Exercise 1 – Integral representation of the Euler beta function

In class we discussed the Veneziano amplitude, which can be expressed in terms of the Euler beta function

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \ . \tag{1}$$

For $\mathrm{Re}(x) > 0$ and $\mathrm{Re}(y) > 0$, the Euler beta function has the integral representation

$$B(x,y) = \int_0^1 dt\, t^{x-1}(1-t)^{y-1} \ , \tag{2}$$

which arises in the scattering amplitude of four open string tachyons in flat space. Prove this integral representation using the integral representation of the Euler gamma function ($\mathrm{Re}(x) > 0$)

$$\Gamma(x) = \int_0^\infty du\, u^{x-1} e^{-u} \ . \tag{3}$$

Hint: One way to proceed is to start with the integral representation of $\Gamma(x)\Gamma(y)$ and making the change of variables $u = a^2$ in (3). Then go over to polar coordinates.

Exercise 2 – Equations of motion for a charged point particle

Consider the variation of the action

$$S = -m \int_{\mathcal{P}} ds + q \int_{\mathcal{P}} d\tau A_\mu(x(\tau)) \frac{dx^\mu}{d\tau}(\tau), \tag{4}$$

where

$$ds^2 = -\eta_{\mu\nu}\ dx^\mu(\tau)\ dx^\nu(\tau), \tag{5}$$

under a variation $\delta x^\mu(\tau)$ of the particle trajectory. $\mathcal{P}$ is the worldline of the particle and the integral along it amounts to an integral from τ_i to τ_f when the worldline is parameterised by τ.

Show that the variation of the first term gives

$$\delta S = -\int_{\tau_i}^{\tau_f} d\tau \delta x^\mu(\tau) \eta_{\mu\nu} \frac{d}{d\tau}\left(m \frac{dx^\nu}{ds} \right) \tag{6}$$

and that the final equation of motion is

$$\frac{dp_\mu}{d\tau} = q F_{\mu\nu} \frac{dx^\nu}{d\tau}, \tag{7}$$

where

$$\begin{aligned} p_\mu &= m u_\mu = m \frac{dx_\mu}{ds} \ , && (8) \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \ . && (9) \end{aligned}$$

Hint: Begin the second part of your calculation by explaining why

$$\delta A_\mu(x(\tau)) = \frac{\partial A_\mu}{\partial x^\nu}(x(\tau))\delta x^\nu(\tau). \tag{10}$$

Exercise 3 – Constraints

a) In class we derived the Hamiltonian equations in the presence of constraints, i.e.

$$\begin{aligned}
\dot{q}^n &= \frac{\partial H}{\partial p_n} + u^i \frac{\partial \phi_i}{\partial p_n} \,, \\
\dot{p}_n &= -\frac{\partial H}{\partial q^n} - u^i \frac{\partial \phi_i}{\partial q^n} \,, \\
\phi_i(q,p) &= 0 \,.
\end{aligned} \tag{11}$$

Show that they can be derived from the variational principle

$$\delta \int_{t_1}^{t_2} (\dot{q}^n p_n - H - u^i \phi_i) = 0 \tag{12}$$

for arbitrary variations $\delta q^n, \delta p_n, \delta u^i$ subject only to the restriction $\delta q^n(t_1) = 0, \delta q^n(t_2) = 0$.

b) Show that the Poisson-bracket of two first class constraints is again a first class constraint.