MINKOWSKI FUNCTIONALS IN COSMOLOGY

A unifying approach to higher–order statistics of large–scale structure

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1. Introduction

Ever since the first high–quality galaxy catalogues became available, statistics of large–scale structure has been an important tool to gain insight into the underlying physical processes and to allow for comparison of observations and predictions, both through theory and numerical experiment.

One of the first methods used is the two–point correlation function[11] which remains the most widely used statistics today. However, the inability to grasp important, non–Gaussian features of the large–scale structure was quickly realised. The necessity of incorporating higher–order statistics into the analysis has led to a variety of methods ranging from the void probability function[19] to genus statistics[9].

Like their predecessors, Minkowski functionals incorporate correlations of any order[8] and supply morphological, i.e. topological and geometrical information on a point distribution. In addition, they can be derived from a few simple requirements and in this sense characterise morphology in a complete and unique way.

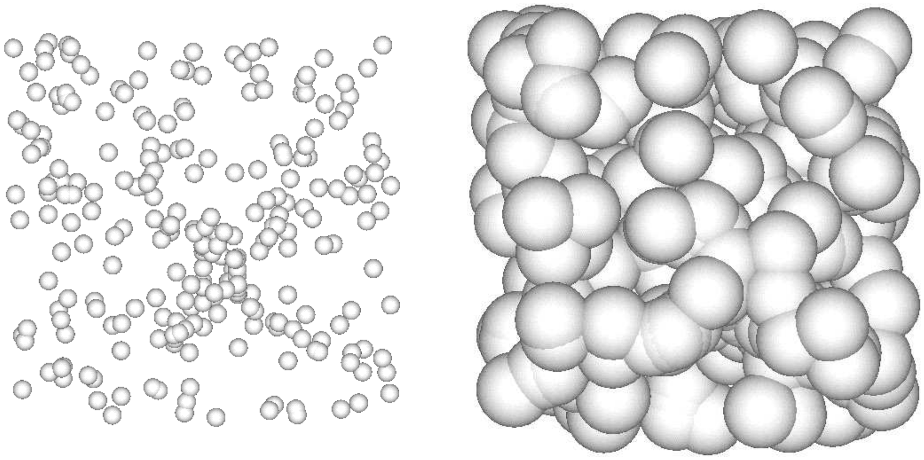

Figure 1. Boolean grain model, where the same set of points generates widely different patterns of balls. To the left, small balls form a pattern consisting of many disconnected parts, while to the right, almost all volume is covered by large balls forming an entangled network of tunnels and cavities, even though the underlying point distribution is the same.

In this talk we will give a brief outline of the mathematical background of Minkowski functionals before we present some cosmological applications.

2. Mathematical Background

Minkowski functionals were known for a long time[10] in integral geometry, a field of mathematics dealing with motions and measures of bodies in space (for an introductory overview of the field, consult Weil[18] or Schneider[16]). However, their applicability to cosmological datasets was not realised until recently[7]. While we restrict ourselves to a brief overview of the underlying concepts, more material can be found in works by the LMU cosmology group[14, 15].

Consider the set of points $\{\mathbf{x}_i\}$, $i = 1 \ldots N$ supplied, for example, by the positions of galaxies in a catalogue. A standard way to measure its morphological properties is the Boolean grain model, which is widely used in statistical physics[6]. Each point becomes the centre of a ball $B_r(\mathbf{x}_i)$ of radius r, so neighbouring points form a connected pattern of intersecting balls. The morphological properties of the union set A_r, with

$$A_r = \bigcup_{i=1}^{N} B_r(\mathbf{x}_i), \tag{1}$$

depend on the radius r (see Figure 1) which can therefore be used as a

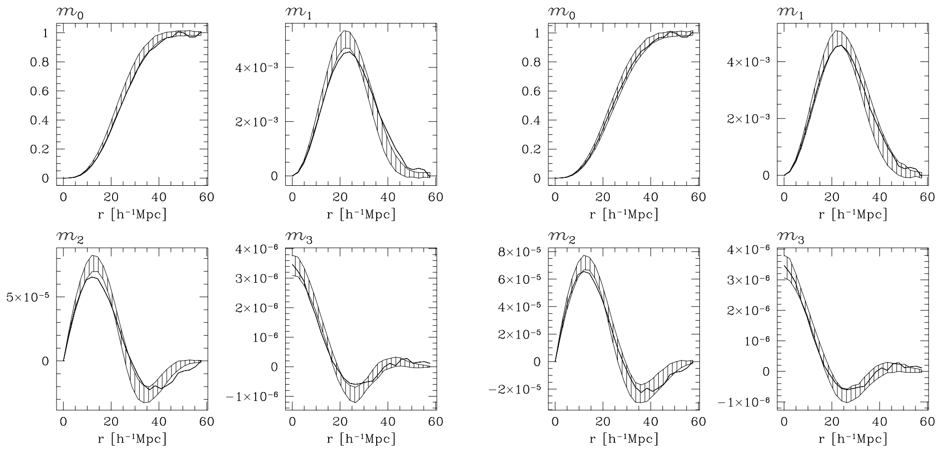

Figure 2. These panels show the Minkowski functionals of a sample of Abell/ACO clusters compared to the results of numerical simulations. The four panels on the left show the four Minkowski functionals of the Abell/ACO sample (solid line) compared to a standard cold dark matter simulation (shaded area; the shading gives an estimate of cosmic variance). Obviously, SCDM fails to reprduce the observed distribution of galaxy clusters, while adding a cosmological constant (right panels; again the solid line gives the Abell/ACO sample) yields compatible but still not perfect results.

diagnostic parameter.

In order to measure the morphological properties of this union set A_r in a quantitative fashion, we want to determine real numbers $M(A_r)$ that completely characterise the topology and geometry of A_r. Hadwiger's theorem[2] tells us that for a body in three dimensions, this can be done by evaluating only four independent quantities M_0, M_1, M_2 and M_3, the so–called *Minkowski functionals*. They can be interpreted as quantities known from differential geometry, namely the volume, the surface area, the integrated mean curvature and the Euler characteristic. Measured as functions of radius, they give a complete and unique characterisation of the point process under consideration.

3. Applications

3.1. CATALOGUES OF GALAXIES AND CLUSTERS

In analysing catalogues of galaxies or clusters, several issues arising from observational restrictions have to be adressed. We would like to mention three of them.

Redshift space distortions are an issue of ongoing research. For the samples

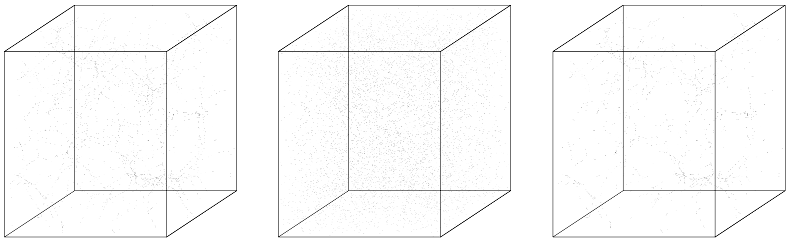

Figure 3. The three boxes in this figure illustrate structure determination using Minkowski functionals. The left box shows a sample of 3,000 points forming a filamentary structure. In the middle box 7,000 Poisson points have been added to mock a mixture of filaments and field galaxies. The right box shows the points identified as filamentary by investigating the behaviour of each point's partial Minkowski functionals. Obviously, the general features of the original structure have been recovered.

analysed so far, comparison of real and redshift space in numerical simulations showed no influence that would alter the significance of any of our results.

Today's surveys are magnitude limited which necessarily makes them inhomogeneous. By using a volume–limited subsample homogeneity in the number density is attained at the price of throwing away a considerable proportion of the points in the survey. A way to overcome this is the construction of a series of volume–limited samples of varying depth.

The boundary effects due to limitations both in depth and in sky coverage can be completely removed by applying the principal kinematical formula[12]. The detailed prescription we have to follow can be found in [3], and in several other publications. It turns out to be equivalent to a procedure described earlier for the void probability function[5], which has thus been validated rigorously using integral geometry.

As an example Figure 2 displays some results of a comparison of the Abell/ACO cluster catalogue by Kerscher et al.[4].

3.2. STRUCTURE DETERMINATION

It is also possible to define local quantities which for each point in a sample describe the local morphology nearby. We can then compare measured values of these so–called *partial Minkowski functionals* with theoretical expectations which are known in exact form for various types of structure, such as filaments, walls and field galaxies, and determine which type of local structure fits best.

Figure 3 shows first results of structure determination in a mock sample generated from a Voronoi tesselation[17]. Knowing where filaments were placed, we are able to assess the performance of the method – almost two thirds of the points in filaments have been recovered, with a contamination over two times less than the desired signal.

Compared to other methods (see for example Sathyaprakash et al.[13] or Davé et al.[1] for a comparison of several existent methods), this is certainly a promising result. A thorough evaluation of the performance of Minkowski functionals in structure determination is subject of ongoing research.

4. Outlook

Minkowski functionals are still emerging as tools for the analysis of large–scale structure. However, we have presented a number of interesting applications. An article by Martin Kerscher in this volume explains the main results of an analysis of the IRAS 1.2Jy catalogue, while numerical studies of the evolution both of clusters of galaxies and of large–scale structure are on the way. Furthermore, the field of integral geometry itself offers a number of possibly useful prospects, such as two–dimensional Minkowski functionals and Quermaßvectors.

Another issue of interest is to establish a connection between cosmic dynamics and Minkowski functionals, which would yield direct insight into the underlying physical processes. But even without this, Minkowski functionals offer a compelling and powerful approach to higher–order statistics of large–scale structure.

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