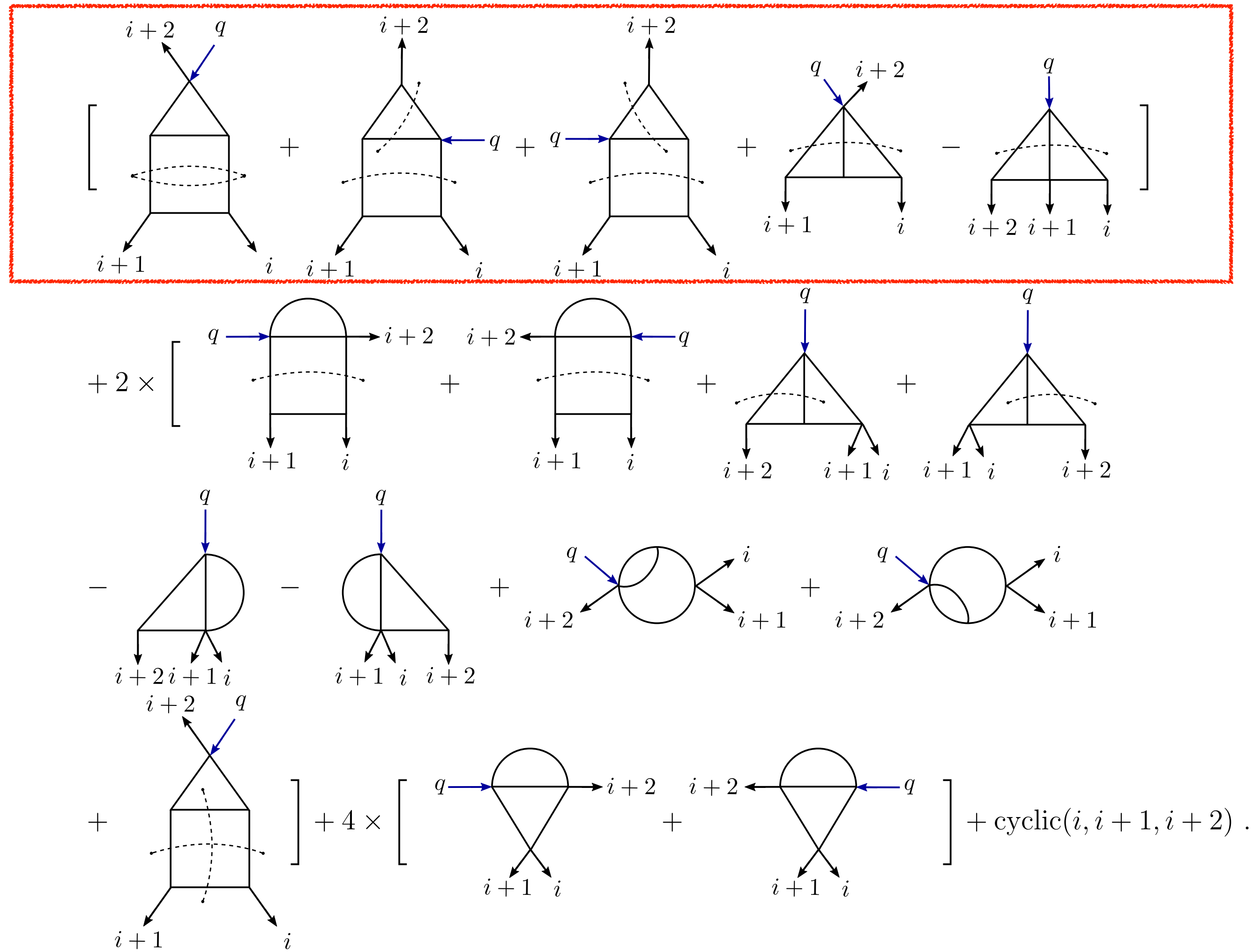
- What does "simple" mean:
  - $\langle\, X\, Y\, Z \mid \mathrm{Tr}\,(X\,\{Y\,,Z\})\ \mid 0\,\rangle$ (half-BPS) is maximally transcendental
  - equal to $\langle\, X\, X\, X \mid \mathrm{Tr}\,(X^3) \mid 0\,\rangle$ (discussed earlier)
  - $\langle\, X\, Y\, Z \mid \mathrm{Tr}\,(X\, Z\, Y) \mid 0\,\rangle$ has NO maximally transcendental piece
    - transcendentality equal to $3, 2, 1$ and 0 (rational terms) only

- A cute observation in the $SU(2)$ spin chain

(Loebbert, Nandan, Sieg, Wilhelm, Yang)

  - highest transcendentality of a "term" is $4 - s$ where $s = \#$ of shufflings
  - same happens here for $\langle\, X\, Y\, Z \mid \mathrm{Tr}\,(X\, Z\, Y)\ \mid 0\,\rangle$
  - one shuffling, hence transcendentality $3, 2, 1$ and rational

- Result for the remainder in terms of integral functions:



- first line corresponds to the half-BPS form factor
- dotted lines correspond to numerators in the integral functions
- presence of sub-bubbles points at UV divergences

- Remainder can be decomposed as

$$\boxed{\mathcal{R}^{(2)}{}_{X[Y,Z]} = \mathcal{R}^{(2)}{}_{\mathrm{BPS}} + \mathcal{R}^{(2)}{}_{\text{non-BPS}}}$$ where

$$\begin{aligned}
\mathcal{R}^{(2)}_{\mathrm{BPS}} &= F^{(2)}_{\mathcal{O}_{\mathrm{BPS}}}(\epsilon) - \frac{1}{2}\left(F^{(1)}_{\mathcal{O}_{\mathrm{BPS}}}(\epsilon)\right)^2 - f^{(2)}(\epsilon)\ F^{(1)}_{\mathcal{O}_{\mathrm{BPS}}}(2\epsilon) - C^{(2)}\ , \\
\mathcal{R}^{(2)}_{\text{non-BPS}} &= F^{(2)}_{\mathcal{O}_{\text{offset}}}(\epsilon) - F^{(1)}_{\mathcal{O}_{\text{offset}}}\left(\frac{1}{2}F^{(1)}_{\mathcal{O}_{\text{offset}}} + F^{(1)}_{\mathrm{BPS}}\right)(\epsilon) - f^{(2)}(\epsilon)\ F^{(1)}_{\mathcal{O}_{\text{offset}}}(2\epsilon)
\end{aligned}$$

- recall that $\mathcal{O}_{\mathrm{BPS}} := \ \mathrm{Tr}\,(X\{Y\,,Z\})\ ,\qquad \mathcal{O}_{\text{offset}} := \ -2\,\mathrm{Tr}\,(X\,Z\,Y)$
- BDS remainder free of IR but not UV divergences
- $\mathcal{R}^{(2)}{}_{\mathrm{BPS}}$ computed earlier, transcendentality-4 function

$$\begin{aligned}
\mathcal{R}^{(2)}_{\mathrm{BPS}} &= \frac{3}{2}\,\mathrm{Li}_4(u) - \frac{3}{4}\,\mathrm{Li}_4\left(-\frac{uv}{w}\right) + \frac{3}{2}\log(w)\,\mathrm{Li}_3\left(-\frac{u}{v}\right) - \frac{1}{16}\log^2(u)\log^2(v) \\
&\quad - \frac{\log^2(u)}{32}\left[\log^2(u) - 4\log(v)\log(w)\right] - \frac{\zeta_2}{8}\log(u)[5\log(u) - 2\log(v)] \\
&\quad - \frac{\zeta_3}{2}\log(u) - \frac{7}{16}\zeta_4 + \mathrm{perms}\,(u,v,w)
\end{aligned}$$

- Focus now on the new part, i.e. $\mathcal{R}^{(2)}{}_{\text{non-BPS}}$

$$\mathcal{R}^{(2)}_{\text{non-BPS}} = \frac{c}{\epsilon} + \sum_{i=0}^{3} \mathcal{R}^{(2)}_{\text{non-BPS};3-i}$$

  - $c = 18 - \pi^2$ this is the UV pole, π^2 "spurious" $-f^{(2)}(\epsilon)\; F^{(1)}_{\mathcal{O}_{\text{offset}}}(2\epsilon)$
  - "18" will enter the mixing matrix

$$\mathcal{R}^{(2)}_{\text{non-BPS};3} = 2\Big[\text{Li}_3(u) + \text{Li}_3(1-u)\Big] - \frac{1}{2}\log^2(u)\log\frac{vw}{(1-u)^2} + \frac{2}{3}\log(u)\log(v)\log(w) + \frac{2}{3}\zeta_3 + \text{perms}\,(u,v,w)$$

$$\mathcal{R}^{(2)}_{\text{non-BPS};2} = -12\Big[\text{Li}_2(1-u) + \text{Li}_2(1-v) + \text{Li}_2(1-w)\Big] - 2\log^2(uvw) + 36\,\zeta_2$$

$$\mathcal{R}^{(2)}_{\text{non-BPS};1} = -12\,\log(uvw)\;,$$

$$\mathcal{R}^{(2)}_{\text{non-BPS};0} = 126$$

  - transcendentality < 4, hence only classical polylogarithms

- Summary so far:
  - leading transcendental part of $\langle\, X\, Y\, Z \mid \mathrm{Tr}\,(X\ [Y, Z]) \mid 0\, \rangle$ same as for the half-BPS case $\langle\, X\, X\, X \mid \mathrm{Tr}\,(X^3) \mid 0\, \rangle$!

- Future goal: compare to $\langle\, g\ g\ g \mid\ \mathrm{Tr}\, F^3\, |0\rangle$
  - conjecture: maximally transcendental part computed by the form factor of the half-BPS operator $\mathrm{Tr}\,(X^3)$? This would parallel the situation for $\mathrm{Tr}\, F^2$ in QCD vs $\mathrm{Tr}\,(\phi_{12})^2$ in N=4 SYM…
  - if the conjecture is true, then…
  - … half-BPS operators in N=4 SYM have a prominent role in QCD!
  - Understand multiplet structure for $\mathrm{Tr}\, F^3$
  - Same one-loop anomalous dimension of $\mathrm{Tr}\,(X[Y\,, Z])$

An $SU(2) \Leftrightarrow SU(2|3)$ sector connection

or are we missing a trivial Ward identity?

- An intriguing connection with the remainder densities in the $SU(2)$ spin chain (Loebbert, Nandan, Sieg, Wilhelm, Yang)
- Contrast the two sectors:
  - $SU(2)$: two bosons, X and Y (scalars). Closed, no length change
  - $SU(2|3)$: $\phi_{12}=X$, $\phi_{23}=Y$, $\phi_{31}=Z$ and $\psi_{123;\alpha}$, $\alpha=1, 2$. Closed, length change
- LNSWY computed the two-loop spin-chain Hamiltonian
  - "open", equivalent to removing the trace (form factor of a product of fields, without the trace)
  - involves three sites at two loops
  - finite parts expressed in terms of remainder densities

- Interaction range 2 and 3 processes:
  - Range 2: 1. $XX \to XX$, 2. $XY \to XY$, 3. $XY \to YX$
  - Range 3: 1. $XXX \to XXX$, 2. $XXY \to XXY$, 3. $XYX \to XYX$,
    4. $XXY \to XYX$, 5. $XYX \to XXY$, 6. $XXY \to YXX$

- Focus on range 3
  - there are only 3 independent processes/remainder densities

$$\left(R_i^{(2)}\right)_{XXX}^{XXX}, \quad \left(R_i^{(2)}\right)_{XXY}^{XYX}, \quad \left(R_i^{(2)}\right)_{XXY}^{YXX}$$

  - i denotes the site
  - each remainder depends on $u_i = \dfrac{s_{ii+1}}{s_{ii+1i+2}}, v_i = \dfrac{s_{i+1i+2}}{s_{ii+1i+2}}, w_i = \dfrac{s_{ii+2}}{s_{ii+1i+2}}$
  - no particular symmetry in the u_i, v_i and w_i

- We find the following relations:

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};3} = -\sum_{S_3} \left(R_i^{(2)}\right)^{XYX}_{XXY}\Big|_3 \; + \; 6\,\zeta_3 \;,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};2} = -\sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_2 \; + \; 5\pi^2 \;,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};1} = -\sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_1 \;,$$

$$\frac{1}{2}\mathcal{R}^{(2)}_{\text{non-BPS};0} = -\sum_{S_3} \left[\left(R_i^{(2)}\right)^{XYX}_{XXY} - \left(R_i^{(2)}\right)^{YXX}_{XXY}\right]\Big|_0$$

  - $(R_i)|_m$ indicates the transcendentality-m part
  - S_3 denotes sum over all six permutations of $(u\ v, w)$

- Universality of form factors across different sectors?
  - or is there a trivial explanation for this result?

$SU(2|3)$ dilatation operator

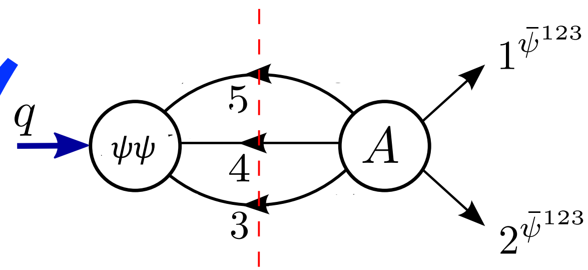
- Resolve mixing
$$\begin{pmatrix} \mathcal{O}_F^{\mathrm{ren}} \\ \mathcal{O}_B^{\mathrm{ren}} \end{pmatrix} = \begin{pmatrix} \mathcal{Z}_F{}^F & \mathcal{Z}_F{}^B \\ \mathcal{Z}_B{}^F & \mathcal{Z}_B{}^B \end{pmatrix} \begin{pmatrix} \mathcal{O}_F \\ \mathcal{O}_B \end{pmatrix}$$
  - $\mathcal{O}_{\mathrm{B}} := \mathrm{Tr}\,(X\,[Y\,,\,Z])$ and $\mathcal{O}_{\mathrm{F}} := (1/2)\,\mathrm{Tr}\,(\psi\psi)$

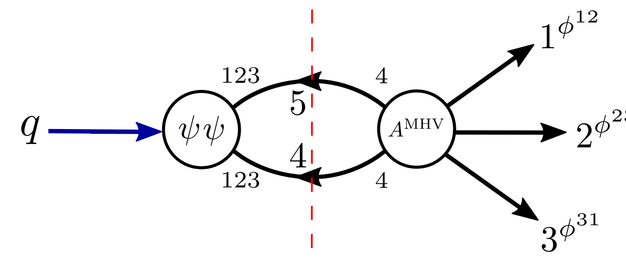
- Extract mixing matrix from requesting finiteness of the renormalised form factors
  - $\langle\, X\,Y\,Z\,|\;\; \mathrm{Tr}\,(X\,[Y\,,\,Z])\;\;\; |\,0\,\rangle$
  - $\langle\, X\,Y\,Z\,|\;\; \mathrm{Tr}\,(\psi\psi)\;\;\; |\,0\,\rangle$ (IR finite, starts at one loop)
  - $\langle\, \psi\psi\,|\;\; \mathrm{Tr}\,(X\,[Y\,,\,Z])\;\;\; |\,0\,\rangle$ (IR finite, starts at two loops)
  - $\langle\, \psi\psi\,|\;\; \mathrm{Tr}\,(\psi\psi)\;\;\; |\,0\,\rangle$
- Dilatation operator $\delta\mathcal{D} = -\mu_R \dfrac{\partial}{\partial \mu_R} \log \mathcal{Z}$

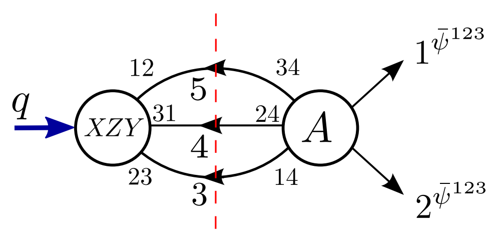
Up to two loops we have (schematically):

$g \times g^3 \times N^2 \sim a^2$ $\qquad$ $g^3 \times N \sim a\, g$

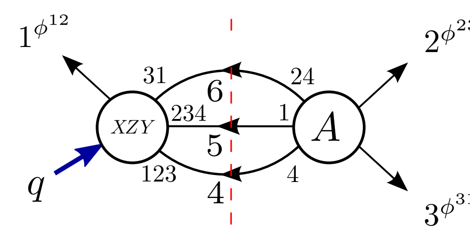
$\log \mathcal{Z} \sim$







1 loop +



$g^3 \times N^2 \sim a^2 / g$ $\qquad$ $g^2 \times N \sim a$ $\qquad$ $g \times g^3 \times N^2 \sim a^2$

$(a \sim g^2 N)$

$$\sim \frac{1}{\epsilon} \begin{pmatrix} \mathcal{O}(a^2) & \mathcal{O}(a\, g) \\ \mathcal{O}(a^2/g) & \mathcal{O}(a) + \mathcal{O}(a^2) \end{pmatrix}$$

- Result for $\log(\mathcal{Z})$:

$$\log(\mathcal{Z}) = \begin{pmatrix} a^2(\mu_R)\,\frac{6}{\epsilon} & -a(\mu_R)\cdot g\,\frac{6}{\epsilon} \\ -\frac{a^2(\mu_R)}{g}\cdot\frac{6}{\epsilon} & a(\mu_R)\cdot\frac{6}{\epsilon} - a^2(\mu_R)\cdot\frac{18}{\epsilon} \end{pmatrix}$$

  - running 't Hooft coupling: $a(\mu_R) := \frac{g^2 N e^{-\epsilon\gamma}}{(4\pi)^{2-\epsilon}}\left(\frac{\mu_R}{\mu}\right)^{-2\epsilon}$

- two-loop dilatation operator:

$$\delta\mathcal{D} = \lim_{\epsilon\to 0}\left[-\mu_R\frac{\partial}{\partial\mu_R}\log(\mathcal{Z})\right] = 12\times\begin{pmatrix} 2a^2 & -a\,g \\ -2\,\frac{a^2}{g} & a - 6\,a^2 \end{pmatrix}$$

  - 't Hooft coupling $a := \frac{g^2 N}{(4\pi)^2}$

- Next: eigenvalues and eigenvector

- Eigenvalues:
  - $\gamma_{\mathrm{BPS'}} = 0, \qquad \gamma_{\mathcal{K}'} = 12\, a - 48\, a^2 + \ldots.$
  - one further BPS combination, one descendent of the Konishi. Results in agreement with Beisert '03

- Eigenvectors:

$$\begin{cases} \mathcal{O}_{\mathrm{BPS}'} = \mathcal{O}_F + g\,\mathcal{O}_B \\ \mathcal{O}_{\mathcal{K}'} = \mathcal{O}_B - \dfrac{gN}{8\pi^2}\,\mathcal{O}_F \end{cases}$$

  - recall that $\mathcal{O}_{\mathrm{B}} := \mathrm{Tr}\,(X\,[Y\,,Z])$ and $\mathcal{O}_{\mathrm{F}} := (1/2)\;\mathrm{Tr}\,(\psi\psi)$
  - $X = \phi_{12}\,,\;\; Y = \phi_{23},\quad Z = \phi_{31}\quad \psi := \psi_{123}$
  - agrees with Bianchi et al, Eden
  - BPS combination can also be obtained by explicitly acting with supersymmetry generators on $\mathrm{Tr}\,(\phi_{12}\,\phi_{12})$ (Intriligator & Skiba)

- Other research direction: derive the dilatation operator from amplitudes techniques (no time to discuss this!)
    - complete two-loop dilatation operator still not known
    - amplitudes symmetries (Yangian) could play an important role
    - one-loop approach in Brandhuber, Heslop, GT, Young ‘15

Summary

- Form factors in N=4 SYM appear in several interesting contexts
  - connection to Higgs amplitudes in QCD
  - possibly true also for higher-dimensional operators describing the corrections to the infinite top-mass approximation
  - can be used to compute the dilatation operator of the theory
- Can the connection between Higgs amplitudes in QCD and form factors in N=4 SYM be made (more) systematic?
- Universality of form factors across different sectors?